The spine that was no spine

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THE SPINE THAT WAS NO SPINE

by Alexandra PETTET and Juan SOUTO *)

ABSTRACT. Let $\mathcal{T}_n$ be the Teichmüller space of flat metrics on the n-dimensional torus $\mathbf{T}^n$ and identify $\mathrm{SL}_n\mathbf{Z}$ with the corresponding mapping class group. We prove that the subset $\mathcal{Y}$ consisting of those points whose systoles generate $\pi_1(\mathbf{T}^n)$ is, for $n \geq 5$, not contractible. In particular, $\mathcal{Y}$ is not an $\mathrm{SL}_n\mathbf{Z}$-equivariant deformation retract of $\mathcal{T}_n$.

1. INTRODUCTION

For $n \geq 2$ let $\mathcal{T}_n$ be the Teichmüller space of flat metrics with unit volume on the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. To be more precise, $\mathcal{T}_n$ is the set of equivalence classes of unit volume flat metrics on $\mathbf{T}^n$, where two metrics ρ and ρ' are equivalent if there is an orientation preserving diffeomorphism $\phi \in \mathrm{Diff}_+(\mathbf{T}^n)$ homotopic to the identity with $\rho' = \phi^*\rho$. We consider on the Teichmüller space $\mathcal{T}_n$ the topology in which two classes of flat metrics ρ and ρ' are close if there is a diffeomorphism $\phi \in \mathrm{Diff}_+(\mathbf{T}^n)$ homotopic to the identity such that ρ' and $\phi^*\rho$ are close as tensors.

Every element $A \in \mathrm{SL}_n\mathbf{Z}$ induces an orientation preserving diffeomorphism $A \in \mathrm{Diff}_+(\mathbf{T}^n)$ which is said to be *linear*. We obtain thus a right action of $\mathrm{SL}_n\mathbf{Z}$ on $\mathcal{T}_n$:

$$\mathcal{T}_n \times \mathrm{SL}_n\mathbf{Z} \to \mathcal{T}_n\,, \quad (\rho, A) \mapsto A^*\rho$$

which is properly discontinuous. There exists a finite index subgroup Γ of $\mathrm{SL}_n\mathbf{Z}$ which acts freely; in particular, the contractibility of $\mathcal{T}_n$ implies that for any such subgroup Γ, the quotient $\mathcal{T}_n/\Gamma$ is an Eilenberg-MacLane space of type $K(\Gamma, 1)$.

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The *systole* $\mathrm{syst}(\rho)$ of a point $\rho \in \mathcal{T}_n$ is the length of the shortest homotopically essential geodesic in the flat torus $(\mathbf{T}^n, \rho)$. Let $\mathcal{S}(\rho)$ be the set of homotopy classes of geodesics in $(\mathbf{T}^n, \rho)$ with length $\mathrm{syst}(\rho)$; the elements in $\mathcal{S}(\rho)$ are known as the *systoles* of $(\mathbf{T}^n, \rho)$. Ash [1] proved that the systole function

$$\mathcal{T}_n \to (0, \infty), \quad \rho \mapsto \mathrm{syst}(\rho)$$

is an $\mathrm{SL}_n \mathbf{Z}$-equivariant topological Morse function, and so it is not surprising that it can be used to construct a particularly nice $\mathrm{SL}_n \mathbf{Z}$-equivariant *spine*, i.e., deformation retract, of $\mathcal{T}_n$. More precisely, the following result was proved, in a different language and much greater generality, by Ash [2]:

THEOREM 1.1 (Ash). *The subset $\mathcal{X}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates a finite index subgroup of $\pi_1(\mathbf{T}^n)$ is an $\mathrm{SL}_n \mathbf{Z}$-equivariant spine of $\mathcal{T}_n$.*

A flat torus whose systoles generate a finite index subgroup of the fundamental group is said to be *well-rounded*; hence Ash's spine $\mathcal{X}$ is known as the *well-rounded retract*. Observe that the well-rounded retract $\mathcal{X}$ is homeomorphic to a CW-complex with the same dimension as the virtual cohomological dimension $\mathrm{vcdim}(\mathrm{SL}_n \mathbf{Z}) = \frac{n(n-1)}{2}$ of $\mathrm{SL}_n \mathbf{Z}$.

