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THE Σ^n-CONJECTURE FOR METABELIAN GROUPS

by Robert BIERI

Let G be a finitely generated group, m the $\mathbf{Z}$-rank of its abelianization G/G', and A a G-module. The *geometric invariant* $\Sigma^n(G;A)$ is a conical subset of the $\mathbf{R}$-vector space $\mathrm{Hom}(G,\mathbf{R}^{\mathrm{add}}) \cong \mathbf{R}^m$, which collects information on the finiteness properties of A, when regarded as a module over certain subrings $\Lambda \subseteq \mathbf{Z}G$. Recall that A is said to be *of type* FP_n over Λ if A admits a resolution by free Λ-modules which are finitely generated in all dimensions $\leq n$. To say that G is *of type* FP_n means that the trivial G-module $\mathbf{Z}$ is of type FP_n. Following [4] we consider, for each homomorphism $\chi\colon G \to \mathbf{R}^{\mathrm{add}}$, the submonoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$, and define

DEFINITION 13.1.

$$\Sigma^n(G;A) := \{\chi \mid A \text{ is of type } \mathrm{FP}_n \text{ over the monoid ring } \mathbf{Z}G_\chi\},$$

in particular,

$$0 \in \Sigma^n(G;A) \Leftrightarrow \Sigma^n(G;A) \neq \varnothing \Leftrightarrow A \text{ is of type } \mathrm{FP}_n \text{ as } \mathbf{Z}G\text{-module.}$$

It is of considerable interest to have information on these invariants, as they allow one to find all normal subgroups $N \lhd G$ of type FP_n, with $Q = G/N$ abelian.

THEOREM 13.2 ([4]). *$N \lhd G$, with G/N abelian, is of type FP_n, if and only if $\chi(N) = 0$ implies $\chi \in \Sigma^n(G;\mathbf{Z})$.*

Properties of $\Sigma^n(G;A)$ are often easier to state in terms of its complement in $\mathrm{Hom}(G,\mathbf{R})$. We will use the following notation: if X and Y are subsets of $\mathrm{Hom}(G,\mathbf{R})$ then $X^c := \mathrm{Hom}(G,\mathbf{R}) - X$ stands for the complement and $X + Y := \{x + y \mid x \in X,\ y \in Y\}$ for their sum.

The group G is *metabelian* if it contains a normal subgroup $A \lhd G$ with both A and $Q := G/A$ abelian. A is then a Q-module via conjugation in G. Since the group G is finitely generated, we have $\Sigma^0(G;\mathbf{Z}) = \mathrm{Hom}(G,\mathbf{R})$. In the metabelian case one observes that $\Sigma^1(G;\mathbf{Z})$ depends only on the Q-module A, and that its complement is contained in the linear subspace $\mathrm{Hom}(Q,\mathbf{R}) \subseteq \mathrm{Hom}(G,\mathbf{R})$; in fact,

$$\Sigma^1(G;\mathbf{Z})^c = \Sigma^0(Q;A)^c .$$

$\Sigma^0(Q;A)^c$ is fairly well understood. By [2] it is a *closed rational polyhedral cone* (i.e., it can be described in terms of finitely many inequalities with integer coefficients). In principle it can effectively be constructed, by Groebner-basis techniques, from a presentation of the Q-module A.

Examples show that the higher invariants $\Sigma^n(G;\mathbf{Z})$ are, in general, independent of $\Sigma^1(G;\mathbf{Z})$; not so in the metabelian case:

CONJECTURE 13.3 (Σ^n-Conjecture). *If G is a finitely generated metabelian group, and $n > 0$, then*

$$\Sigma^n(G;\mathbf{Z})^c = \bigcup_{1\le k\le n} \underbrace{\left(\Sigma^1(G;\mathbf{Z})^c + \ldots + \Sigma^1(G;\mathbf{Z})^c\right)}_{k \text{ copies}} .$$

Note that this contains the older

CONJECTURE 13.4 (FP_n-Conjecture). *If G is a finitely generated metabelian group, and $n > 0$, then*

$$G \text{ is of type } \mathrm{FP}_n \iff 0 \notin \underbrace{\Sigma^1(G;\mathbf{Z})^c + \ldots + \Sigma^1(G;\mathbf{Z})^c}_{n \text{ copies}} .$$

The power of these conjectures (and the partial result on low dimensions and special classes of groups) lies in the fact that neither of the two inclusions is easy. It has a number of intriguing consequences like:

- If a metabelian group G is of type FP_n, so is every homomorphic image of G.
- If $A \lhd G$, $Q := G/A$ as above then whether or not G is of type FP_n depends only on the Q-module A.
- If $A \lhd G$, $Q := G/A$ as above and $P := \{(x,y) \mid xA = yA\} \le G \times G$ is the (untwisted) fiber product then G is of type FP_n if and only if P is of type FP_n.

The Conjectures have been open for twenty years. Progress on the Σ^n-Conjecture in special situations was always triggered by progress on the FP_n-Conjecture [5], [1], [9], [6], [13], [3]. The Σ^n-Conjecture has been settled, by Holger Meinert [12], in the case when G has finite Prüfer rank and by Dessislava Kochloukova [10] when A is torsion with Krull dimension 1. Kochloukova [11] and Harlander–Kochloukova [8] established it when $n = 2$ and, in the semi-direct product case, for $n = 3$ [7].

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