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COMPETING PRIME ASYMPTOTIC DENSITIES IN $\mathbf{F}_q[X]$: A DISCUSSION

by Christian BALLOT

ABSTRACT. We present a discussion on various prime asymptotic densities in polynomial rings $\mathbf{F}_q[X]$ in order to decide which one best emulates the usual concept of prime natural density in $\mathbf{Z}$.

0. INTRODUCTION

Let Π denote the set of rational primes $2, 3, 5, 7, 11, \ldots$ and let S be a subset of Π. The *Dirichlet density* δ of S is defined as

$$\delta = \delta(S) = \lim_{s \to 1+} \frac{\sum_{p \in S} p^{-s}}{\sum_{p \in \Pi} p^{-s}}, \qquad \text{provided the limit exists.}$$

For n an integer ≥ 1, denote by $\Pi(n)$ the number of primes $\leq n$ and by $S(n)$ the number of primes $\leq n$ that are in S. The *(prime) natural density* d of S is defined as

$$\text{(0.1)} \qquad d = d_{\mathbf{Z}}(S) = \lim_n S(n)/\Pi(n), \qquad \text{provided the limit exists.}$$

Let us focus on various facets of the relationship that exists between these two kinds of densities.

In general it is known that if $d(S)$ exists then $\delta(S)$ exists, with $\delta(S) = d(S)$ (see for instance [Des], Chap. 8). The converse may be false. But sets of primes in arithmetic progressions, or sets of primes that split completely in some normal number field, not only have a Dirichlet density, but also a natural density ([Des], Chap. 8 and [Pra], Chap. 5). More generally, any set of primes defined by an Artin symbol prescription, as specified in the Chebotarev density theorem, has a Dirichlet and a natural density ([Nar], Theorem 7.10*).

So let S be a set of primes in the ring $\mathbf{Z}$ of rational integers. Suppose we make a theoretical calculation, say using the Chebotarev density theorem or some of its corollaries, that shows that S possesses a Dirichlet density δ. It is often reassuring to compute successive ratios $S(n)/\Pi(n)$ and see that they do seem to approach δ. And in our experience, if the theoretical calculation of δ is correct, relatively small sets of primes often suffice to yield a close agreement to δ, thereby providing a practical check on the theory. In other words, the (prime) natural density d of S often turns out to be a better computational tool than the Dirichlet density.

From what was said above, sets of primes having a Dirichlet density and no natural density are somewhat unusual. There is an interesting example due to Bombieri. Indeed, Serre ([Ser], p. 76) reports that Bombieri showed him a proof that the set of rational primes whose first decimal digit is 1 has Dirichlet density $\log_{10} 2$, but no natural density.

In 1958, Sierpiński [Sie] raised the question of the proportion of primes p for which 2 has even order (mod p). Several authors tackled the question before Hasse [Ha] fully settled the Sierpiński question: this set of primes has Dirichlet density $17/24$. And it is not hard to show that this set also has a natural density (necessarily the same). The integer 2 may be replaced by a general $a \in \mathbf{Z}$, $|a| \geq 2$ and various densities be obtained, so the Sierpiński question yields sets having both Dirichlet and natural densities.

A similar situation holds for Artin's conjecture, but modulo some Riemann hypotheses. It is then known that sets of primes having a prescribed integer as a primitive root have both a Dirichlet and a natural density.

This paper is concerned with finding a notion of (prime) natural density in the ring $\mathbf{F}_q[X]$, where $\mathbf{F}_q$ is the finite field with q elements, with properties that best match those described above for the ring $\mathbf{Z}$. By "best", we mean that the selected notion must be as consistent as possible: properties of prime natural density in $\mathbf{Z}$, both in its computational aspects and in its relationship to Dirichlet density, have to be preserved. Also, should a set of primes in $\mathbf{Z}$ possess, or not, a natural density, then we expect its most obvious analogue in $\mathbf{F}_q[X]$ to have, respectively not to have, a natural density. In particular, the Sierpiński question, an $\mathbf{F}_q[X]$-analogue of which we treated in [Bal], the Bombieri example mentioned above, sets of primes related to Artin's conjecture and the *Chebotarev sets*, i.e. sets of primes to which the Chebotarev density theorem directly applies, are used as special guides. In fact, four types of asymptotic densities defined in relation to sets of primes in $\mathbf{F}_q[X]$ are considered. Two of these notions are shown to be equivalent, so that essentially three distinct notions are being studied and compared. Of course, each of these three notions is conceptually sound, i.e. applying the principle by which it is defined to the ring $\mathbf{Z}$ yields the common prime natural density (0.1) in $\mathbf{Z}$.

The paper contains three sections and a conclusion. The main discussion is in Section 1. We have decided to retain in part the order of ideas as they occurred to us rather than to give a more concise and less naïve re-written account. The technical lemmas that sustain the discussion have most likely appeared in other contexts. They are often elementary results of classical analysis, but since their proofs are short and to the point, we included them in our text. We also hope that this will appeal to a broad readership. Occasionally we will point to a reference that could have been used to supersede our original result. One of the asymptotic natural densities studied turns out to be equivalent to the Dirichlet density. Section 2 is dedicated to proofs of this theorem. Therefore we are left with essentially two notions of asymptotic density that differ from the Dirichlet density. Section 3 examines the relationships that sets of primes related to Artin's conjecture and Chebotarev sets have to these remaining two asymptotic densities. The results concerned with the comparison of our various densities appear at several places in the paper, but are gathered in Theorem A in the conclusion.

It is possible to grasp the main ideas of the paper and avoid the technical lemmas. Read Definition 1.3 where the four asymptotic densities are defined, skim through Discussions 1.5, 1.9, 1.15 and Section 3, and read Theorem A and the conclusion.

In the analogy between $\mathbf{F}_q[X]$ and $\mathbf{Z}$, we consider the set of monic polynomials as the analogue of the set $\mathbf{N}$ of natural numbers, and the set I of monic irreducible polynomials as the analogue of the set Π of rational primes. The degree of a polynomial P is written $\deg P$ and the size of the quotient ring $\mathbf{F}_q[X]/P$, i.e. the *norm of* P, is denoted by $|P|$. If S is a set of primes of $\mathbf{F}_q[X]$, then we define S_n as the number of primes in S of degree n. Thus I_n denotes the number of monic irreducible polynomials of degree n in $\mathbf{F}_q[X]$. The classical prime number theorem says that $\pi(x)$, the number of rational primes $p \leq x$, is asymptotic to $x/\log x$ as $x \to \infty$. Here we will often use the fact that $I_n \sim q^n/n$. This result is seen as an analogue of the classical prime number theorem and is called the *prime number theorem for polynomials.* Note that q^n is the number of monic polynomials of degree n in $\mathbf{F}_q[X]$ and that putting $x = q^n$ we have $I_n \sim x/\log_q x$. See [Ro], p. 14 for further details.

We recall here that a non-zero integer a is said to be a *primitive root* of a prime p if the class of a (mod p) generates the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$. *Artin's primitive root conjecture* is that any non-square integer a different from -1 is a primitive root for infinitely many primes p. The conjecture is still unproved for any given such a. However, conditionally to Riemann hypotheses, more was proved by Hooley since for such a's the set of primes p having a as a primitive root is not only infinite but has a *positive* natural density. Similarly a polynomial A in $\mathbf{F}_q[X]$ is said to be *a primitive root* of a prime $P \in \mathbf{F}_q[X]$ if the powers of A (mod P) cover all of the cyclic group $(\mathbf{F}_q[X]/P)^*$. For instance X is a primitive root of the prime $X^2 + X + 1 \in \mathbf{F}_2[X]$.

In the famous paper [Bi] in which Bilharz proves an equivalent of Artin's conjecture for function fields, modulo the function-field Riemann hypothesis (not yet proved at that time), Bilharz shows (pp. 490–492) that if the set of primes having a given polynomial as a primitive root has a Dirichlet density, it may not have a natural density, where *natural density* is defined as

$$\lim_{n\to\infty} \frac{S(n)}{I(n)}, \quad \text{where } S(n) = \sum_{k=1}^{n} S_k \text{ and } I(n) = \sum_{k=1}^{n} I_k. \tag{0.2}$$

Bilharz attributes this interesting observation to Davenport. In [Len], p. 203, Lenstra recalls this same observation and adds that his results when applied to the number field case remain valid if Dirichlet density is replaced by natural density, but that this is not true in the function field case.

