A NOTE ON THE CONNECTIVITY OF CERTAIN COMPLEXES ASSOCIATED TO SURFACES

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ABSTRACT. This note is devoted to a trick which yields almost trivial proofs that certain complexes associated to topological surfaces are connected or simply connected. Applications include new proofs that the complexes of curves, separating curves, nonseparating curves, pants, and cut systems are all connected for genus $g \gg 0$. We also prove that two new complexes are connected: one involves curves which split a genus $2g$ surface into two genus g pieces, and the other involves curves which are homologous to a fixed curve. The connectivity of the latter complex can be interpreted as saying the "homology" relation on the surface is (for $g \geq 3$) generated by "embedded/disjoint homologies". We finally prove that the complex of separating curves is simply connected for $g \geq 4$.

1. INTRODUCTION

Let Σ_g be a genus g surface and $\text{Mod}(\Sigma_g)$ be the *mapping class group* of Σ_g, that is, the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g (see [11] for a survey of $\text{Mod}(\Sigma_g)$). An important theme in the study of $\text{Mod}(\Sigma_g)$ and its subgroups is the close relationship between algebraic properties of $\text{Mod}(\Sigma_g)$ (e.g. cohomology, finiteness properties, automorphisms, etc.) and the structure of 1-submanifolds of Σ_g. The combinatorics of these 1-submanifolds have been encoded in the structure of a number of simplicial complexes, such as the curve complex and the pants complex. A key property of these complexes is that they are often highly connected. In this paper, we discuss a general trick which yields simple proofs that complexes of this sort are connected or simply connected; in many cases this is sufficient for the applications.

In the past, these sorts of theorems have been proven using a variety of tools, such as curve surgery (see, e.g., [3, 6, 14, 15, 19, 21]), parametrized Morse theory (see, e.g., [7, 10]), and Teichmüller theory (see, e.g., [2, 4, 17]). We instead exploit the basic combinatorial group-theoretic properties of $\text{Mod}(\Sigma_g)$ and its subgroups, deducing that complexes are connected from the structure of generating sets and deducing that they are simply connected from relations. Of course, we may be accused of circular reasoning, as the standard construction of generators and relations for $\text{Mod}(\Sigma_g)$ involves investigating connected and simply connected complexes upon which it acts! The point of this paper is that this only needs to be done once: as soon as generators and relations for $\text{Mod}(\Sigma_g)$ are found, one can prove that essentially any complex upon which $\text{Mod}(\Sigma_g)$ acts in a reasonable way is connected or simply connected by a formal, finitely checkable (and in practice quite easy) process.

The first complex we will examine is the complex of curves (introduced by Harvey in [5]), together with two of its subcomplexes.

DEFINITION 1.1.   The *complex of curves* $\mathcal{C}(\Sigma_g)$ is the simplicial complex whose simplices are sets $\{c_1, \ldots, c_k\}$ of non-trivial isotopy classes of simple closed curves on Σ_g which can be realized disjointly. The *complex of separating curves* $\mathcal{C}_{\text{sep}}(\Sigma_g)$ and the *complex of nonseparating curves* $\mathcal{C}_{\text{nosep}}(\Sigma_g)$ are the full subcomplexes of $\mathcal{C}(\Sigma_g)$ spanned by separating and nonseparating curves, respectively.

We will give a simple, unified proof of the following theorem, which for $\mathcal{C}(\Sigma_g)$ and $\mathcal{C}_{\text{nosep}}(\Sigma_g)$ is due to Lickorish [13] (though he did not use this language) and for $\mathcal{C}_{\text{sep}}(\Sigma_g)$ is due to Farb and Ivanov [3]. Other proofs of the connectedness of $\mathcal{C}_{\text{sep}}(\Sigma_g)$ can be found in [14] and [15].

THEOREM 1.2. $\mathcal{C}(\Sigma_g)$ and $\mathcal{C}_{\text{nosep}}(\Sigma_g)$ are connected for $g \geq 2$, while $\mathcal{C}_{\text{sep}}(\Sigma_g)$ is connected for $g \geq 3$.

In fact, our trick allows us to achieve rather precise control over the topology of the curves which appear in our complexes. For instance, consider the following complex.

DEFINITION 1.3.   Let $\mathcal{C}_{\text{half}}(\Sigma_{2g})$ be the simplicial complex whose simplices are sets $\{c_1, \ldots, c_k\}$ of isotopy classes of simple closed curves on Σ_{2g} which satisfy the following two conditions.

- Each c_i separates Σ_{2g} into two genus g subsurfaces.

- For $i \neq j$, the geometric intersection number $i(c_i, c_j)$ is *minimal* among such curves. This minimality means the following: if $g = 1$ then $i(c_i, c_j) = 4$, while if $g \geq 2$ then $i(c_i, c_j) = 2$.

We will prove the following theorem, answering a question posed to the author by Schleimer (who proved the theorem for $g = 1$ in [19]).

THEOREM 1.4. $\mathcal{C}_{\text{half}}(\Sigma_{2g})$ *is connected for* $g \geq 1$.

We next investigate the cut system and pants graphs, which were introduced by Hatcher and Thurston in [7].

