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TWO PROBLEMS CONCERNING POLYCYCLIC SPACES

by Oliver BAUES

A finite space (CW-complex) X is called a *nilpotent space* if the fundamental group $\pi_1(X)$ acts nilpotently on the homotopy groups of X. In particular, $\pi_1(X)$ is finitely generated nilpotent itself. Sullivan proved that the group of *homotopy equivalences* of a nilpotent space is, modulo finite kernels, commensurable with an *arithmetic group*.

Natural examples for nilpotent spaces are aspherical spaces X with torsion-free, finitely generated nilpotent fundamental group. For such spaces X, the statement about the homotopy equivalences corresponds to a purely group theoretic result on the *outer automorphism group* of $\pi_1(X)$.

For aspherical spaces with finitely generated nilpotent fundamental group natural *compact smooth model spaces* exist. These spaces are traditionally called *nilmanifolds*. Among all smooth manifolds representing a given nilpotent aspherical homotopy type, nilmanifolds are characterised by their distinctive geometric properties, for example, the existence of almost flat Riemannian metrics. Surprisingly, there do exist also *exotic smooth models* in a nilpotent aspherical homotopy type, which are then *not* diffeomorphic to any nilmanifold.

Quite close to nilmanifolds, but less well understood, are *solvmanifolds* and their finite *geometric* quotients, which are called *infrasolv-manifolds*. By definition, a solvmanifold is a homogeneous space for a solvable Lie group. Generalising nilmanifolds (which admit a transitive action of a nilpotent Lie group), these smooth manifolds do provide natural compact smooth models for aspherical manifolds with a torsion-free *polycyclic-by-finite* fundamental group.

Here come two problems, which are in the realm of the above ideas.

The first concerns the existence of "good" geometric structures on smooth aspherical compact manifolds with solvable fundamental group. (Note that, in this case, the fundamental group is necessarily a torsion-free polycyclic group.)

PROBLEM 1. A recent result states that a compact aspherical *Kähler manifold* with solvable fundamental group is (*diffeomorphic to*) an infra-nilmanifold, and it is also finitely covered by a *standard torus*. On the other hand, it is well known that there exist many solv- and nilmanifolds (not necessarily covered by a torus) which admit a complex manifold structure.

QUESTION 7.1. *Given any aspherical compact complex manifold with solvable fundamental group, is it diffeomorphic to an infra-solvmanifold ?*

The second problem concerns Sullivan's arithmeticity result for nilpotent spaces.

PROBLEM 2. As proved recently, the outer automorphism group of any polycyclic-by-finite group, and, hence, also the group of homotopy equivalences of any aspherical space with a polycyclic-by-finite fundamental group *is* an arithmetic group. Hence, we ask:

QUESTION 7.2. *Does Sullivan's arithmeticity result for nilpotent spaces carry over to a (suitable) more general class of* polycyclic spaces *?*

For more background on Problem 1, see [1], and the references therein. For Problem 2, see [2] and also the references there.

REFERENCES

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[2] BAUES, O. and F. GRUNEWALD. Automorphism groups of polycyclic-by-finite groups and arithmetic groups. *Publ. Math. Inst. Hautes Études Sci. 104* (2006), 213–268.

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