From a geometric point of view, that the systoles generate a finite index subgroup of $\pi_1(\mathbf{T}^n)$ seems an unnecessarily relaxed condition. We say that a flat torus is *extremely well-rounded* if its systoles generate the full group $\pi_1(\mathbf{T}^n)$; the set of all such tori we denote by $\mathcal{Y}$. Notice that $\mathcal{Y}$ is also a CW-complex of dimension $\frac{n(n-1)}{2}$. The authors were led to wonder whether $\mathcal{Y}$ could be an $\mathrm{SL}_n \mathbf{Z}$-equivariant deformation retract of $\mathcal{T}^n$ as well. For $n = 2$, 3 and 4, this is known; for these cases the sets $\mathcal{X}$ and $\mathcal{Y}$ coincide [8, 10]. The goal of this note is to show that this fails to be true for $n \geq 5$.

THEOREM 1.2. *For $n \geq 5$, the subset $\mathcal{Y}$ of $\mathcal{T}_n$ consisting of extremely well-rounded points, i.e., those points ρ with the property that $\mathcal{S}(\rho)$ generates $\pi_1(\mathbf{T}^n)$, is not contractible and hence is not an $\mathrm{SL}_n \mathbf{Z}$-equivariant spine.*

In order to prove Theorem 1.2, we make use of the well-known identification between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbf{R}$. We discuss this identification in Section 2. For the convenience of the reader, we also sketch briefly the proof of Theorem 1.1 in Section 3. Now let Γ be a torsion free finite index subgroup of $\mathrm{SL}_n \mathbf{Z}$.

The action of Γ on S_n is free and hence the quotient $M_\Gamma = S_n/\Gamma$ is a manifold. Borel and Serre [5] constructed a compact manifold $\overline{M}_\Gamma$ with boundary $\partial\overline{M}_\Gamma$ whose interior is homeomorphic to M_Γ. In Section 4 we briefly describe how to construct non-trivial homology classes in $H_{\frac{n(n-1)}{2}}(M_\Gamma)$ and $H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$. These classes are then used in Section 5 to show that whenever Γ is as above and is contained in the kernel of the standard homomorphism $\mathrm{SL}_n \mathbf{Z} \to \mathrm{SL}_n \mathbf{Z}/2\mathbf{Z}$, the inclusion $\mathcal{Y}/\Gamma \to M_\Gamma$ is not surjective on the $\frac{n(n-1)}{2}$-homology; Theorem 1.2 follows.

Recently, after completion of this paper, the authors [9] extended Theorem 1.2, proving that in fact the well-rounded $\mathcal{X}$ retract does not contain any proper, closed, $\mathrm{SL}_n \mathbf{Z}$-invariant, contractible subset.

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2. GENERALITIES

We begin by fixing some notation that will be used in the sequel. We denote by $\{e_1, \dots, e_n\}$ and $\langle \cdot, \cdot \rangle$ the standard basis and scalar product on $\mathbf{R}^n$. If v and A are a vector and a matrix, we let ${}^t v$ and ${}^t A$ denote their transposes. Using this notation, $|v| = \sqrt{{}^t v v}$ is the standard euclidean norm on $\mathbf{R}^n$. If $\mathcal{S}$ is a subset of a group, we denote by $\langle \mathcal{S} \rangle$ the subgroup generated by $\mathcal{S}$; for example, $\mathbf{Z}^n = \langle \{e_1, \dots, e_n\} \rangle$. If $\mathcal{S}$ is a subset of a euclidean vector space, we denote by $\langle \mathcal{S} \rangle_{\mathbf{R}}$ the $\mathbf{R}$-linear subspace generated by $\mathcal{S}$ and by $\langle \mathcal{S} \rangle_{\mathbf{R}}^{\perp}$ its orthogonal complement. We will sometimes use the same symbol to denote both an equivalence class and a representative of the equivalence class. For example, we may use the same notation for an element in $\mathrm{SL}_n \mathbf{R}$, and for the corresponding element in the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}$, or in the even smaller quotient $S_n / \mathrm{SL}_n \mathbf{Z}$. We will consistently denote the homology class corresponding to a cycle β by $[\beta]$. All the homology groups considered below will have coefficients in the field $\mathbf{Z}/2\mathbf{Z}$ of two elements.