DEFINITION 1.5. A *cut system* on Σ_g is a set $\{c_1, \ldots, c_g\}$ of isotopy classes of simple closed curves on Σ_g which can be realized disjointly with $\Sigma_g \setminus (c_1 \cup \cdots \cup c_g)$ connected (see Figure 1.d). Two cut systems $\{c_1, \ldots, c_g\}$ and $\{c'_1, \ldots, c'_g\}$ differ by an *elementary move* if there is some $1 \leq i \leq k$ such that $i(c_i, c'_i) = 1$ and $c_j = c'_j$ for $j \neq i$. The *cut system graph* $\mathcal{CT}(\Sigma_g)$ is the graph whose vertices are cut systems on Σ_g and whose edges correspond to elementary moves between cut systems.

DEFINITION 1.6. For $g \geq 2$, a *pants decomposition* of Σ_g is a maximal simplex $\{c_1, \ldots, c_k\}$ of $\mathcal{C}(\Sigma_g)$ (see Figure 1.e). Observe that $k = 3g - 3$ and that cutting Σ_g along the c_i results in a collection of 3-holed spheres (the "pairs of pants"). Two pants decompositions $\{c_1, \ldots, c_k\}$ and $\{c'_1, \ldots, c'_k\}$ differ by an *elementary move* if there is some $1 \leq i \leq k$ such that for $j \neq i$ we have $c_j = c'_j$ and such that $i(c_i, c'_i)$ is *minimal* among such curves. This minimality means the following (see Figure 1.g): if S is the component of Σ_g cut along $c_1 \cup \cdots \cup c_{i-1} \cup c_{i+1} \cup \cdots \cup c_k$ containing c_i, then $i(c_i, c'_i) = 2$ if S is a 4-holed sphere and $i(c_i, c'_i) = 1$ if S is a 1-holed torus. The *pants graph* $\mathcal{P}(\Sigma_g)$ is the graph whose vertices are pants decompositions of Σ_g and whose edges correspond to elementary moves between pants decompositions.

REMARK 1.7. Hatcher and Thurston in fact considered $\mathcal{CT}(\Sigma_g)$ and $\mathcal{P}(\Sigma_g)$ with a number of 2-cells attached to render them simply connected. We will make no use of these 2-cells.

We will give a new proof of the following theorem of Hatcher and Thurston, which for $\mathcal{CT}(\Sigma_g)$ is Theorem 1.1 of [7] and for $\mathcal{P}(\Sigma_g)$ is contained in the appendix of [7].

THEOREM 1.8. $\mathcal{CT}(\Sigma_g)$ *is connected for* $g \geq 1$, *while* $\mathcal{P}(\Sigma_g)$ *is connected for* $g \geq 2$.

Next, we will use the action of the Torelli subgroup of the mapping class group (defined below) to prove the following theorem, which elucidates the nature of the homology relation on a surface. It says that this relation is generated by "embedded homologies" (in the statement of this theorem and throughout this paper, when we say that two simple closed unoriented curves are *homologous*, we mean that they can be oriented in such a way that they are rendered homologous).

THEOREM 1.9.   *Fix* $g \geq 3$, *and let* γ *and* γ' *be homologous non-trivial simple closed curves on* Σ_g. *Then there exists a sequence*

$$\gamma = \gamma_1, \gamma_2, \ldots, \gamma_k = \gamma'$$

of non-trivial simple closed curves on Σ_g *such that for* $1 \leq i < k$ *the curves* γ_i *and* γ_{i+1} *are disjoint and there exists an embedded subsurface* $S_i \hookrightarrow \Sigma_g$ *with* $\partial S_i = \gamma_i \sqcup \gamma_{i+1}$ *(in particular,* γ_i *and* γ_{i+1} *are homologous).*

REMARK 1.10.   This theorem is false for $g = 2$, as there exist no subsurfaces S of Σ_2 such that ∂S consists of two simple closed curves which are nonseparating and nonisotopic on Σ_2.

Finally, we will show that our methods can be extended to prove that various complexes are simply connected. As an example, we prove the following.

THEOREM 1.11.   *For* $g \geq 4$, *the complex* $\mathcal{C}_{\text{sep}}(\Sigma_g)$ *is simply connected.*

REMARK 1.12.   Hatcher and Vogtmann [8] have proven a much stronger theorem which says that $\mathcal{C}_{\text{sep}}(\Sigma_g)$ is $\lfloor \frac{g-3}{2} \rfloor$-connected. Their result, however, does not imply Theorem 1.11 for $g = 4$.

NOTATION 1.13. Let $P_1, P_2 \ldots, P_k$ be a sequence of paths in a simplicial complex X each of which begins and ends in the 0-skeleton $X^{(0)}$ (we allow degenerate paths P_i consisting of single vertices). For all $1 \le i < k$, let q_i be the terminal point of P_i and p_{i+1} be the initial point of P_{i+1}, and assume that $\{q_i, p_{i+1}\} \in X^{(1)}$. Thus either $q_i = p_{i+1}$ or $\{q_i, p_{i+1}\}$ is a 1-simplex. We then denote the path which first traverses P_1, then P_2, etc. by $P_1 - P_2 - \ldots - P_k$.

2. CONNECTIVITY

Our trick for proving that complexes are connected is contained in the following easy lemma.

