The extended quotient

L'Enseignement Mathématique (2) 54 (2008), 23–26

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THE EXTENDED QUOTIENT

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Let Γ be a finite group acting on a (topological space X or) an affine variety X. The quotient variety (or quotient topological space) X/Γ is obtained by collapsing each orbit to a point. For $x \in X$, Γ_x denotes the stabilizer group of x, that is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\},$$

and $c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x.

The *extended quotient* is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$, this is an affine variety and is a sub-variety of $\Gamma \times X$. Moreover, Γ acts on $\widetilde{X}$ as follows

$$\begin{aligned} \Gamma \times \widetilde{X} &\longrightarrow \widetilde{X} \\ (g,(\gamma,x)) &\mapsto g(\gamma,x) = (g\gamma g^{-1}, gx) \qquad \text{where } g \in \Gamma,\ (\gamma,x) \in \widetilde{X}. \end{aligned}$$

The extended quotient, denoted by $X//\Gamma$, is given by $\widetilde{X}/\Gamma$, the ordinary quotient of $\widetilde{X}$ by the above action of Γ.

The extended quotient is an affine variety (or a topological space). The evident projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ passes to quotient spaces to give a map $\rho \colon X//\Gamma \to X/\Gamma$. The map ρ is the projection of the extended quotient onto the ordinary quotient.

Let G be a reductive p-adic group (examples are $\mathrm{GL}(n,F)$ and $\mathrm{SL}(n,F)$, where F is any finite extension of the p-adic numbers $\mathbf{Q}_p$). Let V be a vector space over the complex numbers $\mathbf{C}$.

DEFINITION 8.1. A representation $\phi \colon G \to \mathrm{Aut}_{\mathbf{C}}(V)$ of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

We will denote by $\widehat{G}$ the set of equivalence classes of smooth irreducible representations of G. One of the main problems in the representation theory of p-adic groups (which is closely related to the local Langlands conjecture) is to describe $\widehat{G}$.

The *Hecke algebra* of G, denoted by $\mathcal{H}G$, is the convolution algebra of all complex-valued locally-constant compactly-supported functions $f\colon G \to \mathbf{C}$. Then $\widehat{G}$ is in bijection with $\operatorname{Prim}\mathcal{H}G$, the set of primitive ideals in $\mathcal{H}G$. On $\operatorname{Prim}\mathcal{H}G$ there is the Jacobson topology. Hence we may consider each connected component of the primitive ideal space. Typically there will be countably many of these connected components.

Let $\mathbf{C}^\times$ denote the (complex) affine variety $\mathbf{C} - \{0\}$.

DEFINITION 8.2. A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbf{C}^\times \times \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times .$$

J. Bernstein assigns to each $\alpha \in \pi_0(\operatorname{Prim}\mathcal{H}G)$ a complex torus T_α and a finite group Γ_α acting on T_α.

He then forms the quotient variety T_α/Γ_α and proves that there is a surjective map (the infinitesimal character)

$$\pi_\alpha\colon X_\alpha \twoheadrightarrow T_\alpha/\Gamma_\alpha .$$

The set X_α is the connected component of $\operatorname{Prim}\mathcal{H}G$ corresponding to α. In Bernstein's work X_α is a set (i.e. is only a set) so π_α is a map of sets, which is surjective, finite-to-one and generically one-to-one.

CONJECTURE 8.3. *There is a certain resemblance between*

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \Big\downarrow{\scriptstyle\rho_\alpha} & \textit{and} & \Big\downarrow{\scriptstyle\pi_\alpha} \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Here ρ_α is (as above) the projection of the extended quotient onto the ordinary quotient.

For the precise conjecture, see papers of A.-M. Aubert, P. Baum and R. Plymen [1] and [2], but we now explain what is meant by *resemblance*.

For each $\alpha \in \pi_0(\mathrm{Prim}\,\mathcal{H}G)$ there exists (conjecturally) a bijection

$$\nu_\alpha \colon T_\alpha//\Gamma_\alpha \longrightarrow X_\alpha$$

such that:

- In the possibly non-commutative diagram

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & \xrightarrow{\ \nu_\alpha\ } & X_\alpha \\ {\scriptstyle \rho_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\alpha} \\ T_\alpha/\Gamma_\alpha & \xrightarrow[\ I\ ]{} & T_\alpha/\Gamma_\alpha \end{array}$$

  the bijection $\nu_\alpha \colon T_\alpha//\Gamma_\alpha \longrightarrow X_\alpha$ is continuous, where $T_\alpha//\Gamma_\alpha$ has the Zariski topology, X_α the Jacobson topology, and the composition

$$\pi_\alpha \circ \nu_\alpha \colon T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

  is a morphism of algebraic varieties.

- For each $\alpha \in \pi_0(\mathrm{Prim}\,\mathcal{H}G)$, there is an algebraic family

$$\theta_t \colon T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

  of morphisms of algebraic varieties, with $t \in \mathbf{C}^\times$, such that $\theta_1 = \rho_\alpha$ and $\theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$, where q is the order of the residue field of the p-adic field F over which G is defined, and π_α is the infinitesimal character of Bernstein.

This conjecture is true for $\mathrm{GL}(n, F)$, where n is any positive integer and F is any finite extension of the p-adic numbers $\mathbf{Q}_p$, see [3].

REFERENCES

[1] AUBERT, A.-M., P. BAUM and R. PLYMEN. The Hecke algebra of a reductive p-adic group: a geometric conjecture. In: *Noncommutative Geometry and Number Theory* (K. Consani and M. Marcolli, eds.) Aspects of Math. Vol. 37, Vieweg, Wiesbaden, 2006, 1–34.

[2] AUBERT, A.-M., P. BAUM and R. PLYMEN. Geometric structure in the representation theory of p-adic groups. *C. R. Acad. Sci. Paris. Sér. I Math. 345* (2007), 573–578.

[3] BRODZKI, J. and R. PLYMEN. Geometry of the smooth dual of $\mathrm{GL}(n)$. *C. R. Acad. Sci. Paris Sér. I Math. 331* (2000), 213–218.

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