These platitudes out of the way, we recall briefly the identification between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbf{R}$. If ρ is a flat metric on $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ with unit volume $\mathrm{vol}(\mathbf{T}^n, \rho) = 1$, the universal cover $\mathbf{R}^n$ is a complete flat manifold with respect to the induced metric $\tilde{\rho}$. In particular, there is an orientation preserving isometry

$$\phi\colon (\mathbf{R}^n, \tilde{\rho}) \to (\mathbf{R}^n, \langle \cdot, \cdot \rangle).$$

The action by deck transformations of the fundamental group $\pi_1(\mathbf{T}^n)$ on $(\mathbf{R}^n, \tilde{\rho})$ is isometric. Conjugating this action by ϕ we obtain an action of $\pi_1(\mathbf{T}^n) = \mathbf{Z}^n$ on $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$, also by isometries. It follows from a classical result of Bieberbach [11] that the group $\phi \pi_1(\mathbf{T}^n) \phi^{-1}$ is a group of translations of $\mathbf{R}^n$. In other words, the isometry ϕ induces a homomorphism

$$\mathbf{Z}^n \to \mathbf{R}^n, \quad \gamma \mapsto \{x \mapsto (\phi \circ \gamma \circ \phi^{-1})(x)\}$$

with discrete and cocompact image. Any such homomorphism is the restriction to $\mathbf{Z}^n$ of an element in $\mathrm{SL}_n \mathbf{R}$. Different choices for the isometry ϕ yield homomorphisms which differ by post-composition with an orthogonal transformation of $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$, and hence elements in $\mathrm{SL}_n \mathbf{R}$ which differ by left-multiplication with an element in SO_n. Thus, to every flat metric on $\mathbf{T}^n$ we can associate a well-defined point in the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbf{R}$. Moreover, equivalent flat metrics on $\mathbf{T}^n$ induce the same point in S_n. We have thus a well-defined map

$$\mathcal{T}_n \to S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbf{R}. \tag{2.1}$$

The map (2.1) is a homeomorphism. Observe that under the identification (2.1), the action of $\mathrm{SL}_n \mathbf{Z}$ on $\mathcal{T}_n$ given in the introduction corresponds to the action on S_n by right multiplication.

As defined in the introduction, the systole $\mathrm{syst}(\rho)$ of a point $\rho \in \mathcal{T}_n$ is the length of the shortest non-trivial geodesic in $(\mathbf{T}^n, \rho)$ and $\mathcal{S}(\rho)$ is the set of homotopy classes of geodesics of length $\mathrm{syst}(\rho)$. Under the identification (2.1), for $A \in \mathrm{SL}_n \mathbf{R}$ we have

$$\mathrm{syst}(A) = \min_{v \in \mathbf{Z}^n, v \neq 0} |Av|$$

and

$$\mathcal{S}(A) = \{v \in \mathbf{Z}^n, |Av| = \mathrm{syst}(A)\}.$$

In particular, Ash's spine $\mathcal{X}$ of well-rounded tori and the complex $\mathcal{Y}$ of extremely well-rounded tori, considered in Theorem 1.2, are given by:

$$\begin{aligned}
\mathcal{X} &= \{\rho \in \mathcal{T}_n \mid \langle \mathcal{S}(\rho) \rangle \text{ has finite index in } \pi_1(\mathbf{T}^n)\} \\
&= \{A \in S_n \mid \langle \mathcal{S}(A) \rangle \text{ has finite index in } \mathbf{Z}^n\} \\
\mathcal{Y} &= \{\rho \in \mathcal{T}_n \mid \langle \mathcal{S}(\rho) \rangle = \pi_1(\mathbf{T}^n)\} \\
&= \{A \in S_n \mid \langle \mathcal{S}(A) \rangle = \mathbf{Z}^n\}.
\end{aligned}$$

As was also mentioned in the introduction, Ash [1] proved that the systole function

$$\mathcal{T}_n \to (0, \infty), \quad \rho \mapsto \operatorname{syst}(\rho)$$

is an $\mathrm{SL}_n \mathbf{Z}$-equivariant topological Morse function. Here we will only use that the systole function is proper when considered as a function on $S_n / \mathrm{SL}_n \mathbf{Z}$.

MAHLER'S COMPACTNESS THEOREM. *For every $\epsilon > 0$, the set of those $A \in S_n / \mathrm{SL}_n \mathbf{Z}$ with $\operatorname{syst}(A) \geq \epsilon$ is compact.*

Computations are simpler with matrices than with flat metrics, so in the sequel we will mainly work in the symmetric space S_n.

3. THE WELL-ROUNDED RETRACT

In this section we discuss briefly the proof of Theorem 1.1. See [2] for a complete proof of a more general version of this theorem.