In this paper we challenge the idea that prime natural density should be defined as above. Were other equally sound definitions of prime natural density considered? Our own doubts about this came from studying the Sierpiński question for $\mathbf{F}_q[X]$ in [Ba1], where we found that the successive ratios $S(n)/I(n)$ in (0.2) do not converge to the Dirichlet density given by the theory. For instance, the set $\{p \in \mathbf{Z};\ p \mid 2^n + 1 \text{ for some } n\}$ has Dirichlet and natural densities equal to 17/24 [Ha], whereas the set $\{P \in \mathbf{F}_3[X];\ P \mid X^n + 1 \text{ for some } n\}$ also has Dirichlet density 17/24, but no natural density as defined in (0.2). However it has d_3-natural density 17/24 (see Definition 1.3).

Identifying a notion of asymptotic density in $\mathbf{F}_q[X]$ such that sets of primes, defined in analogy to sets of primes in the classical setting, possess a density is not just a mind game: *it has consequences*. In fact, the present paper led us to discover an elementary method for computing some densities that, unlike the classical case, avoids any use of either algebraic means or the Chebotarev density theorem (see [Ba1], Section 4, and [Ba2]). This was done with explicit error terms and we foresee more consequences, in particular if the results of [Ba2] can be proved in more generality.

However, from the simple point of view that we adopt in this paper, sets of primes have, or don't have, a given form of density, regardless of error terms. But it is worth mentioning that the strongest form of density we consider here (the d_1-density; see Definition 1.3) can be split into further relevant categories depending essentially on the asymptotic size of the error term. Such types of prime densities in $\mathbf{F}_q[X]$ have been compared to each other in [Ca], Prop. IV.2.

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1. MAIN DISCUSSION

First we recall

DEFINITION 1.1. A set S of primes in $\mathbf{F}_q[X]$ is said to *have Dirichlet density* δ if the limit below exists and

$$\lim_{s\to 1^+} \frac{\sum_{P\in S} |P|^{-s}}{\sum_{P\in I} |P|^{-s}} = \delta\,.$$

To sort out various plausible competing notions of natural density, we will make use of the set T of all primes of even degree in $\mathbf{F}_q[X]$.

PROPOSITION 1.2. *The set T has Dirichlet density* $\delta(T) = \dfrac{1}{2}$.

Proof. Note that for $s > 1$,

$$\sum_{n\geq 1} \frac{1}{n}\frac{1}{q^{n(s-1)}} = -\log(1-q^{1-s}) \sim \log\frac{1}{s-1} \qquad \text{as } s\to 1^+$$

(l'Hôpital's rule will do). Replacing q by q^2 above yields $\log\dfrac{1}{s-1} \sim \sum_{n\geq 1}\dfrac{1}{n}\dfrac{1}{q^{2n(s-1)}}$. Putting $v_n(s) = \dfrac{1}{n}\dfrac{1}{q^{n(s-1)}}$, the previous equivalence says that $\sum_{n\geq 1} v_{2n}(s) \sim \dfrac{1}{2}\log\dfrac{1}{s-1}$. Put $u_n(s) = I_n/q^{ns}$. Since $I_n \sim q^n/n$, the sequences of functions $u_n(s)$ and $v_n(s)$ are uniformly asymptotic to each other, i.e. there exists a sequence, say ε_n, independent of s, such that $\varepsilon_n \to 0$ and for all s, $u_n(s) = (1+\varepsilon_n)v_n(s)$. Since both $\sum_{n\geq 1} v_n(s)$ and $\sum_{n\geq 1} v_{2n}(s)$ tend to $+\infty$ as $s \to 1^+$ and the functions $v_n(s)$ are all continuous at 1 and positive, we have

$$\sum_{n\geq 1} u_n(s) \sim \sum_{n\geq 1} v_n(s) \quad \text{and} \quad \sum_{n\geq 1} u_{2n}(s) \sim \sum_{n\geq 1} v_{2n}(s) \quad \text{as } s \to 1^+ .$$

Therefore

$$\frac{\sum_{P\in T} |P|^{-s}}{\sum_{P\in I} |P|^{-s}} = \frac{\sum_{n\geq 1} u_{2n}(s)}{\sum_{n\geq 1} u_n(s)} \sim \frac{\sum_{n\geq 1} v_{2n}(s)}{\sum_{n\geq 1} v_n(s)} \sim \frac{1}{2} \quad \text{as } s \to 1^+ ;$$

thus $\delta(T) = 1/2$. □

A priori *at least* three or four kinds of prime natural density can reasonably come to mind. We define four of them.

DEFINITION 1.3 (Of four kinds of prime natural density). Let S be any set of primes in $\mathbf{F}_q[X]$. We say that

i) *S has a d_1-density* if there is a real number $d_1(S)$ such that

$$\lim_n \frac{S_n}{I_n} = d_1(S) \text{ or, equivalently, such that } \lim_n \frac{nS_n}{q^n} = d_1(S); \tag{1.2}$$

ii) *S has a d_2-density* if there is a real number $d_2(S)$ such that

$$\lim_n \Big(\sum_{k=1}^{n} S_k\Big) / \Big(\sum_{k=1}^{n} I_k\Big) = d_2(S) \tag{1.3}$$

or, equivalently, such that $\lim_n \big(\sum_{k=1}^n S_k\big)/\big(\sum_{k=1}^n \frac{q^k}{k}\big) = d_2(S)$;

iii) *S has a d_3-density* if there is a real number $d_3(S)$ such that

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \frac{S_n}{I_n} = d_3(S) \tag{1.4}$$

or, equivalently, such that $\lim_N \frac{1}{N} \sum_{n=1}^N \frac{nS_n}{q^n} = d_3(S)$; and

iv) *S has a d_4-density* if there is a real number $d_4(S)$ such that

$$\lim_N \Big(\sum_{n=1}^{N} S_n/q^n\Big) \Big/ \Big(\sum_{n=1}^{N} I_n/q^n\Big) = d_4(S) \tag{1.5}$$

or, equivalently, such that $\lim_N \frac{1}{\log N} \sum_{n=1}^N S_n/q^n = d_4(S)$.

Each d_i, $1 \leq i \leq 4$, is defined as the limit of some sequence of ratios r_N as $N \to \infty$. We will call such a ratio r_N an *approximant* to d_i (r_N being the approximant of order N).

REMARKS. i) These asymptotic prime densities d_1, d_2, d_3 and d_4 can be referred to respectively as *local*, *global* (or *cumulative*), *average* and *Dirichlet average*.

ii) The second equality in (1.2) comes from the fact that $I_n \sim q^n/n$.

iii) The second expression for d_2 in (1.3) holds because $\sum_{k=1}^n I_k \sim \sum_{k=1}^n q^k/k$. Apply Lemma 1.6 with $u_k = I_k$ and $v_k = q^k/k$ to see this.

iv) The validity of the second equality in (1.4) can be deduced from Lemma 1.4 below.

v) Since $I_n/q^n \sim 1/n$, Lemma 1.6 applied to $u_n = I_n/q^n$ and $v_n = 1/n$ yields $\sum_{n=1}^N I_n/q^n \sim \sum_{n=1}^N 1/n \sim \log N$, whence the second equality for $d_4(S)$ in (1.5).

LEMMA 1.4. *Let x_n and y_n be bounded sequences of non-negative real numbers. Define for all $N \geq 1$, the arithmetic means $\bar{x}_N = N^{-1}\sum_{n=1}^N x_n$ and $\bar{y}_N = N^{-1}\sum_{n=1}^N y_n$. Suppose there is a sequence ε_n converging to 0 such that $x_n = (1+\varepsilon_n)y_n$ for all $n \geq 1$. Then the arithmetic means $\bar{x}_N$ and $\bar{y}_N$ are either both convergent or both divergent; in case of convergence they share the same limit.*

Proof. By hypothesis, there is a sequence $\varepsilon_n \to 0$ such that $x_n = (1+\varepsilon_n)y_n$. Therefore $\bar{x}_N = \bar{y}_N + N^{-1}\sum_{n=1}^N \varepsilon_n y_n$, and hence

$$|\bar{x}_N - \bar{y}_N| \leq B \cdot N^{-1} \sum_{n=1}^N |\varepsilon_n| \to 0 \quad \text{as } N \to \infty,$$

where B is an upper bound for y_n. $\square$

DISCUSSION 1.5: Evidence in favour of d_1 and d_2.

