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Autor: Valette, Alain

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SHORT EXACT SEQUENCES AND A-T-MENABILITY

by Alain VALETTE

A locally compact, σ-compact group is *a-T-menable*, or has *the Haagerup property*, if it admits a metrically proper isometric action on a Hilbert space. The class of a-T-menable groups is a huge class, containing amenable groups, free groups, surface groups, Coxeter groups, and much more... (see [1] for more information on that class). The interest of this class stems from a remarkable result by N. Higson and G. Kasparov [3], that a-T-menable groups satisfy the strongest possible form of the Baum–Connes conjecture, namely the Baum–Connes conjecture with coefficients.

Given an interesting class of groups, it is a natural question to ask whether it is stable under short exact sequences. For a-T-menability, this is well known *not* to be the case: e.g. $\mathbf{Z}^2$ and $\mathrm{SL}_2(\mathbf{Z})$ are a-T-menable, but the semi-direct product $\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z})$ is not, because of the relative property (T) with respect to the normal subgroup.

QUESTION 62.1. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of locally compact groups, with N and Q a-T-menable. Under which conditions is G a-T-menable?*

For example, this is known to be the case if Q is amenable, as shown in [1]. Let us single out the case of central extensions in Question 62.1:

CONJECTURE 62.2. *Let $1 \to Z \to G \to Q \to 1$ be a central extension. If Q is a-T-menable, then so is G.*

Some evidence for that conjecture appears in [1] (in particular the case of the universal cover of $SU(n,1)$).

When H, Q are (non-trivial) countable groups, recall that the *wreath product* $H \wr Q$ is the semi-direct product $N \rtimes Q$, where $N = \oplus_Q H$ is a direct sum of copies of H indexed by Q, and Q acts on N by shifting indices.

QUESTION 62.3. *Assume that H and Q are a-T-menable. Is $H \wr Q$ a-T-menable ?*

As a particular case of Question 62.3, we single out what seems to be the first case to look at:

QUESTION 62.4. *Let $\mathbf{F}_2$ denote the free group on 2 generators, and let H be a (non-trivial) finite group. Is $H \wr \mathbf{F}_2$ a-T-menable ?*

The interest of these questions stems from a result of M. Neuhauser: if H, Q are a-T-menable, then $H \wr Q$ has no infinite subgroup with the relative property (T) (see Theorem 1.1 in [4]). So if the answer to these questions is negative, this would provide new examples of countable groups which are *not* a-T-menable, and do not contain any infinite subgroup with the relative property (T) (the first examples — certain S-arithmetic lattices — have been constructed by Y. de Cornulier ([2], Remarks 1.15 and 4.10)).

ADDED IN PROOF. Questions 62.3 and 62.4 have been solved affirmatively by Y. de Cornulier, Y. Stalder and the author. For the special case in 62.4, together with applications in harmonic analysis, see 'Proper actions of lamplighter groups associated with free groups', *C. R. Acad. Sci. Paris Math. 346* (2008), 173–176. For the general case in 62.3, see 'Haagerup properties and wreath products', paper in preparation.

REFERENCES

[1] CHERIX, P.-A., M. COWLING, P. JOLISSAINT, P. JULG and A. VALETTE. *Groups with the Haagerup Property.* Progress in Math. *197.* Birkhäuser, 2001.

[2] DE CORNULIER, Y. Kazhdan and Haagerup properties in algebraic groups over local fields. *J. Lie Theory 16* (2006), 67–82.

[3] HIGSON, N. and G. KASPAROV. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math. 144*(2001), 23–74.

[4] NEUHAUSER, M. Relative property (T) and related properties of wreath products. *Math. Z. 251* (2005), 167–177.

Alain Valette

Institut de Mathématiques — Université de Neuchâtel
Rue Emile Argand 11
CH-2007 Neuchâtel, Switzerland
e-mail: alain.valette@unine.ch