THEOREM 1.1 (Ash). *The subset $\mathcal{X}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates a finite index subgroup of $\pi_1(\mathbf{T}^n)$ is an $\mathrm{SL}_n \mathbf{Z}$-equivariant spine of $\mathcal{T}_n$.*

Recall that given $\rho \in \mathcal{T}_n$, we denote by $\langle \mathcal{S}(\rho) \rangle$ the subgroup $\pi_1(\mathbf{T}^n)$ generated by the shortest non-trivial geodesics in $(\mathbf{T}^n, \rho)$. Identifying $\pi_1(\mathbf{T}^n)$ with $\mathbf{Z}^n$ we see that the subgroup $\langle \mathcal{S}(\rho) \rangle$ is a free abelian group with rank in $\{1, \ldots, n\}$. Moreover, $\operatorname{rank}\langle \mathcal{S}(\rho) \rangle = n$ if and only if $\langle \mathcal{S}(\rho) \rangle$ has finite index in $\pi_1(\mathbf{T}^n)$. For $k = 1, \ldots, n$ consider the set $\mathcal{X}_k$ of those points $\rho \in \mathcal{T}_n$ for which we have $\operatorname{rank}\langle \mathcal{S}(\rho) \rangle \geq k$. We have thus the following chain of nested $\mathrm{SL}_n \mathbf{Z}$-invariant subspaces:

$$\mathcal{X} = \mathcal{X}_n \subset \mathcal{X}_{n-1} \subset \cdots \subset \mathcal{X}_1 = \mathcal{T}_n .$$

In order to prove Theorem 1.1 it suffices to show that for $k = 1, \ldots, n-1$ the space $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n \mathbf{Z}$-equivariant spine of $\mathcal{X}_k$. In order to see that this is the case, we use freely the identification (2.1) discussed above between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbf{R}$.

Under this identification, a point $A \in S_n$ belongs to $\mathcal{X}_k \setminus \mathcal{X}_{k+1}$ if and only if the set $\mathcal{S}(A)$ generates a rank k subgroup of $\mathbf{Z}^n$. Equivalently, $\mathcal{S}(A)$ generates a k-dimensional $\mathbf{R}$-linear subspace $\langle \mathcal{S}(A) \rangle_{\mathbf{R}}$ of $\mathbf{R}^n$. Given $A \in \mathcal{X}_k$ and $\lambda \in \mathbf{R}$, consider the one-parameter family of linear maps

$$T_A^\lambda \in \mathrm{SL}_n \mathbf{R}, \qquad T_A^\lambda(v) = \begin{cases} e^{(n-k)\lambda} v & \text{for } v \in A\langle \mathcal{S}(A) \rangle_{\mathbf{R}} \\ e^{-k\lambda} v & \text{for } v \in (A\langle \mathcal{S}(A) \rangle_{\mathbf{R}})^\perp \end{cases} \tag{3.1}$$

where $(A\langle \mathcal{S}(A) \rangle_{\mathbf{R}})^\perp$ is the orthogonal complement in $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ of the image under A of $\langle \mathcal{S}(A) \rangle_{\mathbf{R}}$.

Now $T_A^0 A = A$, and if $A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1}$, there is some λ positive with $T_A^\lambda A \in \mathcal{X}_{k+1}$. For $A \in \mathcal{X}_k$ let $\tau(A) \geq 0$ be maximal such that

$$T_A^\lambda A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1} \quad \text{for all } \lambda \in \big(0, \tau(A)\big).$$

By definition $\tau(A) = 0$ for $A \in \mathcal{X}_{k+1}$. The function $A \mapsto \tau(A)$ is continuous on $\mathcal{X}_k$, which implies that

$$[0,1] \times \mathcal{X}_k \to \mathcal{X}_k, \quad (t, A) \mapsto T_A^{t\tau(A)} A \tag{3.2}$$

is continuous as well. By definition, this homotopy is $\mathrm{SL}_n \mathbf{Z}$-equivariant, starts with the identity, and ends with a projection of $\mathcal{X}_k$ to $\mathcal{X}_{k+1}$. This proves that $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n \mathbf{Z}$-equivariant spine of $\mathcal{X}_k$ for $k = 1, \ldots, n-1$, concluding the sketch of the proof of Theorem 1.1.

REMARK 3.1. Something must be done to verify the continuity of (3.2), as the map

$$\mathbf{R} \times \mathcal{X}_k \to \mathrm{SL}_n \mathbf{R}, \quad (\lambda, A) \mapsto T_A^\lambda A$$

itself is not continuous. The key point is that this map is continuous on $\mathbf{R} \times (\mathcal{X}_k \setminus \mathcal{X}_{k+1})$, and by definition $\tau(A) = 0$ for $A \in \mathcal{X}_{k+1}$.