LEMMA 2.1. *Consider a group G acting upon a simplicial complex X. Fix a basepoint $v \in X^{(0)}$ and a set S of generators for G. Assume the following hold.*

1. *For all $v' \in X^{(0)}$, the orbit $G \cdot v$ intersects the connected component of X containing v'.*

2. *For all $s \in S^{\pm 1}$, there is some path P_s in X from v to $s \cdot v$.*

Then X is connected.

Proof. Consider $v' \in X^{(0)}$. By Condition 1, there is some $g \in G$ together with a path P from $g \cdot v$ to v'. Write g as a word $s_1 \cdots s_k$ in $S^{\pm 1}$. Then

$$P_{s_1} - s_1 P_{s_2} - \ldots - s_1 s_2 \cdots s_{k-1} P_{s_k} - P$$

is a path from v to v'. $\square$

Proof of Theorems 1.2, 1.4 and 1.8. Let

$$S = \{T_{\alpha_1}, T_{\delta_1}, \ldots, T_{\alpha_g}, T_{\delta_g}, T_{\beta_1}, \ldots, T_{\beta_{g-1}}\}$$

be the Dehn twists about the curves in Figure 1.a; Lickorish proved that S generates $\mathrm{Mod}(\Sigma_g)$ (see [11, §4] for the definition of a Dehn twist and a discussion of Lickorish's theorem). For each complex in question, we will pick a basepoint v and verify that the two conditions of Lemma 2.1 are satisfied for the action of the mapping class group upon the complex. We emphasize that in each proof the basepoint v and all other ancillary curves are chosen to intersect the (already fixed) generators in simple ways. In particular, they depend on the choice of generators.
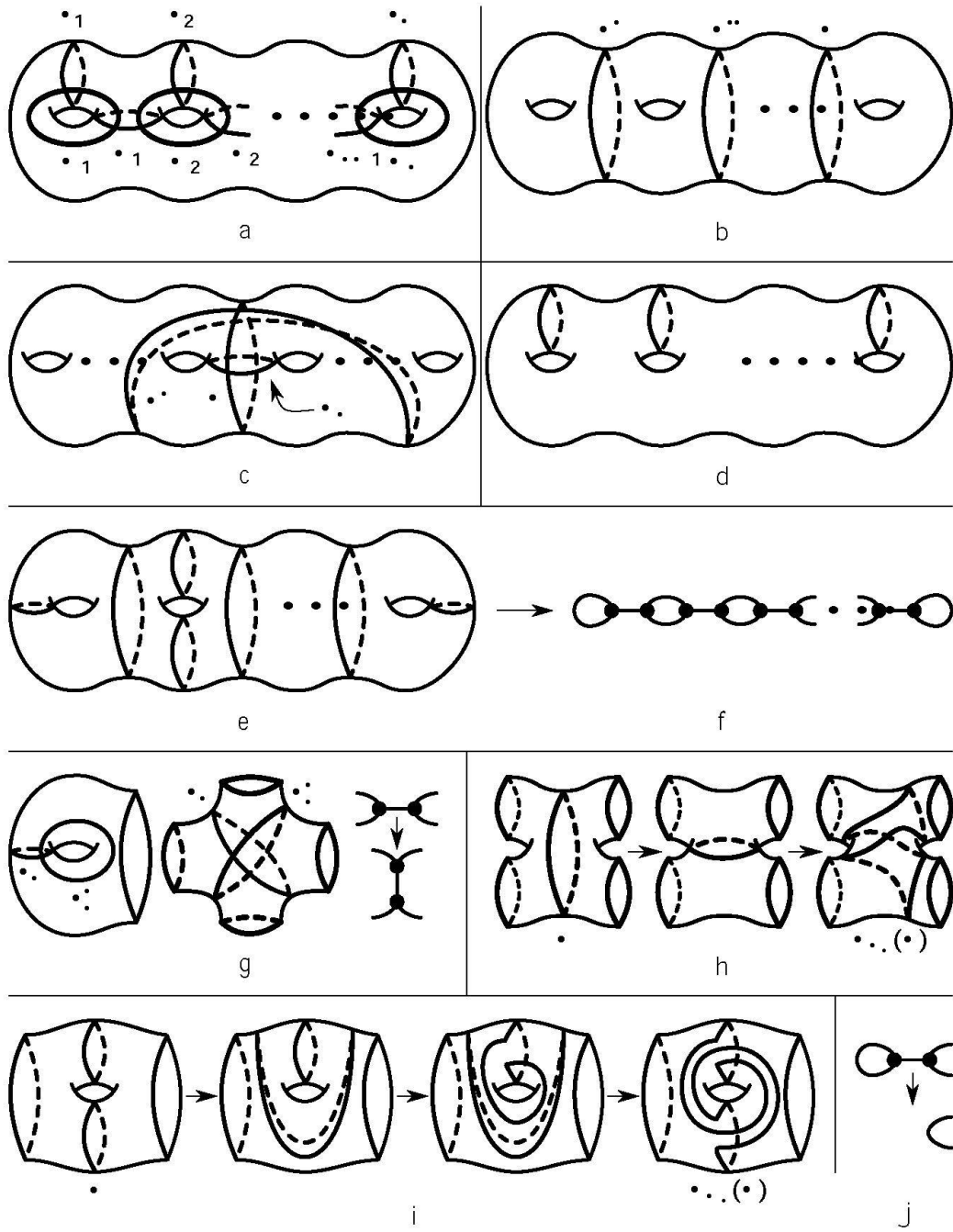
FIGURE 1