Several analogues to classical density results are true with respect to the d_1-density. This is for instance the case of Dirichlet's theorem on primes in arithmetic progressions. Indeed, if A, M are two relatively prime elements in $\mathbf{F}_q[X]$, with $\deg M \geq 1$, then the set S of all primes of the form $A + Mx$, $x \in \mathbf{F}_q[X]$, satisfies

$$\lim_{n\to\infty} \frac{S_n}{I_n} = \frac{1}{\Phi(M)}, \tag{1.6}$$

where $\Phi(M)$ is the number of non-zero elements in $\mathbf{F}_q[X]$ of degree less than $\deg M$ and prime to M. The Dirichlet density theorem for primes in $\mathbf{F}_q[X]$ states that S possesses a *Dirichlet density* equal to $\bigl(\Phi(M)\bigr)^{-1}$. Rosen ([Ro], p. 40) writes that (1.6) constitutes a *natural density* analogue of the Dirichlet density theorem. That is, one may consider d_1-density as a rightful polynomial analogue to rational prime natural density as defined in (0.1).

The prime natural density defined in (0.1) is akin to the notion of *integer natural density*, which for a set M of natural numbers is defined, provided the limit exists, as

$$\lim_n \frac{M(n)}{n},$$

where $M(n)$ counts natural numbers in M that are $\leq n$. Many classic *integer* natural density results, such as the prime number theorem which states that the ratio $\Pi(n)/n$ is asymptotic to $1/\log n$, or the fact that as $n \to \infty$, the ratio of the number of square-free integers $\leq n$ to n has limit $6/\pi^2 = 1/\zeta(2)$, have beautiful corresponding statements in $\mathbf{F}_q[X]$ stated in terms of the ratios S_n/q^n having appropriate asymptotics or limits (see [Ro], p. 14). Here S is a set of monic polynomials in $\mathbf{F}_q[X]$ ($S = I$ or $S =$ set of square-free monic polynomials in the two former examples). The ratio S_n/q^n is the number of monic polynomials of degree n in S divided by the total number of monic polynomials of degree n in $\mathbf{F}_q[X]$. And taking $\lim_n S_n/q^n$ is akin to the d_1-prime natural density notion. Thus, the fact that $\lim_n S_n/q^n$ makes a good $\mathbf{F}_q[X]$-analogue of the $\mathbf{Z}$-notion of integer natural density suggests that the d_1-density might be a fruitful $\mathbf{F}_q[X]$-analogue of the $d_{\mathbf{Z}}$-prime natural density.

However, a priori, the d_2-density seems more faithful than the d_1-density to the original definition of natural density, since it is the *cumulative* asymptotic proportion of primes in S among all primes up to a certain size. The following two lemmas will help to show that S has a d_1-density if and only if it has a d_2-density, with $d_1(S) = d_2(S)$ if S has such densities. Thus it makes sense to choose the d_1 rather than the d_2-definition since it is simpler.

LEMMA 1.6. *Let $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ be sequences of real numbers satisfying*

i) $v_n > 0 \quad \forall n \geq 1$,

ii) $V_n = \sum_{k=1}^n v_k \to \infty$ *as* $n \to \infty$.

Then $u_n/v_n \to \ell \in \mathbf{R}$ *as* $n \to \infty \implies a_n \to \ell$ *as* $n \to \infty$, *where* $a_n = \sum_{k=1}^n u_k / \sum_{k=1}^n v_k$.

Proof. Let $\varepsilon > 0$. By hypothesis, $\exists\, n_0 \geq 1$, $\forall n \geq n_0$, $|u_n - \ell v_n| < \varepsilon v_n$. Put $C = \sum_{k=1}^{n_0-1} |u_k - \ell v_k|$. Then for $n \geq n_0$, we have

$$|a_n - \ell| \leq V_n^{-1} \sum_{k=1}^{n} |u_k - \ell v_k| \leq V_n^{-1}\big[C + \varepsilon \sum_{k=n_0}^{n} v_k\big] \leq C/V_n + \varepsilon .$$

Since $V_n \to \infty$, we get $\limsup |a_n - \ell| < \varepsilon$, $\forall \varepsilon > 0$. Therefore $\lim a_n = \ell$. □

REMARK. For $v_k = 1$, $\forall k$, Lemma 1.6 is the Cesàro (or arithmetic) mean theorem.

The next lemma is a converse of Lemma 1.6 which is valid provided the rate of growth of the v_n-sequence is fast enough so that v_n is at least comparable in size to V_{n-1}.

LEMMA 1.7. *Using the notation of Lemma* 1.6 *we assume that*

i) a_n *converges to some* $a \in \mathbf{R}$,

ii) $\dfrac{V_{n-1}}{v_n}$ *is a bounded sequence.*

Then the sequence u_n/v_n *converges to* a.

Proof. Note that for $n \geq 2$ we have

$$\begin{aligned} V_{n-1}(a_n - a_{n-1}) &= (V_n - v_n)a_n - V_{n-1}a_{n-1} \\ &= (V_n a_n - V_{n-1}a_{n-1}) - v_n a_n \\ &= u_n - v_n a_n . \end{aligned}$$

Hence $\dfrac{u_n}{v_n} - a_n = \dfrac{V_{n-1}}{v_n}(a_n - a_{n-1}) \to 0$ as $n \to \infty$, which yields the conclusion. □

PROPOSITION 1.8. *Let S be a set of primes of $\mathbf{F}_q[X]$. Then $d_1(S)$ exists if and only if $d_2(S)$ exists. And $d_1(S) = d_2(S)$ in case either density exists.*

Proof. Take $u_n = S_n$ and $v_n = q^n/n$. Assume $d_1(S)$ exists and apply Lemma 1.6 to deduce that $d_2(S)$ exists and is equal to $d_1(S)$. For the converse, by Lemma 1.7, all that is needed is to show that V_{n-1}/v_n is bounded. Since, for any prime power q, both functions $x \to q^x/x$ and $x \to q^x/x^2$ are strictly increasing on the interval $[3, +\infty) \subset (2/\log q, +\infty)$ we can write, for any integer $n \geq 4$,

$$V_{n-1} = \sum_{k=1}^{n-1} \frac{q^k}{k} \leq q + \frac{q^2}{2} + \sum_{k=3}^{n-1} \int_k^{k+1} \frac{q^t}{t}\, dt \leq q^2 + \int_3^n \frac{q^t}{t}\, dt ,$$

and integrating by parts

$$\int_3^n \frac{q^t}{t}\,dt = \frac{1}{\log q}\frac{q^t}{t}\Big]_3^n + \frac{1}{\log q}\int_3^n \frac{q^t}{t^2}\,dt \le \frac{1}{\log q}\frac{q^n}{n} + \frac{n-3}{\log q}\cdot\frac{q^n}{n^2} \le \frac{2}{\log q}\frac{q^n}{n}\,.$$

Hence $\dfrac{V_{n-1}}{v_n} \le \dfrac{n}{q^{n-2}} + \dfrac{2}{\log q} \le 1 + \dfrac{2}{\log q} \le 4\,.$ □

DISCUSSION 1.9: Why consider d_3 and d_4 ?