We conclude this section with a couple of additional remarks about the structure of the well-rounded retract $\mathcal{X}$ and a computation of the virtual cohomological dimension of $\mathrm{SL}_n \mathbf{Z}$.

It is not difficult to prove that $\mathcal{X}_k$ is a co-dimension $k-1$ semi-algebraic set given by a locally finite collection of inequalities and quadratic algebraic equations. Hence $\mathcal{X}$ is homeomorphic to a CW-complex of dimension

$$\dim(\mathcal{X}) = \dim S_n - (n-1) = \frac{n(n-1)}{2}.$$

It is also easy to see that the well-rounded retract $\mathcal{X}$ is cocompact, although $\mathcal{X}_k$ is not cocompact for $k < n$.

The symmetric space S_n is contractible, hence so is $\mathcal{X}$. In particular, if Γ is a subgroup of $\mathrm{SL}_n\mathbf{Z}$ which acts freely on S_n, then $\mathcal{X}/\Gamma$ is an Eilenberg-MacLane space of type $K(\Gamma, 1)$, giving us the following upper bound on its cohomological dimension:

$$\mathrm{cdim}(\Gamma) \leq \dim(X) = \frac{n(n-1)}{2}.$$

The group $\mathrm{SL}_n\mathbf{Z}$ contains subgroups Γ of finite index which are torsion free and thus act freely on S_n. This yields the upper bound

$$\mathrm{vcdim}(\mathrm{SL}_n\mathbf{Z}) \leq \frac{n(n-1)}{2}$$

for the virtual cohomological dimension of $\mathrm{SL}_n\mathbf{Z}$. One can see that the upper bound is sharp as follows: Let N be the $\frac{n(n-1)}{2}$-dimensional subgroup of $\mathrm{SL}_n\mathbf{R}$ consisting of upper triangular matrices with units in the diagonal. The intersection $N \cap \mathrm{SL}_n\mathbf{Z}$ is a cocompact subgroup of N; hence for Γ as above $N/(N \cap \Gamma)$ is a closed manifold of dimension $\frac{n(n-1)}{2}$. The group N is contractible, hence $N/(N \cap \Gamma)$ is an Eilenberg-MacLane space of type $K(N \cap \Gamma, 1)$. Thus we have

$$\mathrm{cdim}(\Gamma) \geq \mathrm{cdim}(N \cap \Gamma) = \dim(N/(N \cap \Gamma)) = \frac{n(n-1)}{2}.$$

This implies that $\mathrm{vcdim}(\mathrm{SL}_n\mathbf{Z}) = \frac{n(n-1)}{2}$.

In the next section we will give an elementary argument to prove that the homology class $[N/(N \cap \Gamma)] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ is non-trivial.

4. Some topology

As mentioned some lines above, $\mathrm{SL}_n\mathbf{Z}$ contains a torsion free subgroup of finite index, and any such subgroup acts not only discretely, but also freely on S_n; hence the quotient $M_\Gamma = S_n/\Gamma$ is a manifold. Borel and Serre [5] proved that M_Γ is homeomorphic to the interior of a compact manifold $\overline{M}_\Gamma$ with boundary $\partial\overline{M}_\Gamma$. Identifying $\overline{M}_\Gamma$ with the complement of an open regular neighborhood of $\partial\overline{M}_\Gamma$, we consider the former as a submanifold of M_Γ in the sequel.

REMARK 4.1. Grayson [7] gave a construction of $\overline{M}_\Gamma$ directly as a submanifold of M_Γ, giving a new proof of some of Borel's and Serre's results. If we are only interested in constructing a compactification $\overline{M}_\Gamma$ as above, we can do the following: For $A \in \mathrm{SL}_n\mathbf{R}$ the series $\sum_{v\in\mathbf{Z}^n} e^{-|Av|}$ converges, and its value depends only on the class of A in S_n. In particular, the function

$$F\colon S_n \to \mathbf{R}, \quad F(A) = \sum_{v\in\mathbf{Z}^n} e^{-|Av|}$$

is well-defined, smooth, and descends to a function $f\colon M_\Gamma \to \mathbf{R}$. The function f is proper, and there is some constant L which bounds above the critical values of f. This implies that $f^{-1}[L,\infty)$ is a product, hence we can set $\overline{M}_\Gamma = f^{-1}[0,L]$.

