Two CAT(0) group questions

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TWO CAT(0) GROUP QUESTIONS

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We say a group G acts *geometrically* on a complete, proper, geodesic metric space X if G acts properly discontinuously and cocompactly by isometries on X. If G acts geometrically on a CAT(0) space, then G is called a CAT(0) *group*. Recall that if G acts geometrically on a δ-hyperbolic metric space, then G is word hyperbolic.

For G word hyperbolic, the following facts are well known and can be found in [5]. Of course, many careful proofs have been written down in other places, most notably [1] and [4].

1. Any Cayley graph of G is δ-hyperbolic using the corresponding word metric.

2. The *boundary* of G, denoted ∂G, is well-defined up to homeomorphism — i.e., if G acts geometrically on spaces X and Y, then X and Y are quasi-isometric and this quasi-isometry extends to an (equivariant) homeomorphism of boundaries $\partial X \to \partial Y$.

3. G acts as a convergence group on ∂G.

4. G satisfies the Tits Alternative.

5. Any finite index subgroup and any finite extension of G are again word hyperbolic.

The two questions here involve the last two facts listed above, but for CAT(0) groups as opposed to word hyperbolic groups. If G is word hyperbolic, it is easy to see that any finite index subgroup or any finite extension of G is again word hyperbolic. Indeed, any such group is quasi-isometric to G and thus inherits word hyperbolicity via the quasi-isometry.

If G is a CAT(0) group acting on a CAT(0) space X and H is a finite index subgroup of G, then H is again a CAT(0) group. Indeed, any subgroup of G will again act properly discontinuously and by isometries. Since H is of finite index, H will also act cocompactly. But if K is a finite extension of G, then the question remains:

QUESTION 56.1. *Suppose K is a finite extension of a* CAT(0) *group G. Is K also a* CAT(0) *group?*

The main problem here is that there is no geometric construction that models the group theoretic finite extension. It is still the case that K and G are quasi-isometric groups, but there is no natural candidate for a CAT(0) space for K to act on. Well, that isn't quite true... there is a candidate. Suppose G is a finite index normal subgroup of K of index D and suppose G acts on a topological space X. A construction of Serre gives an action of K on the direct product of D copies of X (this construction can be found in [3]). In our setting, if G acts geometrically on X, then Serre's construction will produce a properly discontinuous and isometric action of K on the product of D copies of X (which is still CAT(0) using the product metric). The problem is finding a convex subspace on which K acts cocompactly.

The second question concerns the Tits Alternative for CAT(0) groups. Recall that a group G satisfies the *Tits Alternative* if for every subgroup H of G, either H is virtually solvable or H contains a free subgroup of rank 2.

If G is word hyperbolic, then G satisfies the Tits Alternative. This fact was first proved in [5]. The beauty of this result is that the proof is quite simple using the action of the group G on its boundary ∂G. The proof goes like this: suppose H is an infinite subgroup of G and consider the closure $\overline{H}$ of H inside $G \cup \partial G$. The *limit set* of H, denoted $\mathcal{L}(H)$, is $\overline{H} \cap \partial G$. One first shows that $|\mathcal{L}(H)| \geq 2$ — this follows from the fact that any infinite subgroup of G must contain an element of infinite order [7]. If $|\mathcal{L}(H)| = 2$, then H is virtually $\mathbf{Z}$. Otherwise, there must be two infinite order elements $a, b \in H$ with $\mathcal{L}(\langle a \rangle) \cap \mathcal{L}(\langle b \rangle) = \varnothing$. Using the dynamics of the action on ∂G, one can do a ping-pong argument using carefully chosen open sets around the limit points of these two cyclic subgroups to show that powers of a and b generate an F_2 in H.

For a CAT(0) group G acting on X, one could try to use the action of G on ∂X. However, this is not a convergence group action. For example, every element of $\mathbf{Z} \oplus \mathbf{Z}$ acts trivially on $\partial \mathbf{E}^2 \equiv S^1$ which cannot happen in a convergence group action. The most recent result of interest here is from M. Sageev and D. Wise for groups acting properly on CAT(0) cube complexes, see [6]. If such a group G has a bound on the order of finite subgroups then any subgroup either contains F_2 or is virtually a finitely generated abelian subgroup. If G does not have a bound on the order of finite subgroups, then the conclusion does not hold. Thus the following general question remains open:

QUESTION 56.2. *Does the Tits Alternative hold for G if G is a* CAT(0) *group?*

This question is still open even if G admits a geometric action on a CAT(0) manifold X.

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