The set T of Proposition 1.2 has a Dirichlet density and clearly no d_1-density, since the approximants to $d_1(T)$, S_n/I_n, are alternatively 0 and 1. The complement of T in I is $S_q(0)$, in the notation of [Ba1], and it was shown there (see the proof of Theorem 3.1) that primes in $S_q(0)$ can be described by splitting conditions in a normal finite extension of the field of fractions $\mathbf{F}_q(X)$ of $\mathbf{F}_q[X]$. As mentioned in the introduction, sets of rational primes determined in this manner have not only a Dirichlet, but also a natural density. Hence we would expect T to have a natural density in $\mathbf{F}_q[X]$. Again T has no d_1-density, but clearly T has a d_3-density and $d_3(T) = 1/2$. In fact T is not exceptional in this respect. For integers $k \ge 0$, the sets $S_q(k) = \{P \in \mathbf{F}_q[X];\ P \text{ prime and } \deg P \equiv 2^k \pmod{2^{k+1}}\}$ and their subsets $O_q(k) = \{P \in S_q(k);\ \text{order of } X \pmod P \text{ is odd}\}$ have Dirichlet and d_3-densities, but no d_1-density. These sets arose in [Ba1] while counting primes P in $\mathbf{F}_q[X]$ for which the order of $X \pmod P$ is even (or odd). This counting problem is an analogue of the Sierpiński question mentioned in the introduction. Mimicking the method of Hasse [Ha], we found that $S_q(k)$ and $O_q(k)$ are in a precise way (see Remark 3.4 of [Ba1]) $\mathbf{F}_q[X]$-analogues of the Hasse sets $S(k)$ and $O_a(k)$, where for each integer $k \ge 1$, $S(k) = \{p \in \mathbf{N};\ p \text{ prime and } p \equiv 1 + 2^k \pmod{2^{k+1}}\}$ and $O_a(k) = \{p \in S(k);\ \text{order of } a \pmod p \text{ is odd}\}$, with $a \in \mathbf{Z} \setminus \{0, \pm 1\}$. These sets of rational primes have a Dirichlet density, but it is not hard to prove that they also have a natural density.

Moreover, d_3-density is compatible with the notions of d_1 or d_2-density, since by the Cesàro mean theorem, any set having a d_1-density has a d_3-density. And as we shall prove, any set with a d_3-density has a Dirichlet density. So there is a better match between Dirichlet δ and d_3-densities than between d_1 and δ-densities. In fact there is an even better match between d_4 and δ-densities since, as we shall prove, sets with a d_3-density d have a d_4-density equal to d, and sets with a d_4-density d have a Dirichlet density equal to d.

As generalizations of the $d_{\mathbf{Z}}$-notion of natural density in use for sets of primes in the ring $\mathbf{Z}$ (see (0.1)), we are about to show that the d_3 and d_4-definitions are as consistent as the d_2-notion. Rather than giving each prime the same weight, regardless of its norm, as is done in d_2-approximants, in d_3 and d_4-approximants *each norm* is given a weight between 0 and 1 proportional to the number of primes having that norm. For sets in $\mathbf{Z}$, both points of view coincide, because there is only one (monic) integer for each norm, and therefore either 0 or 1 prime having a given norm.

For sets S of rational primes, one way to interpret the approximant of order N to $d_{\mathbf{Z}}(S)$, $r_N = S(N)/\Pi(N)$, is that $S(N)$ is the sum of weights s_n associated to each norm n up to N, where

$$s_n = \frac{\text{number of primes of norm } n \text{ in } S}{\text{number of (monic) integers of norm } n \text{ in } \mathbf{Z}}. \tag{1.7}$$

Of course s_n is either $1/1$ if n is a prime in S, or $0/1$ if not. The denominator $\Pi(N)$ of r_N is interpreted identically but with $S = \Pi$, the set of all rational primes.

Carrying over this interpretation to $\mathbf{F}_q[X]$ yields $s_n = S_n/q^n$ and $r_N = \sum_{n=1}^{N}(S_n/q^n)/\sum_{n=1}^{N}(I_n/q^n)$, the approximant of order N to d_4!

In practice, to estimate $d_{\mathbf{Z}}(S)$ we often consider only approximants of prime order. That is, given $N \geq 1$, we gather the list $p_1 = 2$, $p_2 = 3, \ldots, p_N$ of the N smallest primes, test each p_k for membership in S, count how many lie in S and divide that count by N, since N measures how many lie in Π. Non-prime natural integers are ignored, and this suggests another interpretation in which only norms containing at least one prime are assigned a weight

$$\omega_n = \frac{\text{number of primes in } S \text{ of norm } p_n}{\text{number of primes in } \Pi \text{ of norm } p_n}. \tag{1.8}$$

Note that we do have $S(p_N) = \sum_{n=1}^{N} \omega_n$. In $\mathbf{F}_q[X]$, every norm contains primes. Therefore, S being a set of primes in $\mathbf{F}_q[X]$ and following (1.8), we have $\omega_n = S_n/I_n$, which yields $r_N = \sum_{n=1}^{N}(S_n/I_n)/\sum_{n=1}^{N}(I_n/I_n) = \frac{1}{N}\sum_{n=1}^{N}\omega_n$, the approximant of order N to d_3.

Some of the claims made in the foregoing discussion will now be proved.

LEMMA 1.10. *Let u_n be a sequence of real numbers such that U_N/N converges to some $d \in \mathbf{R}$ as $N \to \infty$, where $U_N = \sum_{n=1}^{N} u_n$. Then*

$$r_N = \frac{1}{\log N} \sum_{n=1}^{N} \frac{u_n}{n} \quad \textit{converges to } d \textit{ as } N \to \infty.$$

Proof. For a function f of class C^1 in $[1, N]$, we have the integration by parts formula

$$\sum_{n=1}^{N} u_n f(n) = \sum_{n=1}^{N} u_n \{ f(N) - \int_n^N f'(t) dt \} = U_N f(N) - \int_1^N f'(t) U_t \, dt,$$

where $U_t = \sum_{1 \le n \le t} u_n$ (see [Ap], Theorem 4.2). So

$$\sum_{n=1}^{N} \frac{u_n}{n} = \frac{U_N}{N} + \int_1^N \frac{U_t}{t^2} dt. \tag{1.9}$$

Since $U_t \sim td$ and $\int_1^N t^{-1} dt = \log N \to \infty$ as $N \to \infty$, we may write

$$\int_1^N \frac{U_t}{t^2} dt = d \int_1^N \frac{\big(1 + o(1)\big)}{t} dt = d\big(1 + o(1)\big) \log N,$$

and by (1.9) we get $(\log N)^{-1} \sum_{n=1}^{N} (u_n/n) = O\big((\log N)^{-1}\big) + d\big(1 + o(1)\big) \to d$ as $N \to \infty$. Note that for $d = 0$, one should replace d by $o(1)$ in the above calculation. □

PROPOSITION 1.11. *Let S be a set of primes in $\mathbf{F}_q[X]$ with a d_3-density. Then $d_4(S)$ exists and $d_4(S) = d_3(S)$.*

Proof. By hypothesis there exists $d \in [0, 1]$ such that

$$\lim_N N^{-1} \sum_{n=1}^{N} (nS_n/q^n) = d.$$

By Lemma 1.10, we have $d = \lim_N (\log N)^{-1} \sum_{n=1}^{N} (S_n/q^n)$, which says that $d_4(S)$ exists and is equal to d. □

To show that a set with d_4-density has Dirichlet density, we first state a lemma on power series that closely resembles Lemma 1.6; the hypotheses are stronger. We intentionally choose a proof that closely mimics the proof of Lemma 1.6.

LEMMA 1.12. *Let $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ be sequences of real numbers satisfying*

i) $v_n > 0 \quad \forall n \geq 1$,

ii) $V_x = \sum_{n\geq 1} v_n x^n \to \infty$ *as* $x \to 1^-$,

iii) *the series* $\sum_{n\geq 1} u_n x^n$ *and* $\sum_{n\geq 1} v_n x^n$ *converge on* $[0,1)$.

Then $u_n/v_n \to \ell \in \mathbf{R}$ *as* $n \to \infty \implies a_x \to \ell$ *as* $x \to 1^-$, *where* $a_x = \sum_{n\geq 1} u_n x^n / \sum_{n\geq 1} v_n x^n$.

