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SOLUBLE GROUPS OF TYPE VF

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Cohomological finiteness conditions for soluble groups are very well understood for torsion-free soluble groups, but there remains a gap for soluble groups with torsion. This can be summarised by the following conjecture:

CONJECTURE 51.1. *Every soluble group G of type* VF *admits a cocompact model for* $\underline{\mathrm{E}}G$.

A group is said to be *of type* VF if it has a finite index subgroup admitting a finite $\mathrm{K}(G,1)$.

Finiteness conditions for soluble groups have been attracting wide attention ever since the celebrated result by Stammbach [9], that for a torsion-free soluble group the homological dimension $\mathrm{hd}\, G$ is equal to the Hirsch length $\mathrm{h}\, G$ of the group. For polycyclic groups the Hirsch length is equal to the cohomological dimension, $\mathrm{cd}\, G$. (For countable groups, the homological dimension and the cohomological dimension differ by at most one.)

This result was extended by Gruenberg, see [1], who showed that a torsion-free nilpotent group is finitely generated if and only if $\mathrm{cd}\, G = \mathrm{h}\, G$. This led to the question, which cohomological finiteness condition describes torsion-free soluble groups with $\mathrm{cd}\, G = \mathrm{h}\, G$.

There were partial answers to this question by numerous authors including Bieri, Gildenhuys and Strebel and it was finally answered by Kropholler [3]. Kropholler's later result [4] meant that this result could be phrased as follows:

THEOREM 51.2 ([3,4]). *Let G be a soluble group. Then the following are equivalent:*

(1) *G is of type* FP_∞,
(2) *G is virtually of type* FP,
(3) $\mathrm{vcd}\, G = \mathrm{h}\, G < \infty$,
(4) *G is virtually torsion-free and constructible.*

The fact that G is constructible implies that G is finitely presented. Thus any torsion-free group satisfying the conditions of the Theorem is *of type* F, i.e. it has a finite $\mathrm{K}(G, 1)$ or equivalently a cocompact model for EG.

In case G has torsion it cannot admit a finite dimensional model for EG, which is equivalent to saying that $\mathrm{cd}\, G = \infty$. We say a G-CW-complex X *is a model for* $\underline{E}G$ if X^H is contractible for all finite $H < G$ and empty otherwise. The cohomological counterpart is Bredon (co)homology. It was shown [8] that for countable groups the Bredon cohomological dimension $\underline{\mathrm{cd}}\, G$ and the Bredon homological dimension $\underline{\mathrm{hd}}\, G$ differ by at most one. By using a spectral sequence of Martínez-Pérez [7], Flores and Nucinkis [2] proved the analogue to Stammbach's result, namely that for soluble groups, $\underline{\mathrm{hd}}\, G = \mathrm{h}\, G$. This led us to pose the following conjecture:

CONJECTURE 51.3. *Let G be a soluble group. Then the following are equivalent:*

(1) *G is of type* $\underline{\mathrm{FP}}_\infty$,
(2) $\underline{\mathrm{cd}}\, G = \mathrm{h}\, G < \infty$,
(3) *G is of type* FP_∞.

$\underline{\mathrm{FP}}_\infty$ denotes the Bredon analogue to FP_∞. It is not hard to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), see [2]. There are, however, examples by Leary and Nucinkis [5] showing that generally groups of type VF do not necessarily admit a cocompact model for $\underline{E}G$, but all available evidence leads us to believe that Conjecture 51.1 still holds for soluble groups. Lück [6] showed that a group admits a model of finite type for $\underline{E}G$ if and only if it is finitely presented of type FP_∞, has finitely many conjugacy classes of finite subgroups and all centralisers of finite subgroups are of type FP_∞. But even with this reduction, an answer to both conjectures remains frustratingly elusive.

REFERENCES

[1] BIERI, R. *Homological Dimension of Discrete Groups.* Queen Mary College Mathematics Notes, London, 1976.

[2] FLORES R. J. and B. E. A. NUCINKIS. On Bredon homology of elementary amenable groups. *Proc. Amer. Math. Soc. 135* (2007), 5–11.

[3] KROPHOLLER, P. H. Cohomological dimensions of soluble groups. *J. Pure Appl. Algebra 43* (1986), 281–287.

[4] —— On groups of type FP_∞. *J. Pure Appl. Algebra 90* (1993), 55–67.

[5] LEARY, I. J. and B. E. A. NUCINKIS. Some groups of type VF. *Invent. Math. 151* (2003), 135–165.

[6] LÜCK, W. The type of the classifying space for a family of subgroups. *J. Pure Appl. Algebra 149* (2000), 177–203.

[7] MARTÍNEZ-PÉREZ, C. A spectral sequence in Bredon (co)homology. *J. Pure Appl. Algebra 176* (2002), 161–173.

[8] NUCINKIS, B. E. A. On dimensions in Bredon homology. *Homology Homotopy Appl. 6* (2004), 33–47.

[9] STAMMBACH, U. On the weak homological dimension of the group algebra of solvable groups. *J. London Math. Soc (2) 2* (1970), 567–570.

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