The various figures needed for the proof of Theorems 1.2, 1.4 and 1.8
(see that proof for more details)

$\mathcal{C}(\Sigma_g)$, $\mathcal{C}_{\mathrm{nosep}}(\Sigma_g)$ *and* $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$: The proofs for these three complexes are similar; we will discuss $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$ and leave the other two to the reader. Our basepoint $v \in \mathcal{C}_{\mathrm{sep}}^{(0)}(\Sigma_g)$ will be the curve indicated in Figure 1.b. The orbit $\mathrm{Mod}(\Sigma_g) \cdot v$ consists of all separating curves which cut off 1-holed tori. Indeed, let w be another separating curve which cuts off a 1-holed torus. The classification of surfaces implies that we get homeomorphic non-connected surfaces when we cut Σ_g along either v or w. Gluing together homeomorphisms between the cut surfaces yields the desired homeomorphism of Σ_g taking v to w (this trick will be used repeatedly; we will call it the *classification of surfaces trick*). Since every separating curve is adjacent (in $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$) to a curve which cuts off a 1-holed torus, Condition 1 follows. To check Condition 2, we will need the curve v' indicated in Figure 1.b. Consider $s \in S^{\pm 1}$. If $s = T_{\beta_{g-1}}^{\pm 1}$, then $v - v' - s \cdot v$ is the desired path. Otherwise, we have $s \in S^{\pm 1}$ but $s \neq T_{\beta_{g-1}}^{\pm 1}$, so $s \cdot v = v$. Condition 2 follows.

$\mathcal{C}_{\mathrm{half}}(\Sigma_{2g})$: Here S is the corresponding generating set for $\mathrm{Mod}(\Sigma_{2g})$. Our basepoint $v \in \mathcal{C}_{\mathrm{half}}^{(0)}(\Sigma_{2g})$ will be the curve indicated in Figure 1.c. If $g \geq 2$, we will also need the ancillary curve v' from the same figure. Now, by the classification of surfaces trick, $\mathrm{Mod}(\Sigma_{2g})$ acts transitively on $\mathcal{C}_{\mathrm{half}}^{(0)}(\Sigma_{2g})$, so Condition 1 is trivial. To check Condition 2, consider $s \in S^{\pm 1}$. If $s = T_{\beta_g}^{\pm 1}$, then for $g = 1$ the vertices v and $s \cdot v$ are adjacent, while for $g \geq 2$, the desired path is $v - v' - s \cdot v$. If instead $s \neq T_{\beta_g}^{\pm 1}$, then $s \cdot v = v$. Condition 2 follows.

$\mathcal{CT}(\Sigma_g)$: Our basepoint $v \in \mathcal{CT}^{(0)}(\Sigma_g)$ will be the cut system indicated in Figure 1.d. By the classification of surfaces trick, $\mathrm{Mod}(\Sigma_g)$ acts transitively on $\mathcal{CT}^{(0)}(\Sigma_g)$, so Condition 1 holds. Also, for $s \in S^{\pm 1}$, either $s \cdot v = v$ or $s \cdot v$ is adjacent to v, so Condition 2 holds.

$\mathcal{P}(\Sigma_g)$: Our basepoint $v \in \mathcal{P}^{(0)}(\Sigma_g)$ will be the pants decomposition indicated in Figure 1.e. We start by verifying Condition 2. Consider $s \in S^{\pm 1}$. If $s = T_{\delta_i}^{\pm 1}$, then $s \cdot v = v$. If $s = T_{\beta_i}$, then Figure 1.h contains the desired path (we only draw the portion of the pants decomposition which changes). A similar path works if $s = T_{\beta_i}^{-1}$. If $s = T_{\alpha_1}^{\pm 1}$ or $s = T_{\alpha_g}^{\pm 1}$, then $s \cdot v$ is adjacent to v. If $s = T_{\alpha_i}$ but $i \neq 1$ and $i \neq g$, then Figure 1.i contains the desired path. A similar path works if $s = T_{\alpha_i}^{-1}$ with $i \neq 1$ and $i \neq g$. Condition 2 follows.

We now verify Condition 1. It is enough to show that $\mathcal{P}(\Sigma_g)/\operatorname{Mod}(\Sigma_g)$ is connected. For each pants decomposition $p = \{c_1, \ldots, c_k\}$ of Σ_g, define a graph $\phi(p)$ as follows (see Figure 1.f). The vertices of $\phi(p)$ are the connected components of Σ_g cut along the c_i (the "pairs of pants"). The edges are in bijection with the curves c_i; the edge corresponding to c_i connects the vertices corresponding to the components on either side of c_i. Thus $\phi(p)$ is a trivalent graph with $2g - 2$ vertices (a loop at a vertex counts as two edges abutting that vertex). It is clear that each such graph comes from a pants decomposition. Moreover, it is not hard to see that for pants decompositions p and p' we have $\phi(p)$ isomorphic to $\phi(p')$ if and only if there is some $f \in \operatorname{Mod}(\Sigma_g)$ such that $p = f \cdot p'$.