Proof. Let $\varepsilon > 0$ and $x \in (0,1)$. By assumption, $\exists n_0 \geq 1$, $\forall n \geq n_0$, $|u_n x^n - \ell v_n x^n| < \varepsilon v_n x^n$. Put $C_x = \sum_{n=1}^{n_0-1} |u_n - \ell v_n|\, x^n$. Then we have

$$|a_x - \ell| \leq V_x^{-1}\big[C_x + \varepsilon \sum_{n\geq n_0} v_n x^n\big] \leq C_1/V_x + \varepsilon\,.$$

By ii), $V_x \to +\infty$ as $x \to 1^-$ so that, ε being arbitrary, $\lim a_x = \ell$ as $x \to 1^-$. $\square$

PROPOSITION 1.13. *Let S be a set of primes in $\mathbf{F}_q[X]$. If S has d_4-density d, then S has Dirichlet density $\delta = d$.*

Proof. Using the notation $s_n = S_n/q^n$, $t_n = I_n/q^n$, $u_n = \sum_{k=1}^n s_k$, $v_n = \sum_{k=1}^n t_k$ and $x = 1/q^{s-1}$, our hypothesis becomes: $u_n/v_n \to d$ as $n \to \infty$. And we wish to show that

$$\sum_{n\geq 1} s_n x^n \Big/ \sum_{n\geq 1} t_n x^n \to d \qquad \text{as } x \to 1^-\,.$$

We first check that u_n and v_n satisfy the hypotheses of Lemma 1.12. For $x \in (0,1)$, $V_x > \sum_{n\geq 1} t_n x^n$. Since $t_n \sim 1/n$, there is some $n_0 \geq 1$ such that $\sum_{n\geq n_0} t_n x^n > 2^{-1} \sum_{n\geq n_0} x^n/n$. But $\sum_{n\geq n_0} x^n/n \to +\infty$ as $x \to 1^-$ since $\sum_{n\geq 1} x^n/n = -\log(1-x) \to +\infty$ as $x \to 1^-$. Hence Lemma 1.12 ii) holds. Since $0 \leq s_n \leq t_n \leq 1/n$ and $\sum_{n\geq 1} x^n/n$ converges on $(0,1)$, the series $\sum_{n\geq 1} s_n x^n$ and $\sum_{n\geq 1} t_n x^n$ converge on $(0,1)$. But for any $x \in (0,1)$, we have

$$(1.10)\quad \sum_{n\geq 1} u_n x^{n+1} = \sum_{n\geq 1} x^n \cdot \sum_{n\geq 1} s_n x^n \quad\text{and}\quad V_x = \sum_{n\geq 1} v_n x^{n+1} = \sum_{n\geq 1} x^n \cdot \sum_{n\geq 1} t_n x^n,$$

so that Lemma 1.12 iii) also holds. Therefore, by (1.10) and Lemma 1.12, we have for all $x \in (0,1)$ that $\sum_{n\geq 1} s_n x^n / \sum_{n\geq 1} t_n x^n$ is equal to the ratio $\sum_{n\geq 1} u_n x^n / \sum_{n\geq 1} v_n x^n$, which converges to d as $x \to 1^-$. $\square$

REMARK. Since $d_3(T) = 1/2$, we now have a second proof of Proposition 1.2.

As we mentioned in the introduction, Bombieri showed Serre a proof that the set of rational primes whose first decimal digit is 1 has Dirichlet density $\log_{10} 2$, but no natural density. Inspired by this example, the next proposition shows that the converse of Proposition 1.11 does not necessarily hold. Note that the set Y defined below is essentially the set of primes whose degree expressed in base 4 has first digit equal to 1.

PROPOSITION 1.14. *The set*

$$Y = \left\{ P \ \textit{prime in} \ \mathbf{F}_q[X] \ ; \ \exists\, n \geq 0,\ 2^{2n} < \deg P \leq 2^{2n+1} \right\}$$

has d_4-density equal to $1/2$, but no d_3-density.

Proof. Let ω_n be the weight function $\omega_n = Y_n/I_n$ defined in (1.8). If Y has a d_4-density then $d_4(Y) = \lim_N r_4(N)$, where, by (1.5), $r_4(N)$ is $(\log N)^{-1} \sum_{n=1}^{N} Y_n/q^n$. We claim that $d_4(Y)$ exists if and only if $\lim (\log N)^{-1} \sum_{n=1}^{N} \omega_n/n$ exists and that in case of existence the previous limit is $d_4(Y)$. Indeed $I_n \sim q^n/n$ means there is an $\varepsilon_n \to 0$ such that $Y_n/q^n = (1+\varepsilon_n)\,\omega_n/n$. Therefore

$$\left| \, r_4(N) - \frac{1}{\log N} \sum_{n=1}^{N} \frac{\omega_n}{n} \, \right| \leq \frac{1}{\log N} \sum_{n=1}^{N} |\varepsilon_n|\, \omega_n/n \leq \frac{1}{\log N} \sum_{n=1}^{N} |\varepsilon_n|/n\,, \tag{1.11}$$

since $\omega_n \in [0,1]$. Now $|\varepsilon_n| \to 0 \implies N^{-1} \sum_{n=1}^{N} |\varepsilon_n| \to 0$, which by Lemma 1.10, implies that $(\log N)^{-1} \sum_{n=1}^{N} |\varepsilon_n|/n \to 0$ and the claim then follows by (1.11).

Thus we have shown that $d_4(Y) = \lim_N X_4(N)/\log\, N$, where $X_4(N) = \sum_{n=1}^{N} \omega_n/n$. Since $\frac{1}{k+1} \leq \int_k^{k+1} \frac{dt}{t} \leq \frac{1}{k}$ for $k \geq 1$, we get for $m \geq 0$

$$\Gamma_m \doteq \sum_{2^{2m}+1}^{2^{2m+1}} \frac{1}{k} = \sum_{2^{2m}}^{2^{2m+1}-1} \frac{1}{k+1} \leq \int_{2^{2m}}^{2^{2m+1}} \frac{dt}{t} = \log 2 \leq \sum_{2^{2m}}^{2^{2m+1}-1} \frac{1}{k} \leq \frac{1}{2^{2m}} + \Gamma_m\,,$$

whence

$$\sum_{m=0}^{n-1} \Gamma_m \leq n \log 2 \quad \text{and} \quad \sum_{m=0}^{n-1} \Gamma_m \geq n \log 2 - \sum_{m=0}^{n-1} \frac{1}{4^m} > n \log 2 - \frac{4}{3}\,. \tag{1.12}$$

By the definition of Y, we have $X_4(2^{2n}) = \sum_{m=0}^{n-1} \Gamma_m$ for $n \geq 1$. Hence, by (1.12), $X_4(2^{2n}) \sim n \log 2$ and $X_4(2^{2n+2}) \sim (n+1)\log 2 \sim n \log 2$. But for any $N \geq 4$, there is a unique $n \geq 1$ with $2^{2n} \leq N < 2^{2n+2}$ and since X_4 is an increasing sequence we have $X_4(N) \sim n \log 2$. Moreover $\log N \sim 2n \log 2$, therefore $\lim_N X_4(N)/\log N = 1/2 = d_4(Y)$.

Let $X_3(N) = \sum_{n=1}^{N} \omega_n$ and assume that $d_3(Y)$ exists. Then $d_3(Y) = 1/2$ and therefore $X_3(N) \sim N/2$ as $N \to \infty$. In particular, $2^{2k} \sim X_3(2^{2k+1}) = X_3(2^{2k+2}) \sim 2^{2k+1}$ as $k \to \infty$ whence $2 = 1$. Therefore, $d_3(Y)$ does not exist. $\square$

REMARK. Both Bombieri's result and Proposition 1.14 can be obtained as special cases of respectively Théorème 2.1 and Corollaire 1.7 of [FuLe].

DISCUSSION 1.15: d_3 is more adequate than d_4.

In Discussion 1.9 we presented reasons why the d_3 and d_4-densities might be more analogous to the usual prime natural density in $\mathbf{Z}$ than the d_1 or d_2-notions. Now we will argue that the d_3-notion provides the best analogy, despite the fact that d_4-density is one step closer to δ-density than is d_3-density.