Borel and Serre constructed the compactification $\overline{M}_\Gamma$ to study homological properties of Γ. We will only need some basic facts, well-known probably to experts and non-experts alike, which we deduce in an elementary way.

Recall that we always consider homology with coefficients in $\mathbf{Z}/2\mathbf{Z}$. By Lefschetz duality there is a non-degenerate pairing

$$\iota\colon H_{\frac{n(n-1)}{2}}(M_\Gamma) \times H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma) \to \mathbf{Z}/2\mathbf{Z}$$

which can be computed as follows. Given homology classes $[\alpha] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ and $[\beta] \in H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$, represent them by cycles α and β in general position. Then $\iota([\alpha],[\beta])$ is just the parity of the cardinality of the set $\alpha\cap\beta$.

REMARK 4.2. This is the simplest version of the Alexander-Whitney product in homology, which dualizes the cup product.

In particular, in order to prove that the $\frac{n(n-1)}{2}$-cycle $\alpha = N/(N\cap\Gamma)$ represents a non-trivial homology class it suffices to find a cycle $\beta \in C_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$ which intersects α transversally at a single point. In order to find such a cycle β we consider the subgroup Δ of $\mathrm{SL}_n\mathbf{R}$ consisting of diagonal matrices with positive entries and the map $\Delta \to M_\Gamma$ which maps every $H \in \Delta$ to its class in $M_\Gamma = \mathrm{SO}_n \backslash \mathrm{SL}_n\mathbf{R}/\Gamma$. By Mahler's compactness theorem, the systole function is proper on $S_n/\mathrm{SL}_n\mathbf{Z}$; since Γ has finite index in $\mathrm{SL}_n\mathbf{Z}$ it is also proper on M_Γ. Then the following lemma implies that the map $\Delta \to M_\Gamma$ is proper as well.

LEMMA 1. *Let $H \in \Delta$ be a diagonal matrix with positive entries. Then* $\operatorname{syst}(H)$ *is the minimum of the entries in the diagonal of H. In particular* $\operatorname{syst}(H) \leq 1$, *with equality if and only if* $H = \mathrm{Id}$.

Proof. Let $a_1, \ldots, a_n$ be the diagonal entries of H, and for the sake of concreteness assume that a_1 is minimal. Then for $v = {}^t(v_1, \ldots, v_n) \in \mathbf{Z}^n$ with, say, $v_i \neq 0$, we have

$$|Av| = \sqrt{a_1^2 v_1^2 + \cdots + a_n^2 v_n^2} \geq |a_i v_i| \geq a_i \geq a_1$$

with equality if, for example, $v_1 = 1$ and $v_2 = \cdots = v_n = 0$. This proves the first claim of the lemma. The second claim follows from the fact that $a_1 \ldots a_n = 1$, so that either some a_i is less than 1 or all of the a_i's are equal to 1.

The proper map $\Delta \to M_\Gamma$ can be considered as a cycle β in $C_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$. We denote by $[\Delta] = [\beta]$ the homology class of β.

LEMMA 2. *Let $A \in N$ be an upper triangular matrix with 1 at the diagonal. Then* $\operatorname{syst}(A) = 1$.

Proof. Given $v = {}^t(v_1, \ldots, v_n) \in \mathbf{Z}^n$, let i be minimal such that $v_j = 0$ for all $j > i$. Then we have that v_i is the i-th coordinate of Av and hence $|Av| \geq |v_i| \geq 1$, with equality when, for example, $v_1 = 1$ and $v_2 = \cdots = v_n = 0$.

The intersection points of the cycles $\alpha = N/(N \cap \Gamma)$ and β in M_Γ correspond bijectively to the set of those $H \in \Delta$ for which there is $A \in \Gamma$ with $HA \in N$. For any such H we have by Lemma 2

$$1 = \operatorname{syst}(HA) = \operatorname{syst}(H)$$

and hence $H = \mathrm{Id}$ by Lemma 1; thus α and β intersect at a single point. Moreover, their intersection is locally modeled by the intersection of the images of Δ and N in S_n, and hence it is transversal; therefore $\iota([\alpha], [\beta]) = 1$. This implies that $[\alpha] = [N/(N \cap \Gamma)]$ and $[\beta] = [\Delta]$ are not homologically trivial.

