Simplicial nonpositive curvature

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SIMPLICIAL NONPOSITIVE CURVATURE

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Classifying spaces for proper actions (that we denote by $\underline{E}G$) which interested Guido Mislin for a long time often arise from geometric considerations. A prime example is the following situation: Let X be a proper $CAT(0)$ geodesic metric space, and let G be a group admitting a properly discontinuous isometric action on X. Then X is $\underline{E}G$. To see this,

(1) one proves a fixed-point theorem for finite group actions on $CAT(0)$ spaces,

(2) one proves convexity properties, hence contractibility of fixed-point sets.

Recently in [2], Jacek Świątkowski and I studied a combinatorial analog of non-positive curvature. Our motivation came from cube complexes, which provide the richest source of high-dimensional $CAT(0)$ spaces. Here the $CAT(0)$ condition (on the geodesic metric for which every cube is a standard Euclidean cube) can be stated as a simple, checkable, combinatorial property of links: they should be flag simplicial complexes.

Then one tries to do the same for simplicial complexes. A condition equivalent to the $CAT(0)$ property for the geodesic metric (for which every simplex is a standard equilateral Euclidean simplex) is unknown (and finding it is probably hard). However there is a simple condition, which we call *systolicity*, that implies many of the consequences of $CAT(0)$, without actually implying $CAT(0)$ (and in high dimensions there are non-systolic triangulations for which geodesic metrics are $CAT(0)$).

The definition is this. Suppose L is a flag simplicial complex. Define the *systole* $sys(L)$ to be the minimum of length(γ), where γ is a full sub-complex of L homeomorphic to S^1 and the length of γ is the number of edges in γ. We say a simplicial complex X is *k-systolic* if it is simply connected and for any simplex σ, the systole of the link of σ is at least k. We say that a simplicial complex X is *systolic* if it is 6-systolic, and that a group G is *systolic* if it acts geometrically on a systolic complex.

Systolicity is a good analog of $CAT(0)$. We have proved significant parts of the $CAT(0)$ package. Alas, the fixed-point theorem is still open.

CONJECTURE 40.1. *A finite group F acting on a systolic complex X by simplicial automorphisms has a fixed point.*

We understand convexity well enough to be able to prove that fixed-point sets X^F are contractible if nonempty. So if Conjecture 40.1 is true, systolic spaces provide geometric models for the *classifying space $\underline{E}G$ for proper actions* of a systolic group G.

There are many examples of systolic spaces (admitting compact quotients) in every dimension, but they are somewhat exotic from the conventional perspective. Three (related) examples of their strange properties are:

(1) Systolic groups, that is fundamental groups of locally systolic spaces, do not contain fundamental groups of nonpositively curved Riemannian manifolds [3].

(2) Boundaries of Gromov hyperbolic systolic groups are *hereditarily aspherical* (every closed subset in ∂X is aspherical in appropriate Čech sense) [4].

(3) A systolic space X is *asymptotically hereditarily aspherical* [3]. This means that for every $r \geq 0$ there exists $R \geq r$ such that for every subcomplex $A \subset X$ the inclusion of Rips' complexes $R_r(A) \to R_R(A)$ induces the zero-map on homotopy groups π_i, for $i \geq 2$.

Study of asymptotic properties of X rather than of topological properties of a strange compactum ∂X is a shift of emphasis Guido should like. And in a sense, doing this, one obtains a more precise information about X.

One may speculate that the above three properties point towards a definition of a "dimension" according to which systolic groups are 2-dimensional. It was Dani Wise who insisted that systolic groups, some of which have large cohomological dimension are "essentially two-dimensional". We have found this to be a useful general guiding principle, and it motivates questions about non-systolic spaces. Here is an example.

Are there restrictions on the "dimension" of the boundary of a $CAT(-1)$ cubical complex? We do know that certain nice compact spaces (e.g. S^n, $n \geq 4$) are not boundaries of $CAT(-1)$ cube complexes (this is related to Vinberg's theorem on the absence of Coxeter groups acting cocompactly on the classical hyperbolic space $\mathbf{H}^n$ for large n, see [1]).

QUESTION 40.2. *What are topological restrictions on boundaries (or on asymptotic properties) of CAT(−1) cubical complexes? Can one find a restriction similar to (asymptotic) hereditary asphericity in the case of systolic spaces?*

A more precise, asymptotic version, using Rips' complex, is this:

QUESTION 40.3. *Let X be a CAT(-1) cube complex. Is it true that for every $r \geq 0$ there exists $R \geq r$ such that for every sub-complex $A \subset X$ the following property holds:*

For every map $f \colon S^k \to R_r(A)$, the composition $S^k \to R_r(A) \to R_R(A)$ factors, up to homotopy, through a 3-dimensional complex.

ADDED IN PROOF. Recently Piotr Przytycki has proved that if F is a finite group acting geometrically on a systolic space X, then there is a vertex in X, whose orbit has diameter at most 5. Equivalently, there is a fixed point for the induced F action on the Rips complex $R_5(X)$. He also proved that if G acts geometrically on a systolic complex X, then $R_5(X)$ is $\underline{E}G$ (see `http://www.mimuw.edu.pl/~pprzytyc/`).

REFERENCES

[1] JANUSZKIEWICZ, T. and J. ŚWIĄTKOWSKI. Hyperbolic Coxeter groups of large dimension. *Comment. Math. Helv. 78* (2003), 555–583.

[2] JANUSZKIEWICZ, T. and ŚWIĄTKOWSKI, J. Simplicial nonpositive curvature. *Publ. Math. Inst. Hautes Études Sci. 104* (2006), 1–85.

[3] JANUSZKIEWICZ, T. and ŚWIĄTKOWSKI, J. Filling invariants in systolic complexes and groups. *Geom. Topol. 11* (2007), 727–758.

[4] OSAJDA, D. Ideal boundary of 7-systolic complexes and groups. *Algebr. Geom. Topol. 8* (2008), 81–99.

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