Now consider an elementary move from $p = \{c_1, \ldots, c_k\}$ to $p' = \{c'_1, \ldots, c'_k\}$. Let c_i be the curve which changes in this move. If $i(c_i, c'_i) = 1$ (so c_i corresponds to a loop in $\phi(p)$; see the left hand part of Figure 1.g), then $\phi(p) = \phi(p')$. If $i(c_i, c'_i) = 2$ (see the central part of Figure 1.g), then $\phi(p)$ is transformed into $\phi(p')$ in the following way (see the right part of Figure 1.g): we first collapse the edge in $\phi(p)$ corresponding to c_i, yielding a vertex of valence 4, which we then "expand" to two vertices of valence 3, each of which abuts two of the edges which once abutted the vertex of valence 4. We will call this an *elementary shift* of the graph. It is not hard to see that any elementary shift of $\phi(p)$ is induced by an elementary move of p.

It is enough, therefore, to prove that if G and G' are trivalent graphs with the same (necessarily even) number of vertices, then G may be transformed into G' by a sequence of elementary shifts. The proof will be by induction on the number k of vertices. The base case $k = 2$ being trivial, we assume that $k > 2$. Since neither G nor G' can be a tree, each must contain a simple closed edge-path. Transform G and G' by elementary shifts so that these closed edge paths are as short as possible. Observe that these minimal-length closed edge paths must be loops: if they were not loops, then we could shorten them by performing elementary shifts which collapse edges in them. Let $\overline{G}$ and $\overline{G}'$ be the result of removing these loops, deleting the resulting valence 1 vertices, and then finally deleting the resulting valence 2 vertices while combining the two edges abutting them into a single edge (see Figure 1.j). By induction we can convert $\overline{G}$ into $\overline{G}'$ by a sequence of elementary shifts. It is easy to see that we can then "lift" this sequence of elementary shifts to G, thus proving the theorem. $\square$

Next, we prove Theorem 1.9.

Proof of Theorem 1.9. This theorem is clearly equivalent to the connected-ness of the following complex for $g \geq 3$.

DEFINITION 2.2. Let $C^\gamma(\Sigma_g)$ denote the full subcomplex of $C(\Sigma_g)$ spanned by curves homologous to γ.

If γ is separating, then $C^\gamma(\Sigma_g) = C_{\text{sep}}(\Sigma_g)$, which is connected by Theorem 1.2. Assume, therefore, that γ is nonseparating, and let $\mathcal{I}(\Sigma_g) \subset \text{Mod}(\Sigma_g)$ (the *Torelli group*) be the kernel of the action of $\text{Mod}(\Sigma_g)$ on $H_1(\Sigma_g; \mathbf{Z})$. We will apply Lemma 2.1 to the action of $\mathcal{I}(\Sigma_g)$ on $C^\gamma(\Sigma_g)$.

To apply Lemma 2.1, we need a basepoint and a generating set. Since γ is nonseparating, the classification of surfaces trick implies that there is a homeomorphism taking γ to the curve v depicted in Figure 2.a. We can therefore assume without loss of generality that γ in fact equals the curve v; this will be our basepoint. It is well known (see, e.g., [18, Lemma 6.2]) that $\mathcal{I}(\Sigma_g)$ acts transitively on the 0-skeleton of $C^\gamma(\Sigma_g)$, so Condition 1 is trivial.
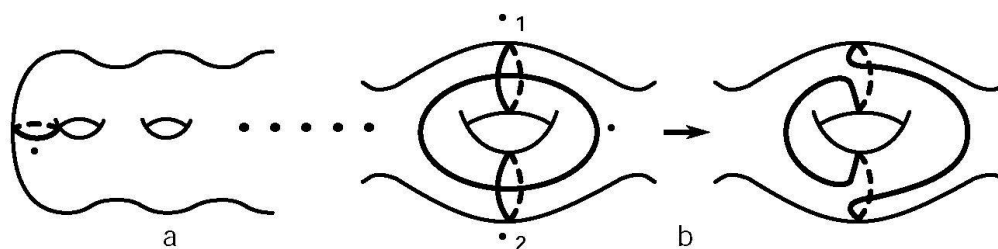

FIGURE 2

a. Our base vertex in $C^\gamma(\Sigma_g)$ b. $T_{\gamma_1} T_{\gamma_2}^{-1}(v)$ is disjoint from v

For the generating set, recall that Johnson [12] proved that $\mathcal{I}(\Sigma_g)$ is finitely generated (for $g \geq 3$; and this is false for $g = 2$ as demonstrated by McCullough-Miller [16]). Our generating set S will be the generating set for $\mathcal{I}(\Sigma_g)$ constructed in [12]. We will need two facts about S. First, S consists of *bounding pair maps*, that is, mapping classes $T_{\gamma_1} T_{\gamma_2}^{-1}$ where the γ_i are disjoint nonseparating curves such that $\gamma_1 \cup \gamma_2$ separates Σ_g. Second, for $T_{\gamma_1} T_{\gamma_2}^{-1} \in S$, either $\gamma_1 \cap v = \gamma_2 \cap v = \varnothing$ or $\gamma_1 \cup \gamma_2 \cup v$ is homeomorphic to the curves pictured on the left hand side of Figure 2.b. These facts imply that for $s \in S^{\pm 1}$, either $s \cdot v = v$ or (as demonstrated by Figure 2.b) $s \cdot v$ is disjoint from v. Condition 2 follows. $\square$

3. SIMPLE CONNECTIVITY

We conclude this paper by proving Theorem 1.11.

