Zeitschrift: L'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: Volumes of ideal simplices in Hilbert's geometry and symmetric spaces
Autor: Bucher-Karlsson, Michelle / Karlsson, Anders
DOI: https://doi.org/10.5169/seals-109886

L'Enseignement Mathématique (2) 54 (2008), 53–55

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VOLUMES OF IDEAL SIMPLICES IN HILBERT'S GEOMETRY AND SYMMETRIC SPACES

by Michelle BUCHER-KARLSSON and Anders KARLSSON

Let X be a bounded convex domain in $\mathbf{R}^n$ endowed with its natural *Hilbert metric* d, where $d(x,y)$ equals the logarithm of the projective cross-ratio of x and y, and the endpoints of the chord through x and y. It is also a Finslerian metric and as such there is a natural notion of volume.

The question is to study the volume of ideal simplices in this geometry. Of particular interest is to find out which admit the minimal or maximal volume, and whether the volume is bounded at all. Two examples:

- Let X be a ball in $\mathbf{R}^n$. Then X with the Hilbert metric is nothing but the n-dimensional hyperbolic space. It is a well-known result of Haagerup and Munkholm in [4] that *the volume of an ideal geodesic simplex is maximal if and only if the simplex is regular*, i.e. any permutation of its vertices can be realized by an isometry.
- Let X be a triangle in $\mathbf{R}^2$. To avoid degeneracies, we restrict our attention to simplices which have their three vertices on the three different 1-faces of X. Colbois, Vernicos and Verovic in [3] showed that, while the volume (or area) of ideal triangles is unbounded, its minimum is attained by the simplex having as vertices the midpoints of the 1-faces of X — and its isometric copies of course. This can be given an elementary proof without any computation and without knowing the exact definition of the invariant volume, once it is observed that those simplices are here again, the regular ones.

Thus, the slogan we would like to advocate here is that "the most regular" simplices have extremal volume.

The symmetric space $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$ has a natural model as a bounded, convex domain in $\mathbf{R}^{n(n+1)/2}$, namely as the positive definite matrices normalized to have trace equal to 1. The action of $\mathrm{SL}(n,\mathbf{R})$ on $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$

is then given by the projective transformations $S \mapsto g \cdot S = (1/\operatorname{tr}(gSg^t))gSg^t$. Note that the Riemannian symmetric metric does not coincide with the Hilbert metric. The Hilbert volume or the Riemannian volume can be used to define a cocycle

$$V\colon (g_0, \dots, g_d) \longmapsto \operatorname{Vol}\langle g_0 \cdot x, \dots, g_d \cdot x\rangle$$

mapping the $(d+1)$-tuple of elements of $\mathrm{SL}(n, \mathbf{R})$, where d denotes the dimension of $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$, to the volume of the convex simplex with vertices the orbit of a fixed point x in $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$. The cocycle V represents the top dimensional generator of the real valued continuous cohomology $H_c^*(\mathrm{SL}(n, \mathbf{R}))$ of $\mathrm{SL}(n, \mathbf{R})$.

QUESTION 17.1. *Is the cocycle V uniformly bounded?*

A positive answer to this question would give a new unified proof of the fact that a compact manifold whose universal cover is isometric to the symmetric space $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ has strictly positive simplicial volume, proven by Thurston in [7] for $n = 2$, by the first-named author of this note for $n = 3$ in [1] and by Lafont and Schmidt for $n \geq 4$ in [6].

The behavior of the Hilbert volume in $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ is mixed, since the latter space contains both isometric copies of the hyperbolic space and of triangles.

QUESTION 17.2. *Which convex simplices in $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ have extremal volume?*

In this general context the reader is encouraged to study in particular section 3.1 in [5]. The understanding of these questions also in the cases of lower-dimensional simplices and volumes may furthermore yield insights, or the solution of, the conjecture on the surjectivity of the comparison map

$$H_{c,b}^*(\mathrm{SL}(n, \mathbf{R})) \to H_c^*(\mathrm{SL}(n, \mathbf{R}))$$

for $\mathrm{SL}(n, \mathbf{R})$ discussed in this volume by Burger, Iozzi, Monod and Wienhard [2].

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Michelle Bucher-Karlsson
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
e-mail: mickar@math.kth.se

Anders Karlsson
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
e-mail: akarl@kth.se