

The extended quotient

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THE EXTENDED QUOTIENT

by Paul BAUM

Let Γ be a finite group acting on a (topological space X or) an affine variety X . The quotient variety (or quotient topological space) X/Γ is obtained by collapsing each orbit to a point. For $x \in X$, Γ_x denotes the stabilizer group of x , that is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\},$$

and $c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x .

The *extended quotient* is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\tilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$, this is an affine variety and is a sub-variety of $\Gamma \times X$. Moreover, Γ acts on \tilde{X} as follows

$$\begin{aligned} \Gamma \times \tilde{X} &\longrightarrow \tilde{X} \\ (g, (\gamma, x)) &\mapsto g(\gamma, x) = (g\gamma g^{-1}, gx) \quad \text{where } g \in \Gamma, (\gamma, x) \in \tilde{X}. \end{aligned}$$

The extended quotient, denoted by $X//\Gamma$, is given by \tilde{X}/Γ , the ordinary quotient of \tilde{X} by the above action of Γ .

The extended quotient is an affine variety (or a topological space). The evident projection $\tilde{X} \rightarrow X$, $(\gamma, x) \mapsto x$ passes to quotient spaces to give a map $\rho: X//\Gamma \rightarrow X/\Gamma$. The map ρ is the projection of the extended quotient onto the ordinary quotient.

Let G be a reductive p -adic group (examples are $\mathrm{GL}(n, F)$ and $\mathrm{SL}(n, F)$, where F is any finite extension of the p -adic numbers \mathbf{Q}_p). Let V be a vector space over the complex numbers \mathbf{C} .

DEFINITION 8.1. A representation $\phi: G \rightarrow \mathrm{Aut}_{\mathbf{C}}(V)$ of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G .

We will denote by \widehat{G} the set of equivalence classes of smooth irreducible representations of G . One of the main problems in the representation theory of p -adic groups (which is closely related to the local Langlands conjecture) is to describe \widehat{G} .

The *Hecke algebra* of G , denoted by $\mathcal{H}G$, is the convolution algebra of all complex-valued locally-constant compactly-supported functions $f: G \rightarrow \mathbf{C}$. Then \widehat{G} is in bijection with $\text{Prim } \mathcal{H}G$, the set of primitive ideals in $\mathcal{H}G$. On $\text{Prim } \mathcal{H}G$ there is the Jacobson topology. Hence we may consider each connected component of the primitive ideal space. Typically there will be countably many of these connected components.

Let \mathbf{C}^\times denote the (complex) affine variety $\mathbf{C} - \{0\}$.

DEFINITION 8.2. A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbf{C}^\times \times \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times .$$

J. Bernstein assigns to each $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$ a complex torus T_α and a finite group Γ_α acting on T_α .

He then forms the quotient variety T_α/Γ_α and proves that there is a surjective map (the infinitesimal character)

$$\pi_\alpha: X_\alpha \twoheadrightarrow T_\alpha/\Gamma_\alpha .$$

The set X_α is the connected component of $\text{Prim } \mathcal{H}G$ corresponding to α . In Bernstein's work X_α is a set (i.e. is only a set) so π_α is a map of sets, which is surjective, finite-to-one and generically one-to-one.

CONJECTURE 8.3. *There is a certain resemblance between*

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \rho_\alpha \downarrow & \text{and} & \downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Here ρ_α is (as above) the projection of the extended quotient onto the ordinary quotient.

For the precise conjecture, see papers of A.-M. Aubert, P. Baum and R. Plymen [1] and [2], but we now explain what is meant by *resemblance*.

For each $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$ there exists (conjecturally) a bijection

$$\nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow X_\alpha$$

such that:

- In the possibly non-commutative diagram

$$\begin{array}{ccc} T_\alpha // \Gamma_\alpha & \xrightarrow{\nu_\alpha} & X_\alpha \\ \rho_\alpha \downarrow & & \downarrow \pi_\alpha \\ T_\alpha / \Gamma_\alpha & \xrightarrow{I} & T_\alpha / \Gamma_\alpha \end{array}$$

the bijection $\nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow X_\alpha$ is continuous, where $T_\alpha // \Gamma_\alpha$ has the Zariski topology, X_α the Jacobson topology, and the composition

$$\pi_\alpha \circ \nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$$

is a morphism of algebraic varieties.

- For each $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$, there is an algebraic family

$$\theta_t: T_\alpha // \Gamma_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$$

of morphisms of algebraic varieties, with $t \in \mathbf{C}^\times$, such that $\theta_1 = \rho_\alpha$ and $\theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$, where q is the order of the residue field of the p -adic field F over which G is defined, and π_α is the infinitesimal character of Bernstein.

This conjecture is true for $\text{GL}(n, F)$, where n is any positive integer and F is any finite extension of the p -adic numbers \mathbf{Q}_p , see [3].

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