LEMMA 3. *If Γ is a torsion free subgroup of* $\mathrm{SL}_n \mathbf{Z}$ *then the classes* $[N/N \cap \Gamma] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ *and* $[\Delta] \in H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ *have intersection*

$$\iota([N/N \cap \Gamma], [\Delta]) = 1$$

and hence are not trivial. □

5. Proof of Theorem 1.2

Taking into account the title of this section, it can hardly be surprising that we now prove:

THEOREM 1.2. *For $n \geq 5$, the subset $\mathcal{Y}$ of $\mathcal{T}_n$ consisting of extremely well-rounded points, i.e., those points ρ with the property that $\mathcal{S}(\rho)$ generates $\pi_1(\mathbf{T}^n)$, is not contractible and hence is not an $\mathrm{SL}_n\mathbf{Z}$-equivariant spine.*

Let all the notation be as in the previous section. As mentioned in the introduction, in order to prove Theorem 1.2 we will show that there is a finite index torsion free subgroup $\Gamma \subset \mathrm{SL}_n\mathbf{Z}$ for which the map

$$H_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma) \longrightarrow H_{\frac{n(n-1)}{2}}(M_\Gamma) \tag{5.1}$$

is not surjective. More precisely, we will show that this is the case for those torsion-free finite-index subgroups Γ contained in the kernel of the homomorphism

$$\mathrm{SL}_n\mathbf{Z} \to \mathrm{SL}_n\mathbf{Z}/2\mathbf{Z}\,. \tag{5.2}$$

Fix such a Γ and let $A \in \mathrm{SL}_n\mathbf{R}$ be the upper triangular matrix which, up to a factor, is the identity on the upper left $(n-1)\times(n-1)$ quadrant and with entries equal to $\frac{1}{2}$ in the last column:

$$A = 2^{-\frac{1}{n}}\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 1 & \dots & 0 & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{1}{2} \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{pmatrix}. \tag{5.3}$$

The assumption that Γ is contained in the kernel of (5.2) implies that every element $B \in \Gamma$ can be written as $B = \mathrm{Id} + B'$, where every entry of B' is even. In particular, we have for any such B that ABA^{-1} has integer entries, so that

$$A\Gamma A^{-1} \subset \mathrm{SL}_n\mathbf{Z}\,.$$

Observe that we have a diffeomorphism

$$S_N \to S_n\,, \quad B \mapsto BA$$

which induces a diffeomorphism

$$\mathcal{A}\colon M_{A\Gamma A^{-1}} \to M_\Gamma\,.$$

The diffeomorphism $\mathcal{A}$ maps the non-trivial (by Lemma 3) homology classes

$$[N/(N\cap(A\Gamma A^{-1}))] \in H_{\frac{n(n-1)}{2}}(M_{A\Gamma A^{-1}})\,, \quad [\Delta] \in H_{n-1}(\overline{M}_{A\Gamma A^{-1}}, \partial\overline{M}_{A\Gamma A^{-1}})$$

to, a fortiori, non-trivial classes with

$$\iota\big(\mathcal{A}_*[\Delta], \mathcal{A}_*([N/(N\cap(A\Gamma A^{-1}))])\big) = 1\,.$$

Observe that the class $\mathcal{A}_*[\Delta] \in H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$ is represented by a cycle supported in $\{HA \mid H\in\Delta\}\cap\overline{M}_\Gamma$. Below we will prove

LEMMA 4. *Assume that $n\geq 5$, that A is the matrix given in* (5.3), *and that $H\in\Delta$ is a diagonal matrix. Then we have:*
- *$A\in\mathcal{X}\setminus\mathcal{Y}$, and*
- *$HA\in\mathcal{X}$ if and only if $H=\mathrm{Id}$.*

Lemma 4 implies that the homologically non-trivial class $\mathcal{A}_*[\Delta]$ is supported by a cycle which does not intersect $\mathcal{Y}/\Gamma$. This implies that the class $\mathcal{A}_*([N/(N\cap(A\Gamma A^{-1}))]) \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ is not represented by any cycle in $C_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma)$. In particular we deduce, as was claimed, that the map (5.1) is not surjective. We can now conclude the proof of Theorem 1.2. If $\mathcal{Y}$ were contractible, then $\mathcal{Y}/\Gamma$ would be an Eilenberg-MacLane space for Γ and the inclusion $\mathcal{Y}/\Gamma \hookrightarrow S_n/\Gamma = M_\Gamma$ a homotopy equivalence, contradicting the lack of surjectivity of (5.1).