Proof of Theorem 1.11. Let

$$S = \{T_{\alpha_1}, T_{\delta_1}, \ldots, T_{\alpha_g}, T_{\delta_g}, T_{\beta_1}, \ldots, T_{\beta_{g-1}}, h\}$$

be the collection of twists about the curves in Figure 1.a together with the *hyperelliptic involution* h (see [11, p. 52] for the definition of h; the need for h will become clear shortly). Also, let v and v' be the curves in Figure 1.b. There is a natural map $\mathrm{Mod}(\Sigma_g) \to \mathcal{C}_{\mathrm{sep}}(\Sigma_g)$ taking g to $g(v)$. Closely examining the proofs of Lemma 2.1 and Theorem 1.2, we see that they say that this map extends to a $\mathrm{Mod}(\Sigma_g)$-equivariant map

$$\phi\colon \mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S) \longrightarrow \mathcal{C}_{\mathrm{sep}}(\Sigma_g).$$

Here $\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S)$ is the *Cayley graph* of $\mathrm{Mod}(\Sigma_g)$, that is, the graph whose vertices are elements of $\mathrm{Mod}(\Sigma_g)$ and where g_2 is connected by an edge to g_1 if $g_2 = g_1 s$ for some $s \in S$. We will prove that the induced map $\phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S), 1) \to \pi_1(\mathcal{C}_{\mathrm{sep}}(\Sigma_g), v)$ is the zero map by examining the images of the loops associated to a set of relations for $\mathrm{Mod}(\Sigma_g)$. We will then show that ϕ_* is surjective; this will allow us to conclude that $\pi_1(\mathcal{C}_{\mathrm{sep}}(\Sigma_g), v) = 0$, as desired.

CLAIM 1. $\phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S), 1) \to \pi_1(\mathcal{C}_{\mathrm{sep}}(\Sigma_g), v)$ *is the zero map.*

Proof of claim. It is well known that we can construct a simply connected complex X from $\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S)$ by attaching discs to the $\mathrm{Mod}(\Sigma_g)$-orbits of the loops associated to any complete set of relations for $\mathrm{Mod}(\Sigma_g)$. We will show that the images in $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$ of the loops associated to these relations are contractible. This will imply that we can extend ϕ to X. Since X is simply connected, we will be able to conclude that ϕ_* is the zero map, as desired.

Now, the loop in $\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S)$ associated to a relation $s_1 \cdots s_k = 1$ with $s_i \in S^{\pm 1}$ is $1 - s_1 - s_1 s_2 - \ldots - s_1 s_2 \cdots s_k$. Since the only elements of $S^{\pm 1}$ which act non-trivially on v are $T_{\beta_{g-1}}^{\pm 1}$, the function ϕ maps the edge $s_1 \cdots s_{i-1} - s_1 \cdots s_i$ to a fixed vertex unless $s_i = T_{\beta_{g-1}}^{\pm 1}$, in which case it maps it to the path $s_1 \cdots s_{i-1}(v) - s_1 \cdots s_{i-1}(v') - s_1 \cdots s_i(v)$. Observe that the only elements of $S^{\pm 1}$ which act non-trivially on v' are $T_{\beta_1}^{\pm 1}$. If none of the s_i equal $T_{\beta_1}^{\pm 1}$, then for all i we would have $s_1 \cdots s_{i-1}(v') = v'$, so ϕ would take the loop associated to the relation $s_1 \cdots s_k = 1$ to a loop of the form

$v_1 - v_2 - \ldots - v_{2l+1}$ with $v_{2i} = v'$ for $1 \le i \le l$. This loop can be contracted to v'.

We therefore only need to worry about relations which involve both $T_{\beta_1}^{\pm 1}$ and $T_{\beta_{g-1}}^{\pm 1}$. By Theorem A.1 from the appendix, we can find a presentation for $\mathrm{Mod}(\Sigma_g)$ whose generators are S and whose only relations involving both $T_{\beta_1}^{\pm 1}$ and $T_{\beta_{g-1}}^{\pm 1}$ are

$$(3.1) \qquad T_{\beta_1}^{-1} T_{\beta_{g-1}}^{-1} T_{\beta_1} T_{\beta_{g-1}} = 1$$

and

$$(3.2) \qquad T_{\alpha_g} T_{\beta_{g-1}} T_{\alpha_{g-1}} \cdots T_{\beta_1} T_{\alpha_1} T_{\delta_1}^2 T_{\alpha_1} T_{\beta_1} \cdots T_{\beta_{g-1}} T_{\alpha_g} h^{-1} = 1.$$

We conclude that we must only check that the ϕ-images of the loops associated to these two relations are contractible. For the relation given in (3.1), it is clear that we can find a separating curve disjoint from every vertex of the associated $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$-loop (for instance, v'' in Figure 1.b is such a curve), so this loop is contractible. For the relation given in (3.2), the associated loop is the following, where we suppress the (trivial) edges $w(v) - ws(v)$ associated to generators $s \in S^{\pm 1}$ not equal to $T_{\beta_{g-1}}^{\pm 1}$:

$$v - T_{\alpha_g}(v') - T_{\alpha_g} T_{\beta_{g-1}}(v)$$
$$- T_{\alpha_g} T_{\beta_{g-1}} T_{\alpha_{g-1}} \cdots T_{\beta_1} T_{\alpha_1} T_{\delta_1}^2 T_{\alpha_1} T_{\beta_1} \cdots T_{\alpha_{g-1}}(v') - v$$
$$= v - v' - T_{\alpha_g} T_{\beta_{g-1}}(v) - v' - v.$$