First from a practical point of view, it is likely that d_3-approximants, $N^{-1}\sum_{n=1}^{N}(S_n/I_n)$, will settle around $\delta(S)$ faster than d_4-approximants $(\log N)^{-1}\sum_{n=1}^{N}(S_n/q^n)$ do. Indeed, both numerators and denominators in d_4-approximants grow on average as the logarithm of corresponding numerators and denominators in d_3-approximants. Hence, if primes in S of small norm do not obey the asymptotic pattern that yields $\delta(S)$, it will take d_4-approximants of much higher order than corresponding d_3-approximants to re-adjust to the asymptotic value. And that means more computing to do.

Also from a conceptual point of view, the d_3-notion is more satisfactory: it seems more natural to make each norm equipollent, i.e. to assign to each norm, or to each degree, a weight potentially equal to that of any other norm. In d_4-approximants the weight of primes in S of degree n is $s_n = S_n/q^n$ and $s_n \leq I_n/q^n < 1/n$ if $n \geq 2$, since for $n \geq 2$ not every element of the finite field $\mathbf{F}_{q^n}$ is of algebraic degree n over $\mathbf{F}_q$ so that $I_n < q^n/n$. Thus larger degrees n are given potential weights that are bounded above by $1/n$ and therefore contribute less than smaller degrees. On the other hand, in d_3-approximants the weight assigned to degree n is $\omega_n = S_n/I_n$, which may reach the value 1 no matter what the size of n is.

Thirdly, as indicated before stating Proposition 1.14, the set Y is an analogue of Bombieri's example. So we would expect Y *not* to have a natural density and to *have* a Dirichlet density (perhaps even equal to $\log_4 2$). And Y has no d_3-density, but has a d_4-density. Therefore it has a Dirichlet density and its value is indeed $1/2 = \log_4 2$.

The Bombieri example points to a simple scaling property associated to the $d_{\mathbf{Z}}$-density:

Let m be an integer ≥ 1 and S be a set of rational primes having a natural density $d > 0$. Then because $\Pi(N)/\Pi(mN) \sim \dfrac{1}{m}$ as $N \to \infty$ one has

$$\frac{S(mN)}{S(N)} \to m \qquad \text{as } N \to \infty\,.$$

The corresponding ratio for a set of primes S in $\mathbf{F}_q[X]$ having positive d_3-density d is

$$\frac{X_3(mN)}{X_3(N)} = \frac{\sum_{n=1}^{mN} \omega_n}{\sum_{n=1}^{N} \omega_n} \sim \frac{mNd}{Nd} = m\,, \qquad \text{where } \omega_n = \frac{S_n}{I_n}\,;$$

this ratio also converges to m as $N \to \infty$. Using this simple property, one sees immediately that the set of rational primes with first decimal digit 1 does not have positive natural density (take $N = 2 \times 10^n$ and $m = 5$ for instance), and Proposition 1.14 uses this property to show that Y does not have a d_3-density. None of the approximants to d_1, d_2 or d_4 shares that property. For d_4 and the set Y, we saw in Proposition 1.14 that if $N = 2^{2n}$, then $X_4(4N)/X_4(N)$ converges to 1 rather than to 4.

Moreover, there is yet another reason to prefer d_3 to d_4 since as we are about to see, Dirichlet density and d_4-density are equivalent notions in $\mathbf{F}_q[X]$.

2. Dirichlet density implies asymptotic Dirichlet average density

THEOREM 2.1. *Let S be a set of primes of $\mathbf{F}_q[X]$ with Dirichlet density $\delta \in [0,1]$. Then S has d_4-density δ.*

We have two radically different proofs of Theorem 2.1. The second proof will be given in detail, while our first proof will only be sketched. It uses a result of Tauber [Tau] as improved further by Littlewood [Li], namely

PROPOSITION 2.2. *If a real power series* $\sum a_n x^n$ *converges to some real number* ℓ *as* $x \to 1^-$ *and* $na_n = O(1)$, *then the series* $\sum a_n$ *converges to* ℓ.

Brief outline of a first proof of Theorem 2.1. Using the notation of Proposition 1.13 and the equivalence of t_n with $1/n$, our hypothesis says that $\delta = \lim_{x\to 1^-} q(x)$, where $q(x)$ is the analytic function on $(-1,1)$ defined by $\sum_{n\geq 1} s_n x^{n-1} / \sum_{n\geq 1}(x^{n-1}/n) = \sum_{n\geq 0} q_n x^n$. To show that Proposition 2.2 applies to the series $q(x)$, we studied the coefficients c_n of the series $\sum_{n\geq 0} c_n x^n = 1/\sum_{n\geq 1}(x^{n-1}/n)$. An integration in the complex plane led to the formula, valid for all $n \geq 2$,

$$c_n = -\int_0^{+\infty} \frac{dt}{(\log^2 t + \pi^2)(1+t)^n}\,. \tag{2.1}$$

Formula (2.1) was used to obtain that all c_n's for $n \geq 1$ are negative. Then, from the relation

$$\sum_{n\geq 0} s_{n+1} x^n \cdot \sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} q_n x^n\,,$$

we got $-\dfrac{1}{n+1} < q_n < \dfrac{1}{n+1}$ and $nq_n = O(1)$. Hence, by Proposition 2.2, $\sum_{n\geq 0} q_n = \delta$. Then it is easy to show that $\sum_{n=1}^N s_n$ is asymptotic to $\delta \sum_{n=1}^N (1/n)$, yielding $\lim_{N\to\infty} \big(\sum_{n=1}^N s_n\big)/\big(\sum_{n=1}^N t_n\big) = \delta$, which says that S has a d_4-density. $\square$

The complete proof of Theorem 2.1 we present is based on observing that d_4-density can also be viewed as an analogue of the notion of logarithmic density for sets of rational primes. If S is a set of primes in $\mathbf{Z}$ then its *logarithmic density* is defined as

$$\lim_{x\to\infty} \frac{\sum_{p\in S,\, p\leq x} p^{-1}}{\sum_{p\leq x} p^{-1}}\,, \quad \text{provided the limit exists.} \tag{2.2}$$

Now for any $x \geq q$, there is a unique natural number $N = N_x$ with

$$q^N \leq x < q^{N+1}\,. \tag{2.3}$$

And one readily checks that if a set S of primes in $\mathbf{F}_q[X]$ has a d_4-density then

$$d_4(S) = \lim_{x\to\infty} \frac{\sum_{P\in S,\, |P|\leq x} |P|^{-1}}{\sum_{|P|\leq x} |P|^{-1}}\,,$$

an obvious analogue to (2.2). This proof is an adaptation of an outline of a proof Carl Pomerance showed us, to the effect that prime Dirichlet density implies logarithmic density for sets of rational primes, a fact which does not appear to be very well known.

Letting S be a set of primes in $\mathbf{F}_q[X]$ and using the notation in (2.3), we begin by stating and proving three short lemmas.

LEMMA 2.3. *We have the estimates*

$$\sum_P |P|^{-s} \sim |\log(s-1)|, \quad \text{as } s \to 1^+,$$

$$\sum_{|P| \le x} |P|^{-1} \sim \log\log x, \qquad \text{as } x \to \infty,$$

and

$$\sum_{|P| \le x} \frac{\log |P|}{|P|} \sim \log x, \qquad \text{as } x \to \infty.$$

Proof. We saw in Proposition 1.2 that $\sum_{n \ge 1} \frac{1}{n} \frac{1}{q^{n(s-1)}} \sim |\log(s-1)|$, as $s \to 1^+$. But $\sum_P |P|^{-s} = \sum_{n\ge 1} \frac{I_n}{q^{ns}} = \sum_{n \ge 1} \frac{I_n}{q^n} \frac{1}{q^{n(s-1)}}$. Applying Lemma 1.12 with $x = q^{1-s}$, $u_n = I_n/q^n$ and $v_n = 1/n$ yields the first estimate since $u_n \sim v_n$. The second estimate is easily seen to hold since, by Lemma 1.6,

$$\log\log x \sim \log\log q^N \sim \log N \sim \sum_{n=1}^{N} \frac{1}{n} \sim \sum_{n=1}^{N} \frac{I_n}{q^n}, \quad \text{as } x \to \infty.$$

But $\sum_{n=1}^{N} I_n/q^n = \sum_{|P|\le x} |P|^{-1}$. The last estimate holds since from $I_n \sim q^n/n$ and Lemma 1.6, we have

$$\sum_{|P|\le x} \frac{\log |P|}{|P|} = \sum_{n=1}^{N} I_n \frac{\log q^n}{q^n} \sim \sum_{n=1}^{N} \frac{q^n}{n} \frac{\log q^n}{q^n} = \sum_{n=1}^{N} \log q \sim \log x. \qquad \square$$

LEMMA 2.4. *The sum* $S_1(x) = \sum_{|P|\le x} |P|^{-1}\left[1 - |P|^{-1/\log x}\right] = O(1)$.

