

# The genus of a group

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### THE GENUS OF A GROUP

by Karl GRUENBERG

Localization methods arise in infinite group theory and also, in a seemingly different incarnation, in integral representations of finite groups. Is there a common generalization ?

Let  $\pi$  be a finite set of primes. A group  $P$  is  $\pi$ -local if  $x \mapsto x^m$  is bijective for all integers  $m$  coprime to  $\pi$ . Every (abstract, discrete) group  $G$  has (essentially) a unique  $\pi$ -localization  $\phi_\pi: G \rightarrow G_\pi$  (meaning  $G_\pi$  is  $\pi$ -local and any homomorphism from  $G$  to a  $\pi$ -local group factors uniquely through  $\phi_\pi$ ). Guido Mislin and Peter Hilton began the study of localizations of finitely generated nilpotent groups that led Guido to introduce the *genus* of a finitely generated nilpotent group [2]. With this as a starting point, we make the following definition (it coincides with Guido's for finitely generated nilpotent groups whose centre has finite index): the *genus*  $\mathcal{G}(G)$  of  $G$  is all isomorphism classes  $[H]$  of groups  $H$  such that  $H$  is finitely generated and residually of finite exponent and  $H_\pi \simeq G_\pi$  for all finite sets  $\pi$ ; write  $H \vee G$ . (If all structural requirements on  $H$  were dropped, then  $\mathcal{G}(G)$  would be infinite for all  $G$ , which would not be a satisfactory situation.)

Guido's paper [2] is concerned with finitely generated nilpotent groups whose centre has finite index. What happens for finitely generated abelian-by-finite groups ? This question was successfully investigated by Niamh O'Sullivan ([3], [4], [5]). Her techniques involve the module version of genus.

Recall that if  $Q$  is a finite group and  $A, B$  are  $\mathbf{Z}Q$ -lattices, then  $A \vee B$  (same genus) means that  $A_\pi \simeq B_\pi$  for all finite sets  $\pi$ . A pointed lattice is a pair  $(A, x)$  where  $x \in H^2(Q, A)$  and  $(A, x) \vee (B, y)$  means there exists a  $Q$ -map  $f: A \rightarrow B$  with finite cokernel of order prime to  $|Q|$  (such maps exist if, and only if,  $A \vee B$ ) and  $f_*(x) = y$ . Let  $\mathcal{G}(A, x)$  denote all isomorphism classes  $[B, y]$  such that  $(A, x) \vee (B, y)$ .

Let  $G$  be a finitely generated abelian-by-finite group, choose  $m$  so that  $A := \langle g^m \mid g \in G \rangle$  is free abelian of finite index in  $G$  and write  $Q = G/A$ . Let  $x$  be the cohomology class of the resulting extension.

(1) *There is a well defined surjective map  $\theta: \mathcal{G}(A, x) \twoheadrightarrow \mathcal{G}(G)$  and  $\mathcal{G}(A, x)$  is finite.*

(2) *There is an explicitly defined subgroup  $J$  of  $\text{Aut}Q$  that acts on  $\mathcal{G}(A, x)$  and  $\theta$  induces a bijection  $\mathcal{G}(A, x)/J \xrightarrow{\sim} \mathcal{G}(G)$ .*

The point here is that the left hand side of (2) is better suited for calculations than is the right hand side: cf. O'Sullivan's papers for explicit examples, including new derivations of some of Guido's results.

*What is the natural level of generalization for this point of view?* For example, do the basic connexions that we have outlined *carry over to the class of polycyclic-by-finite groups*? A relevant fact here is that if  $G$  is such a group, then the number of isomorphism classes of polycyclic-by-finite groups  $H$  in the genus of  $G$  is known to be finite [1].

When Guido wrote his 1974 paper he added some problems at the end; so he clearly thought there was unfinished business here<sup>5</sup>). I rather think that this is still true today.

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<sup>5</sup>) His Problem 1 is solved in C. Casacuberta, C. Cassidy, D. Scevenels, 'On genus and embeddings of torsion-free nilpotent groups of class two', *Manuscripta Math.* 92 (1997) 463–475; his Problem 2 is solved in [5].