This is clearly contractible, so the claim follows. $\square$

CLAIM 2. $\phi_* \colon \pi_1(\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S), 1) \to \pi_1(\mathcal{C}_{\mathrm{sep}}(\Sigma_g), v)$ *is surjective.*

Proof of claim. We first find a sufficient condition for a loop to lie in the image of ϕ_*. Consider any loop $\ell = v_1 - v_2 - \ldots - v_{2n+1}$ in $\mathcal{C}_{\mathrm{sep}}(\Sigma_g)$ with $v_1 = v_{2n+1} = v$. Assume that each v_i is a separating curve which cuts off a 1-holed torus and that for $0 \le i < n$ there exists a simple closed nonseparating curve ϵ_i and some $e_i = \pm 1$ for which $v_{2i+3} = T_{\epsilon_i}^{e_i}(v_{2i+1})$, $i(\epsilon_i, v_{2i+1}) = 2$, and $i(\epsilon_i, v_{2i+2}) = 0$ (for instance, we could have $v_{2i+1} = v$, $v_{2i+2} = v'$, and $\epsilon_i = \beta_{g-1}$). We claim that ℓ is in the image of ϕ_*.

To begin with, it is enough to find some word w in $S^{\pm 1}$ (not necessarily a relation) with the property that ℓ is the image under ϕ_* of the path in $\mathrm{Cay}(\mathrm{Mod}(\Sigma_g), S)$ associated to w. Indeed, we will then have $w(v) = v$. Since $S \setminus \{T_{\beta_{g-1}}\}$ generates the stabilizer in $\mathrm{Mod}(\Sigma_g)$ of v, we can find some word w' in $(S \setminus \{T_{\beta_{g-1}}\})^{\pm 1}$ such that $ww' = 1$; this is the desired relation.

We will prove the existence of w by induction on n (in this part of the proof, we do not assume that ℓ is a loop). The case $n = 0$ being trivial, we assume that $n > 0$. Using the induction hypothesis, we can find a word w_{n-1} such that ϕ_* takes the path associated to w_{n-1} to $v_1 - \ldots - v_{2n-1}$. Observe that

$$i(w_{n-1}^{-1}(v_{2n}), w_{n-1}^{-1}(\epsilon_n)) = i(v_{2n}, \epsilon_n) = 0\,,$$

$$i(w_{n-1}^{-1}(v_{2n}), v) = i(v_{2n}, w_{n-1}(v)) = i(v_{2n}, v_{2n-1}) = 0\,,$$

$$i(w_{n-1}^{-1}(\epsilon_n), v) = i(\epsilon_n, w_{n-1}(v)) = i(\epsilon_n, v_{2n-1}) = 2\,.$$

This implies that there must exist some $f \in \mathrm{Mod}(\Sigma_g)$ with the property that $f(v') = w_{n-1}^{-1}(v_{2n})$, $f(\beta_{g-1}) = w_{n-1}^{-1}(\epsilon_n)$, and $f(v) = v$. Since the stabilizer in $\mathrm{Mod}(\Sigma_g)$ of v is generated by $S \setminus \{T_{\beta_{g-1}}\}$, we can find some word w'' in $(S \setminus \{T_{\beta_{g-1}}\})^{\pm 1}$ for which $w'' = f$. We claim that $w = w_{n-1}w''T_{\beta_{g-1}}^{\epsilon_n}$ works. Indeed, since $w_{n-1}f(v') = v_{2n}$, the path associated to w consists of $v_1 - \ldots - v_{2n-1}$ followed by the path

$$\begin{aligned}
w_{n-1}f(v') - w_{n-1}fT_{\beta_{g-1}}^{\epsilon_n}(v) &= v_{2n} - w_{n-1}(fT_{\beta_{g-1}}^{\epsilon_n}f^{-1})(v) \\
&= v_{2n} - w_{n-1}T_{f(\beta_{g-1})}^{\epsilon_n}(v) \\
&= v_{2n} - w_{n-1}T_{w_{n-1}^{-1}(\epsilon_n)}^{\epsilon_n}(v) \\
&= v_{2n} - w_{n-1}w_{n-1}^{-1}T_{\epsilon_n}^{\epsilon_n}w_{n-1}(v) \\
&= v_{2n} - T_{\epsilon_n}^{\epsilon_n}(v_{2n-1})\,.
\end{aligned}$$

Since $T_{\epsilon_n}^{\epsilon_n}(v_{2n-1}) = v_{2n+1}$, this is the desired path.

Now consider an arbitrary $\ell' \in \pi_1(\mathcal{C}_{\mathrm{sep}}(\Sigma_g), v)$. We claim that we can homotope ℓ' so that it satisfies the above condition. In fact, we will prove more generally that if ℓ' is any (not necessarily closed) path starting at v whose final endpoint corresponds to a curve cutting off a 1-holed torus, then we can homotope it (fixing the endpoints) so that it satisfies all of the above conditions except for the closedness of the path.