It just remains to prove Lemma 4:

Proof of Lemma 4. We start proving that $A\in\mathcal{X}\setminus\mathcal{Y}$. For every vector $v={}^t(v_1,\dots,v_n)\in\mathbf{Z}^n$ we have that

$${}^t(Av) = 2^{-\frac{1}{n}}\left(v_1+\frac{v_n}{2},\dots,v_{n-1}+\frac{v_n}{2},\frac{v_n}{2}\right).$$

If v_n is odd, then $|Av|\geq\frac{\sqrt{n}}{2}2^{-\frac{1}{n}}$. On the other hand, if v_n is even, then every vector has at least length $2^{-\frac{1}{n}}$ with, for example, equality for e_1. This proves that $\mathrm{syst}(A)=2^{-\frac{1}{n}}$, and one can easily see that $\mathcal{S}(A)$ consists of the following $2n$ vectors in $\mathbf{Z}^n$:

$$\pm e_1,\dots,\pm e_{n-1},\pm\left(2e_n-\sum_{i=1}^{n-1}e_i\right).$$

This implies that $\mathcal{S}(A)$ generates the subgroup of $\mathbf{Z}^n$ consisting of vectors whose last coordinate is even. This is a proper subgroup with index 2, hence $A \notin \mathcal{Y}$, but $A \in \mathcal{X}$.

Continuing with the proof of the lemma, let $H \in \Delta$ be a diagonal matrix with positive entries $a_1, \dots, a_n$. When we multiply H and A we obtain:

$$HA = 2^{-\frac{1}{n}} \begin{pmatrix} a_1 & 0 & \dots & 0 & \frac{a_1}{2} \\ 0 & a_2 & \dots & 0 & \frac{a_2}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & \frac{a_{n-1}}{2} \\ 0 & 0 & \dots & 0 & \frac{a_n}{2} \end{pmatrix}. \tag{5.4}$$

For any such HA and $i = 1, \dots, n-1$ we have $|HAe_i| = 2^{-\frac{1}{n}} a_i$. We also have $\left|HA\left(2e_n - \sum_{i=1}^{n-1} e_i\right)\right| = 2^{-\frac{1}{n}} a_n$. This shows that

$$\operatorname{syst}(HA) \leq 2^{-\frac{1}{n}} \min\{a_i \mid i = 1, \dots, n\}. \tag{5.5}$$

Assume from now on that HA belongs to the well-rounded retract $\mathcal{X}$, and recall that this means that the set $\mathcal{S}(HA)$ of those $v \in \mathbf{Z}^n$ with $|HAv| = \operatorname{syst}(HA)$ generates a finite index subgroup of $\mathbf{Z}^n$. In particular, there is a shortest vector $v = {}^t(w_1, \dots, w_n) \in \mathcal{S}(HA)$ with $w_n > 0$. For such a v one has

$$\operatorname{syst}(HA) = |HAv| \geq 2^{-\frac{1}{n}} \frac{w_n}{2} a_n.$$

We deduce then from (5.5) that w_n is either 1 or 2. We claim that $w_n = 2$. Otherwise we have

$$|HAv| \geq \frac{1}{2}\sqrt{a_1^2 + \dots + a_{n-1}^2 + a_n^2} \geq 2^{-\frac{1}{n}} \frac{\sqrt{n}}{2} \min\{a_i \mid i = 1, \dots, n\}$$

contradicting (5.5), as $n \geq 5$. Hence there is a shortest vector with last coefficient $w_n = 2$. Among all these vectors, HAv is minimal if and only if $v = 2e_n$; thus $\operatorname{syst}(HA) = 2^{-\frac{1}{n}} a_n$. The assumption that $HA \in \mathcal{X}$ implies that for $i = 1, \dots, n-1$, there is also some vector v' with $|HAv'| = \operatorname{syst}(HA) = 2^{-\frac{1}{n}} a_n$ and whose i-th coefficient w_i' does not vanish. By the discussion above, the last coefficient of v' must vanish and hence the i-th coefficient of HAv is $2^{-\frac{1}{n}} w_i' a_i$. This implies that $a_i = a_n$. We have proved that if $HA \in \mathcal{X}$ then $H = \mathrm{Id}$.

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Alexandra Pettet

Department of Mathematics
Stanford University
450 Serra Mall, Bldg 380
Stanford, CA 94305-2125
USA
e-mail: apettet@math.stanford.edu

Juan Souto

Department of Mathematics
University of Chicago
5734 University Avenue
Chicago, IL 60637-1514
USA
e-mail: juan@math.uchicago.edu