Proof. Let P be a prime. Using the mean value theorem with the function $f_P(u) = |P|^{-u}$, one gets for $u \ge 0$,

$$1 - |P|^{-u} \le \log |P| \cdot u\,.$$

Therefore we have

$$\begin{aligned} S_1(x) &= \sum_{|P|\le x} |P|^{-1}\Big[1 - f_P\big((\log x)^{-1}\big)\Big] \\ &\le \frac{1}{\log x} \sum_{|P|\le x} \frac{\log |P|}{|P|} = O(1), \end{aligned}$$

by the third estimate in Lemma 2.3. □

LEMMA 2.5. *The sum* $S_2(x) = \sum_{|P|>x} |P|^{-1-1/\log x} = O(1)$.

Proof. Since $I_n \le q^n/n$ for any degree n, and $1 - q^{-1/u} \sim \dfrac{\log q}{u}$, we have as $x \to \infty$

$$\begin{aligned} S_2(x) &= \sum_{n>N} I_n q^{-n-n/\log x} \le \sum_{n>N} \frac{q^{-n/\log x}}{n} \\ &\le \frac{1}{N} \sum_{n>N} q^{-n/\log x} = \frac{1}{N} q^{-(N+1)/\log x} \cdot (1 - q^{-1/\log x})^{-1} \\ &\sim \frac{1}{N} q^{-(N+1)/\log x} (\log q/\log x)^{-1} \le \frac{1}{N} q^{-(N+1)/(N+1)\log q} \, \frac{(N+1)\log q}{\log q} \\ &\le \frac{N+1}{N} q^{-1/\log q} \to 1/e. \end{aligned}$$

Hence $S_2(x) = O(1)$. □

Our second proof of Theorem 2.1. Since $\sum_{|P|\le x} |P|^{-1} \sim \log\log x$, we need to show that

$$\sum_{|P|\le x,\ P\in S} |P|^{-1} \tag{2.4}$$

is asymptotic to $\delta \log\log x$ as $x \to \infty$, if $\delta \ne 0$, and is $o(\log\log x)$, if $\delta = 0$. By hypothesis, the sum of $|P|^{-s}$ for $P \in S$ is asymptotic to $\delta|\log(s-1)|$ as s approaches 1 from the right (or is $o(|\log(s-1)|)$ if $\delta = 0$). Let $s = 1 + 1/\log x$, so that, by hypothesis, the sum of $|P|^{-1-1/\log x}$ for P in S is either asymptotic to $\delta \log\log x$ as x goes to infinity if $\delta \ne 0$, or is $o(\log\log x)$ if $\delta = 0$. Now this sum differs from the sum in (2.4) by $O(1)$. Indeed, the contribution of the gap between $|P|^{-1}$ and $|P|^{-1-1/\log x}$ for $|P|$ up to x is $O(1)$ by Lemma 2.4 and the contribution from $|P| > x$ is also $O(1)$ by Lemma 2.5. The result follows. □

3. Artin's conjecture and natural density

Let a be a non-square integer different from -1. Hooley [Ho] showed, provided the Riemann hypothesis holds for certain number fields, that the set of primes p having a as a primitive root possesses a density. His work is valid for natural and Dirichlet density.

However, as mentioned in the introduction, the argument Bilharz uses at the end of his paper ([Bi], pp. 490–491) shows that, given a prime p and the polynomial $A = X$ of $\mathbf{F}_p[X]$, the set S of primes $P \in \mathbf{F}_p[X]$ having X as a primitive root does not have a natural density, where this density was "naturally" taken to be the d_2-notion, despite the fact that this set has a Dirichlet density.

Let us rewrite the Bilharz-Davenport argument in the next proposition using the equivalent d_1-notion of density and replacing p by any power q of p. The letter ℓ will denote a prime and S the set of primes of $\mathbf{F}_q[X]$ having X as a primitive root.

Proposition 3.1. *The set S of primes in $\mathbf{F}_q[X]$ having X as a primitive root does not have a d_1-density.*

Proof. Note that a prime P of degree n is in S if and only if $X^{q^n-1} \equiv 1 \pmod{P}$ in $\mathbf{F}_q[X]$, but $X^k \not\equiv 1 \pmod{P}$ for any proper natural divisor k of q^n-1. That is, if and only if each of the n roots α of P satisfies $\mathbf{F}_q(\alpha) \simeq \mathbf{F}_{q^n}$, or each such root is one of the $\varphi(q^n-1)$ generators of the group $\mathbf{F}_{q^n}^*$. Hence we have shown that S_n is equal to $\varphi(q^n-1)/n$. Therefore

$$\frac{S_n}{I_n} \sim \frac{(q^n-1)y_n/n}{q^n/n} \sim y_n\,,$$

where y_n is the product $\prod_{\ell | q^n-1}(1-1/\ell)$. To show that S does not have a d_1-density we exhibit two sequences of integers (n_t) and (n_s) such that $y_{n_t} \to 0$ and $y_{n_s} \not\to 0$.

For an integer $t > 1$, define n_t as the product $\prod\limits_{\ell \le t,\, \ell \ne p}$ [order of $q \pmod{\ell}$]. Then

$$y_{n_t} \le \prod_{\ell \le t,\, \ell \ne p} \bigl(1 - 1/\ell\bigr)\,.$$

Since the product $P_x = \prod_{\ell \le x}(1-1/\ell)$ satisfies

$$P_x^{-1} = \prod_{\ell \le x}(1-1/\ell)^{-1} = \prod_{\ell \le x}\sum_{k \ge 0} \ell^{-k} \ge \sum_{n \le x} 1/n \ge \int_1^{[x]+1} u^{-1}du \ge \log x\,,$$

we have $P_x \le (\log x)^{-1}$ and therefore $y_{n_t} = O\bigl((\log t)^{-1}\bigr) \to 0$ as $t \to \infty$.

On the other hand for $n_s = p_s$, the s-th prime distinct from p, primes ℓ dividing $q^{n_s} - 1$ are not overabundant, so that $y_{n_s} \not\to 0$ as $s \to \infty$. Indeed, $\ell \mid q^{p_s} - 1$ implies, for one thing, that either $\ell \mid q - 1$ or $\ell \equiv 1 \pmod{p_s}$, but also that $\ell < q^{p_s}$. So

$$\log y_{n_s} = \sum_{\ell \mid q^{p_s}-1} \log(1 - 1/\ell) \geq - \sum_{\ell \mid q^{p_s}-1} (1/\ell) + O(1),$$

whence

$$\begin{aligned} - \log y_{n_s} &\leq \sum_{\ell \mid q^{p_s}-1} (1/\ell) + O(1) = \sum_{\substack{\ell \equiv 1 \pmod{p_s} \\ \ell \leq q^{p_s}}} (1/\ell) + O(1) \\ &\leq C(\log\log q^{p_s})/\varphi(p_s) + O(1), \end{aligned}$$

for some absolute constant C. We used the fact that the estimate

$$\sum_{\substack{\ell \equiv c \pmod{b} \\ \ell \leq x}} (1/\ell) \ll \frac{\log\log x}{\varphi(b)} \tag{3.1}$$

holds uniformly in, say, the range $x \geq 80$ and $1 \leq c < b < \sqrt{x}$ with c and b coprime. Here $x = q^{p_s}$ and $b = p_s = \log x / \log q < 2\log x < \sqrt{x}$. Note that (3.1) is easily deduced from the Brun-Titchmarsh inequality [MoVa]

$$\pi(x;\ b,\ c) \leq \frac{2}{\varphi(b)} \frac{x}{\log(x/b)},$$

valid for $x \geq 2$, $b < x$ and $\gcd(b, c) = 1$, where $\pi(x;\ b,\ c)$ counts primes congruent to $c \pmod{b}$ that are $\leq x$.

