Proper actions on acyclic spaces

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PROPER ACTIONS ON ACYCLIC SPACES

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Here are a few questions about proper cellular actions of discrete groups G on acyclic spaces. I have deliberately avoided the classifying space for proper G-actions, $\underline{E}G$, partly because some of the questions have already been answered for this space, and partly because I know that some other people will write in this volume about questions concerning $\underline{E}G$. I start with a version of the classic question that was posed by Ken S. Brown [1], p. 226.

QUESTION 48.1. *If G is of finite virtual cohomological dimension, does G act properly on some acyclic space of dimension equal to* $\mathrm{vcd}\, G$ *?*

REMARK 48.2. If $\mathrm{vcd}\, G$ is not equal to 2, then 'acyclic' in the above question can be replaced by 'contractible' without changing the question. The answer is 'yes' when $\mathrm{vcd}\, G = 1$ by a theorem of Martin Dunwoody [2], and Quillen's plus construction can be used to replace an acyclic space of dimension n by a contractible space of dimension equal to the maximum of n and 3.

Brita Nucinkis and I found examples to show that the dimension of the space $\underline{E}G$ can be strictly greater than $\mathrm{vcd}\, G$ [5]. Some of the techniques that we used in [5], including Bredon cohomology, were learned from Guido Mislin.

Secondly, a rather vague question. It is well known that $\mathrm{vcd}\, G$ is finite if and only if G is virtually torsion-free and G acts properly on some finite dimensional contractible space [1].

QUESTION 48.3. *Are there any results concerning group cohomology where virtual torsion-freeness plays a role ? For example, are there any results about* $H^*(G; \mathbf{Z}G)$ *that hold for groups of finite* vcd, *but do not hold for all groups in Peter Kropholler's class* $\mathrm{H}_1\mathfrak{F}$ *?*

Peter Kropholler's class $\mathrm{H}_1\mathfrak{F}$ consists of the groups that admit a proper action on some finite-dimensional contractible CW-complex. (See [3] for further details and for the definition of the larger class $\mathrm{H}\mathfrak{F}$.)

Finally, a few questions concerning the connection between algebraic and topological finiteness conditions. See also [4], [5].

QUESTION 48.4. *If* G *is of type* FP *over a ring* R*, does* G *act cellularly cocompactly on some* R*-acyclic CW-complex* X *with stabilizers whose orders are units in* R *?*

There is an algebraic version of this question too. Define a *projective permutation module* for the group algebra RG to be a direct sum of modules isomorphic to RG/H, where H ranges over the finite subgroups whose orders are units in R. Say that G is *of type* FPP *over* R if there is a finite resolution of R over RG by finitely generated projective permutation modules.

QUESTION 48.5. *If* G *is* FP *over* R*, is* G *necessarily of type* FPP *over* R *?*

For $R = \mathbf{Z}$, this question is equivalent to the famous question of whether every group of type FP is FL.

QUESTION 48.6. *If* G *is* FL *over a prime field* F*, does* G *act freely cellularly cocompactly on some* F*-acyclic CW-complex ?*

REMARK 48.7. There are groups that are FP but not FL over $\mathbf{Q}$, and are FL over $\mathbf{C}$ [4].

Such a group cannot act freely cellularly cocompactly on any $\mathbf{C}$-acyclic CW-complex. It is because of these examples that the previous question is stated only for the fields $\mathbf{Q}$ and $\mathbf{F}_p$.

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