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Autor(en): **Gruenberg, Karl**

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THE GENUS OF A GROUP

by Karl GRUENBERG

Localization methods arise in infinite group theory and also, in a seemingly different incarnation, in integral representations of finite groups. Is there a common generalization ?

Let π be a finite set of primes. A group P is *π-local* if $x \mapsto x^m$ is bijective for all integers m coprime to π. Every (abstract, discrete) group G has (essentially) a unique *π-localization* $\phi_\pi \colon G \to G_\pi$ (meaning G_π is π-local and any homomorphism from G to a π-local group factors uniquely through ϕ_π). Guido Mislin and Peter Hilton began the study of localizations of finitely generated nilpotent groups that led Guido to introduce the *genus* of a finitely generated nilpotent group [2]. With this as a starting point, we make the following definition (it coincides with Guido's for finitely generated nilpotent groups whose centre has finite index): the *genus* $\mathcal{G}(G)$ of G is all isomorphism classes $[H]$ of groups H such that H is finitely generated and residually of finite exponent and $H_\pi \simeq G_\pi$ for all finite sets π; write $H \vee G$. (If all structural requirements on H were dropped, then $\mathcal{G}(G)$ would be infinite for all G, which would not be a satisfactory situation.)

Guido's paper [2] is concerned with finitely generated nilpotent groups whose centre has finite index. What happens for finitely generated abelian-by-finite groups ? This question was successfully investigated by Niamh O'Sullivan ([3], [4], [5]). Her techniques involve the module version of genus.

Recall that if Q is a finite group and A, B are $\mathbf{Z}Q$-lattices, then $A \vee B$ (same genus) means that $A_\pi \simeq B_\pi$ for all finite sets π. A pointed lattice is a pair (A, x) where $x \in H^2(Q, A)$ and $(A,x) \vee (B,y)$ means there exists a Q-map $f \colon A \rightarrowtail B$ with finite cokernel of order prime to $|Q|$ (such maps exist if, and only if, $A \vee B$) and $f_*(x) = y$. Let $\mathcal{G}(A, x)$ denote all isomorphism classes $[B, y]$ such that $(A, x) \vee (B, y)$.

Let G be a finitely generated abelian-by-finite group, choose m so that $A := \langle g^m \mid g \in G \rangle$ is free abelian of finite index in G and write $Q = G/A$. Let x be the cohomology class of the resulting extension.

(1) *There is a well defined surjective map* $\theta\colon \mathcal{G}(A,x) \twoheadrightarrow \mathcal{G}(G)$ *and* $\mathcal{G}(A,x)$ *is finite.*

(2) *There is an explicitly defined subgroup* J *of* $\mathrm{Aut}Q$ *that acts on* $\mathcal{G}(A,x)$ *and* θ *induces a bijection* $\mathcal{G}(A,x)/J \xrightarrow{\sim} \mathcal{G}(G)$.

The point here is that the left hand side of (2) is better suited for calculations than is the right hand side: cf. O'Sullivan's papers for explicit examples, including new derivations of some of Guido's results.

What is the natural level of generalization for this point of view? For example, do the basic connexions that we have outlined *carry over to the class of polycyclic-by-finite groups?* A relevant fact here is that if G is such a group, then the number of isomorphism classes of polycyclic-by-finite groups H in the genus of G is known to be finite [1].

When Guido wrote his 1974 paper he added some problems at the end; so he clearly thought there was unfinished business here[5]). I rather think that this is still true today.

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[5]) His Problem 1 is solved in C. Casacuberta, C. Cassidy, D. Scevenels, 'On genus and embeddings of torsion-free nilpotent groups of class two', *Manuscripta Math.* 92 (1997) 463–475; his Problem 2 is solved in [5].