Now

$$\frac{\log\log q^{p_s}}{\varphi(p_s)} \sim \frac{\log p_s}{p_s} \to 0, \qquad \text{as } s \to \infty,$$

so $-\log y_{n_s} = O(1)$ and therefore $y_{n_s} \not\to 0$ as $s \to \infty$. □

However we show that the same set S has a d_3-density.

PROPOSITION 3.2. *The set S of primes in $\mathbf{F}_q[X]$ having X as a primitive root has a d_3-density.*

Proof. Theorem 3 from the Russian paper [Shp] states that

$$N^{-1} \sum_{n=1}^{N} \frac{\varphi(|ag^n + b|)}{|ag^n + b|} = d + O\big((\log N)/N\big), \tag{3.2}$$

for some positive constant d, where $a, b, g \in \mathbf{Z}$, $g \geq 2$ and $\gcd(ag, b) = 1$.

Taking $a=1$, $g=q$ and $b=-1$, we get that $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N}\varphi(q^n-1)/(q^n-1)$ exists. And since

$$\frac{\varphi(q^n-1)}{q^n-1}\sim\frac{\varphi(q^n-1)/n}{q^n/n}\sim\frac{\varphi(q^n-1)/n}{I_n}=\frac{S_n}{I_n},$$

Lemma 1.4 yields that $N^{-1}\sum_{n=1}^{N}(S_n/I_n)$ converges as $N\to\infty$. Thus $d_3(S)$ exists. $\square$

REMARK. The result of Shparlinski (3.2) is a special case of fairly general asymptotic estimates of averages of arithmetic functions evaluated at consecutive terms of recurrence sequences. See [LuSh].

We end this section with a final remark.

REMARK 3.3 (Chebotarev sets of primes in $\mathbf{F}_q[X]$). Recall that any Chebotarev set of primes in $\mathbf{Z}$ has a natural density. So we ask whether it might be true that any Chebotarev set of primes in $\mathbf{F}_q[X]$ has a d_3-density. By a *Chebotarev set* we mean the set of primes whose Artin symbol is a given conjugacy class $\mathcal{C}$ of the Galois group of some finite Galois extension F over $\mathbf{F}_q(X)$. There is a large class of such extensions, the so-called *geometric extensions*, for which corresponding Chebotarev sets have a d_1-density (see [Ro], Theorem 9.13B, p. 125). But as we will show, the set T is a Chebotarev set and has no d_1-density. So not every Chebotarev set in $\mathbf{F}_q[X]$ has a d_1-density. However even for non-geometric extensions such sets have a d_3-density.

Indeed, if L is the algebraic closure of $\mathbf{F}_q$ in F then all elements of $\mathcal{C}$ restrict to the same element σ^a of the cyclic Galois group of $L/\mathbf{F}_q$, where σ is the automorphism of $L/\mathbf{F}_q$ that sends x to x^q. Now call S the set of primes P of $\mathbf{F}_q[X]$ whose associated Frobenius automorphisms lie in $\mathcal{C}$. Then Proposition 5.16 of [FrJa] shows that $S_n=0$ if $n\not\equiv a \pmod u$, where u is the degree extension $[L:\mathbf{F}_q]$, while if $n\equiv a \pmod u$, then $S_n/I_n\sim|\mathcal{C}|/v$ as $n\to\infty$, where v is the degree extension $[F:L\mathbf{F}_q(X)]$. Therefore $d_3(S)$ exists and is equal to $|\mathcal{C}|/uv$.

For $S=T$, consider a field $F\simeq\mathbf{F}_{q^2}(X)$, so that $L=\mathbf{F}_{q^2}$, $u=2$, $v=1$. Let $\mathcal{C}=\{id\}$ be the conjugacy class of the identity automorphism of $F/\mathbf{F}_q(X)$ so that $a=0$. Then $P\in T$ if and only if $\tau_P=id$, where τ_P is the Frobenius automorphism of P. Indeed, for a prime P of degree n and α a primitive element of $\mathbf{F}_{q^2}/\mathbf{F}_q$, we have $\tau_P(\alpha)=id(\alpha)$ if and only if $\alpha^{|P|}=\alpha^{q^n}=\alpha$, which holds exactly when n is even. So T is a Chebotarev set. Of course we again find that $d_3(T)=1/2$ since $|\mathcal{C}|/uv=1/2$.

4. Conclusion

We summarize our results in a theorem accompanied by the synoptic diagram

$$(d_1 \iff d_2) \implies d_3 \implies (d_4 \iff \delta).$$

THEOREM A. *Let S be a set of primes in $\mathbf{F}_q[X]$. Then the statements below hold true.*

i) *Should S possess any two of the five kinds of densities defined above (d_1, d_2, d_3, d_4 or Dirichlet δ-density), then their values necessarily coincide.*

ii) *S has a d_1-density if and only if S has a d_2-density.*

iii) *If S has a d_1-density, then S has a d_3-density. The converse is false; for instance, the set T of Proposition* 1.2 *has a d_3-density equal to* $1/2$, *but no d_1-density.*

iv) *If S has a d_3-density, then S has a d_4-density. The converse is false; for instance, the set Y of Proposition* 1.14 *has a d_4-density equal to* $1/2$, *but no d_3-density.*

v) *S has a d_4-density if and only if S has a Dirichlet density δ.*

Therefore of the five types of $\mathbf{F}_q[X]$-prime densities considered, three are essentially distinct.

Taking into account the three discussions of Section 1 and the results of Section 3 it is our belief, based on conceptual, computational and qualitative properties that the d_3-notion represents, within the few candidates we examined, the most viable analogue of prime natural density in $\mathbf{Z}$. By *qualitative properties* we mean again that: sets of primes linked to the $\mathbf{F}_q[X]$-analogue of the Sierpiński question, sets related to Artin's conjecture and Chebotarev sets of primes in $\mathbf{F}_q[X]$, have a d_3-density, but do not necessarily have a d_1-density, while our $\mathbf{F}_q[X]$-analogue of the Bombieri example has a Dirichlet density but no d_3-density. So it is tempting to adopt the following terminology: a set of primes in $\mathbf{F}_q[X]$ will be said to have a *(prime) natural density* if it has a d_3-density. A set of primes having a d_1-density should be viewed as having a strong form of natural density, which could be referred to as *uniform* prime natural density. The d_4-density appears to act as a faithful analogue of the logarithmic density in use for sets of rational primes. It can be used as a tool, as it was in Proposition 1.14, to determine whether some set of primes with no natural density has a Dirichlet density, or not.

Other examples should be tested to confirm, or invalidate, d_3 as a rightful analogue of prime natural density in $\mathbf{Z}$.

We conclude by a general question pertaining to the density theory in number systems presented in the book [Bu]. Consider a class $\mathcal{K}$ of finite structures and a property $\mathcal{P}$. Define, for each $n \geq 1$, $\mathcal{K}_n$ as the subset of these structures of size n. Define p_n as the proportion of structures in $\mathcal{K}_n$ that have property $\mathcal{P}$, and P_n as the proportion of structures in $\bigcup_{k=1}^{n} \mathcal{K}_k$ that have property $\mathcal{P}$. In [Bu], the function p_n (resp. P_n) is referred to as the *local* (resp. *global*) *counting function* and is somewhat akin to the d_1 (resp. d_2) approximant of order n. General results [Bu] have been proved in which the existence of a limit for p_n, or P_n, as $n \to \infty$ has been established. Suppose we define an *average counting function* $\bar{p}_n$ as $\frac{1}{n} \sum_{k=1}^{n} p_k$. Are there classes of structures such that neither p_n, nor P_n have a limit law, but $\bar{p}_n$ does ? Can we build a general theory of limit laws based on the function $\bar{p}_n$? This function being similar to the d_3-approximant of order n, Propositions 1.11 and 1.13 of our paper suggest that if $\bar{p}_n$ converges there are general hypotheses under which the associated Dirichlet density also exists.

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