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PUTATIVE RELATION GAPS

by Martin R. BRIDSON and Michael TWEEDALE

Given $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$, the action of the free group $F(\mathcal{A})$ by conjugation on $R = \langle\langle \mathcal{R} \rangle\rangle$ induces an action of Γ on the abelian group $M = R/[R,R]$. It is obvious that the rank of M as a $\mathbf{Z}\Gamma$-module serves as a lower bound on the minimal number of relators that one requires to present Γ on the generators $\mathcal{A}$. This lower bound seems so crude that one cannot imagine it would be sharp in general. And yet, despite sustained attack over many years, not a single example has been established to lend substance to this intuition. The question of whether or not there exists such an example has become known as the *relation gap problem.*

The problem has its origins in the early work of Karl Gruenberg and others on relation modules in the 1960s and 1970s; it first seems to have appeared in print in [5] and occurs twice in different guises in Wall's famous problem list [10], though the name does not enter the literature until 1993 [7]. It belongs to a circle of notoriously hard problems concerning the homotopy properties of finite 2-complexes — the Andrews–Curtis conjecture, Whitehead's asphericity conjecture, the Eilenberg–Ganea conjecture, and the question (resolved by M. Bestvina and N. Brady [1]) of finite presentability versus FP_2.

To clarify our convictions regarding the relation gap problem we state

CONJECTURE 15.1. *There exist finite group presentations with arbitrarily large relation gaps.*

This conjecture is closely related to the $D(2)$ conjecture: if a group Γ with $H^3(\Gamma; \mathbf{Z}\Gamma) = 0$ admits a presentation that both realizes the group's deficiency and has a relation gap, then the $D(2)$ conjecture is false, i.e. there exists a finite 3-complex that looks homologically like a 2-complex, in the sense that it possesses Wall's property $D(2)$, but that does not have the homotopy type of a finite 2-complex. (This result is due to M. Dyer; a published proof can be found in J. Harlander's survey article [8].)

In the remainder of this note, we'll describe two families of groups and indicate why we think that they ought to have relation gaps, making explicit conjectures to that effect. The two families are of a very different nature: the first consists of groups with finite classifying spaces, based on the Bestvina–Brady construction; the second is comprised of virtually free groups and it is the nature of the torsion that dictates the key features of the relation module that we believe lead to a relation gap.

Cyclic coverings and right-angled Artin groups

Let Σ be a connected flag complex with non-trivial, perfect fundamental group, and let G be the associated right-angled Artin group. This group has a presentation with generating set the vertices v_i of Σ, and defining relations asserting that two generators commute if and only if the corresponding vertices in Σ are joined by an edge. Let $\pi\colon G \to \mathbf{Z}$ be the homomorphism sending each v_i to a fixed generator. The kernel of π is the *Bestvina–Brady group* H_Σ, which is FP_2 but not finitely presented [1].

Let $\Gamma_n \subset G$ be the index n subgroup $\pi^{-1}(n\mathbf{Z})$. Notice that H_Σ is the intersection of the Γ_n. We construct in [2] a presentation for Γ_n where the generators S_n are indexed by the vertices of Σ and the $\mathbf{Z}\Gamma$-rank of the relation module is bounded independently of n.

Conjecture 15.2. *The number of relators needed to present Γ_n on the generators S_n goes to infinity as $n \to \infty$, so Γ_n has a relation gap for n sufficiently large.*

Some virtually free examples

We consider groups similar in spirit to ones considered by D. Epstein [4], C. Hog-Angeloni, W. Metzler and M. Lustig [9], and more recently by K. Gruenberg and P. Linnell [6].

Given letters x_m and t_m, let ρ_m be the word

$$\rho_m = (t_m x_m t_m^{-1}) x_m (t_m x_m^{-1} t_m^{-1}) x_m^{-m} .$$

We look at the groups $\Gamma_{m,n} = Q_m * Q_n$, where

$$Q_m = \langle x_m, t_m \mid \rho_m, x_m^{m-1} \rangle$$

and $(m^{m-1} - 1)$ and $(n^{n-1} - 1)$ are coprime.

One of the main attractions we see in these new examples is that one can give a short, transparent and natural proof [3] that the relation module of the obvious presentation of $\Gamma_{m,n}$ can be generated by three elements, namely the images of ρ_m, ρ_n and $x_m^{m-1}x_n^{n-1}$.

The groups $\Gamma = \Gamma_{m,n}$ are virtually free, so $H^3(\Gamma; \mathbf{Z}\Gamma) = 0$. Thus one could find both a relation gap and a counterexample to the $D(2)$ conjecture simply by solving the following concrete problem:

CONJECTURE 15.3. *The kernel of the map $F_4 \to \Gamma$ associated to the presentation*

$$\Gamma = \left\langle x_m, t_m, x_n, t_m \;\middle|\; \rho_m, x_m^{m-1}, \rho_n, x_n^{n-1} \right\rangle$$

is not the normal closure of three elements.

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