

# Two CAT(0) group questions

Autor(en): **Ruane, Kim**

Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **54 (2008)**

Heft 1-2

PDF erstellt am: **23.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-109925>

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## 56

### TWO CAT(0) GROUP QUESTIONS

by Kim RUANE

We say a group  $G$  acts *geometrically* on a complete, proper, geodesic metric space  $X$  if  $G$  acts properly discontinuously and cocompactly by isometries on  $X$ . If  $G$  acts geometrically on a CAT(0) space, then  $G$  is called a CAT(0) *group*. Recall that if  $G$  acts geometrically on a  $\delta$ -hyperbolic metric space, then  $G$  is word hyperbolic.

For  $G$  word hyperbolic, the following facts are well known and can be found in [5]. Of course, many careful proofs have been written down in other places, most notably [1] and [4].

1. Any Cayley graph of  $G$  is  $\delta$ -hyperbolic using the corresponding word metric.
2. The *boundary* of  $G$ , denoted  $\partial G$ , is well-defined up to homeomorphism — i.e., if  $G$  acts geometrically on spaces  $X$  and  $Y$ , then  $X$  and  $Y$  are quasi-isometric and this quasi-isometry extends to an (equivariant) homeomorphism of boundaries  $\partial X \rightarrow \partial Y$ .
3.  $G$  acts as a convergence group on  $\partial G$ .
4.  $G$  satisfies the Tits Alternative.
5. Any finite index subgroup and any finite extension of  $G$  are again word hyperbolic.

The two questions here involve the last two facts listed above, but for CAT(0) groups as opposed to word hyperbolic groups. If  $G$  is word hyperbolic, it is easy to see that any finite index subgroup or any finite extension of  $G$  is again word hyperbolic. Indeed, any such group is quasi-isometric to  $G$  and thus inherits word hyperbolicity via the quasi-isometry.

If  $G$  is a CAT(0) group acting on a CAT(0) space  $X$  and  $H$  is a finite index subgroup of  $G$ , then  $H$  is again a CAT(0) group. Indeed, any subgroup of  $G$  will again act properly discontinuously and by isometries. Since  $H$  is

of finite index,  $H$  will also act cocompactly. But if  $K$  is a finite extension of  $G$ , then the question remains:

QUESTION 56.1. *Suppose  $K$  is a finite extension of a CAT(0) group  $G$ . Is  $K$  also a CAT(0) group?*

The main problem here is that there is no geometric construction that models the group theoretic finite extension. It is still the case that  $K$  and  $G$  are quasi-isometric groups, but there is no natural candidate for a CAT(0) space for  $K$  to act on. Well, that isn't quite true... there is a candidate. Suppose  $G$  is a finite index normal subgroup of  $K$  of index  $D$  and suppose  $G$  acts on a topological space  $X$ . A construction of Serre gives an action of  $K$  on the direct product of  $D$  copies of  $X$  (this construction can be found in [3]). In our setting, if  $G$  acts geometrically on  $X$ , then Serre's construction will produce a properly discontinuous and isometric action of  $K$  on the product of  $D$  copies of  $X$  (which is still CAT(0) using the product metric). The problem is finding a convex subspace on which  $K$  acts cocompactly.

The second question concerns the Tits Alternative for CAT(0) groups. Recall that a group  $G$  satisfies the *Tits Alternative* if for every subgroup  $H$  of  $G$ , either  $H$  is virtually solvable or  $H$  contains a free subgroup of rank 2.

If  $G$  is word hyperbolic, then  $G$  satisfies the Tits Alternative. This fact was first proved in [5]. The beauty of this result is that the proof is quite simple using the action of the group  $G$  on its boundary  $\partial G$ . The proof goes like this: suppose  $H$  is an infinite subgroup of  $G$  and consider the closure  $\bar{H}$  of  $H$  inside  $G \cup \partial G$ . The *limit set* of  $H$ , denoted  $\mathcal{L}(H)$ , is  $\bar{H} \cap \partial G$ . One first shows that  $|\mathcal{L}(H)| \geq 2$  — this follows from the fact that any infinite subgroup of  $G$  must contain an element of infinite order [7]. If  $|\mathcal{L}(H)| = 2$ , then  $H$  is virtually  $\mathbf{Z}$ . Otherwise, there must be two infinite order elements  $a, b \in H$  with  $\mathcal{L}(\langle a \rangle) \cap \mathcal{L}(\langle b \rangle) = \emptyset$ . Using the dynamics of the action on  $\partial G$ , one can do a ping-pong argument using carefully chosen open sets around the limit points of these two cyclic subgroups to show that powers of  $a$  and  $b$  generate an  $F_2$  in  $H$ .

For a CAT(0) group  $G$  acting on  $X$ , one could try to use the action of  $G$  on  $\partial X$ . However, this is not a convergence group action. For example, every element of  $\mathbf{Z} \oplus \mathbf{Z}$  acts trivially on  $\partial \mathbf{E}^2 \cong S^1$  which cannot happen in a convergence group action. The most recent result of interest here is from M. Sageev and D. Wise for groups acting properly on CAT(0) cube complexes, see [6]. If such a group  $G$  has a bound on the order of finite subgroups then any subgroup either contains  $F_2$  or is virtually a finitely generated abelian

subgroup. If  $G$  does not have a bound on the order of finite subgroups, then the conclusion does not hold. Thus the following general question remains open :

QUESTION 56.2. *Does the Tits Alternative hold for  $G$  if  $G$  is a CAT(0) group ?*

This question is still open even if  $G$  admits a geometric action on a CAT(0) manifold  $X$ .

#### REFERENCES

- [1] ALONSO, J.M., T. BRADY, D. COOPER, V. FERLINI, M. LUSTIG, M. MIHALIK, M. SHAPIRO and H. SHORT. Notes on word hyperbolic groups. In : *Group Theory from a Geometrical Viewpoint*, ICTP, Trieste (1990), 3–63. World Scientific, Singapore, 1991.
- [2] BRIDSON, M.R. and A. HAEFLIGER. *Metric Spaces of Non-Positive Curvature*. Grundlehren der mathematischen Wissenschaften 319. Springer-Verlag, Berlin, 1999.
- [3] BROWN, K.S. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer-Verlag, 1982.
- [4] GHYS, E. et P. DE LA HARPE (editors). *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Mathematics 83. Birkhäuser, 1990.
- [5] GROMOV, M. Hyperbolic groups. In : *Essays in Group Theory*, 75–263. Math. Sci. Res. Inst. Publ. 8. Springer-Verlag, New York and Berlin, 1987.
- [6] SAGEEV, M. and D. T. WISE. The Tits alternative for CAT(0) cubical complexes. *Bull. London Math. Soc.* 37 (2005), 706–710.
- [7] SWENSON, E.L. Hyperbolic elements in negatively curved groups. *Geom. Dedicata* 55 (1995), 199–210.

K. Ruane

Mathematics Department  
 Tufts University  
 Medford, MA 02155  
 USA  
*e-mail*: kim.ruane@tufts.edu