We can assume without loss of generality that ℓ' is a simplicial path in the 1-skeleton. It is an easy exercise to see that we can homotope ℓ' so that all of its vertices cut off 1-holed tori, and in addition we can arrange for ℓ' to contain an odd number of vertices and for no two adjacent vertices of ℓ' to be identical. Enumerate the vertices of ℓ' as $v_1 - \ldots - v_{2m+1}$. By induction on m, we can assume that $v_1 - \ldots - v_{2m-1}$ satisfies the desired condition. Now, using standard properties of $\mathrm{Mod}(\Sigma_g)$ we can find a sequence of simple closed curves $\eta_1, \ldots, \eta_k$ and numbers $f_1, \ldots, f_k = \pm 1$ such that $T_{\eta_1}^{f_1} \cdots T_{\eta_k}^{f_k}(v_{2m-1}) = v_{2m+1}$ and such that for $1 \leq i \leq k$ we

have $i(\eta_i, v_{2m-1}) = 2$ and $i(\eta_i, v_{2m}) = 0$. We can then homotope ℓ' (adding "whiskers") so that the path $v_{2m-1} - v_{2m} - v_{2m+1}$ is replaced by

$$v_{2m-1} - v_{2m} - T_{\eta_1}^{f_1}(v_{2m-1}) - v_{2m} - T_{\eta_1}^{f_1} T_{\eta_2}^{f_2}(v_{2m-1}) - v_{2m}$$
$$- \ldots - T_{\eta_1}^{f_1} \cdots T_{\eta_k}^{f_k}(v_{2m-1}) = v_{2m+1},$$

thus proving the claim. $\square$

This completes the proof of Theorem 1.11. $\square$

APPENDIX: A VARIANT ON THE WAJNRYB PRESENTATION

THEOREM A.1.   *For* $g \geq 4$, *the group* $\mathrm{Mod}(\Sigma_g)$ *has a presentation* $\langle S | R \rangle$ *satisfying the following conditions:*

- *S is the set of Dehn twists* $\{T_{\alpha_1}, T_{\delta_1}, \ldots, T_{\alpha_g}, T_{\delta_g}, T_{\beta_1}, \ldots, T_{\beta_{g-1}}\}$ *depicted in Figure 1.a together with the hyperelliptic involution* h.
- *The only relations* $r \in R$ *which involve both* $T_{\beta_1}^{\pm 1}$ *and* $T_{\beta_{g-1}}^{\pm 1}$ *are*

$$T_{\beta_1}^{-1} T_{\beta_{g-1}}^{-1} T_{\beta_1} T_{\beta_{g-1}} = 1$$

and

$$T_{\alpha_g} T_{\beta_{g-1}} T_{\alpha_{g-1}} \cdots T_{\beta_1} T_{\alpha_1} T_{\delta_1}^2 T_{\alpha_1} T_{\beta_1} \cdots T_{\beta_{g-1}} T_{\alpha_g} h^{-1} = 1.$$

Proof. The presentation described in this theorem is a variant of the standard Wajnryb presentation ([20]; see [1] for errata). The generating set for the Wajnryb presentation is

$$S' = \{T_{\alpha_1}, \ldots, T_{\alpha_g}, T_{\beta_1}, \ldots, T_{\beta_{g-1}}, T_{\delta_1}, T_{\delta_2}\}.$$

There are four families of relations. In the notation of [20], the first three are labeled A, B and C. The relation $T_{\beta_1}^{-1} T_{\beta_{g-1}}^{-1} T_{\beta_1} T_{\beta_{g-1}} = 1$ is the only relation from family A (the "braid relations") involving both $T_{\beta_1}^{\pm 1}$ and $T_{\beta_{g-1}}^{\pm 1}$. Families B (the "two-holed torus relation") and C (the "lantern relation") do not involve both $T_{\beta_1}^{\pm 1}$ and $T_{\beta_{g-1}}^{\pm 1}$. The final relation D (as corrected by [1]) says that the hyperelliptic involution h commutes with T_{δ_g}; both h and T_{δ_g} are expressed using rather complicated formulas involving the generators S'. Our relation

$$T_{\alpha_g} T_{\beta_{g-1}} T_{\alpha_{g-1}} \cdots T_{\beta_1} T_{\alpha_1} T_{\delta_1}^2 T_{\alpha_1} T_{\beta_1} \cdots T_{\beta_{g-1}} T_{\alpha_g} h^{-1} = 1$$

expands out Wajnryb's formula for h. As was observed in [1, Remark 1.a], the expression for T_{δ_g} in terms of S' used by Wajnryb is unimportant;

any correct expression will work. Now, using an argument of Humphries [9], for $1 \leq i \leq g-2$ we can express $T_{\delta_{i+2}}$ as a complicated product of elements in

$$\{T_{\alpha_i}, T_{\alpha_{i+1}}, T_{\alpha_{i+2}}, T_{\beta_i}, T_{\beta_{i+1}}, T_{\delta_i}, T_{\delta_{i+1}}\}^{\pm 1}.$$

This allows us eliminate T_{δ_i} from S for $i \geq 3$ by adding relations which do not involve both $T_{\beta_1}^{\pm 1}$ and $T_{\beta_{g-1}}^{\pm 1}$. Our final relation is $[h, T_{\delta_g}] = 1$; since this does not involve either T_{β_1} or $T_{\beta_{g-1}}$, we are done. $\square$

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