A short geometric proof of a conjecture of Fulton

L'Enseignement Mathématique (2) 57 (2011), 103–115

A SHORT GEOMETRIC PROOF OF A CONJECTURE OF FULTON

by Nicolas RESSAYRE

ABSTRACT. We give a new geometric proof of a conjecture of Fulton about the Littlewood-Richardson coefficients. This conjecture was first proved by Knutson, Tao and Woodward using the Honeycomb theory (see [KTW04]). A geometric proof was given by Belkale in [Bel07b]. Our proof is based on the geometry of Horn cones.

1. INTRODUCTION

1.1 THE HORN CONJECTURE

We start with a question first considered by H. Weyl [Wey12] in 1912:

What can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands?

Let $H(n)$ denote the set of n by n Hermitian matrices. For $A \in H(n)$, we denote its spectrum by $\alpha(A) = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$ repeated according to multiplicity and ordered so that $\alpha_1 \geq \cdots \geq \alpha_n$. We set

$$\Delta(n) := \{(\alpha(A), \alpha(B), \alpha(C)) \in \mathbf{R}^{3n} \,:\, A,\, B,\, C \in H(n) \text{ s.t. } A + B + C = 0\}.$$

In 1962, Horn proposed a conjectural answer to Weyl's question. Indeed, Horn conjectured in [Hor62] an inductive description of $\Delta(n)$. We now introduce notation in order to state the Horn conjecture. Set $E(n) = \mathbf{R}^{3n}$, let $E(n)^+$ denote the set of $(\alpha_i, \beta_i, \gamma_i) \in E(n)$ such that $\alpha_i \geq \alpha_{i+1}$, $\beta_i \geq \beta_{i+1}$ and $\gamma_i \geq \gamma_{i+1}$ for all $i = 1, \dots, n-1$. Because of the trace, the points (α, β, γ) in $\Delta(n)$ satisfy $\sum_{i=1}^{n}(\alpha_i + \beta_i + \gamma_i) = 0$. Let $E_0(n)$ denote the hyperplane of $E(n)$ defined by this condition.

Achevé de composer le 28 juin 2011 à 15:18

Let $\mathcal{P}(r,n)$ denote the set of subsets of $\{1,\dots,n\}$ with r elements. To any $I=\{i_1<\cdots<i_r\}\in\mathcal{P}(r,n)$ we usually associate (see Section 4 for details) a partition $\lambda_I=(i_r-r\geq\cdots\geq i_1-1)\in\mathbf{Z}^r_{\geq 0}$. If J and K are two other elements of $\mathcal{P}(r,n)$ then $(\lambda_I,\lambda_J,\lambda_K-2(n-r)1^r)$ belongs to $\mathbf{Z}^{3r}$ and so to $E(r)$. Note that $(\lambda_I,\lambda_J,\lambda_K-2(n-r)1^r)$ belongs to $E(r)^+$ and to $E_0(r)$.

THE HORN CONJECTURE ([Hor62]). *Let $(\alpha_i,\beta_i,\gamma_i)\in E_0(n)\cap E(n)^+$. Then, $(\alpha_i,\beta_i,\gamma_i)$ belongs to $\Delta(n)$ if and only if for every $r=1,\dots,n-1$, for every $(I,J,K)\in\mathcal{P}(r,n)^3$ such that*

$$(\lambda_I,\lambda_J,\lambda_K-2(n-r)1^r)\in\Delta(r)\,, \tag{1}$$

the following inequality holds:

$$\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j+\sum_{k\in K}\gamma_k\leq 0\,. \tag{2}$$

Note that this conjecture implies that $\Delta(n)$ is a closed convex polyhedral cone. This fact is a consequence of convexity results in Hamiltonian geometry (see [Kir84]). The combination of a theorem of Klyachko [Kly98] with a theorem of Knutson-Tao [KT99] implies the truth of this conjecture (see Section 2 for details).

1.2 LITTLEWOOD-RICHARDSON COEFFICIENTS

Recall that the irreducible representations of $\mathrm{GL}_r(\mathbf{C})$ (or $U_r(\mathbf{C})$ if you want to work with a compact Lie group) are indexed by sequences $\lambda=(\lambda_1\geq\cdots\geq\lambda_r)\in\mathbf{Z}^r$ (see for example [FH91, Lecture 6]). Denote by V_λ the representation corresponding to λ. Like any representation of $\mathrm{GL}_r(\mathbf{C})$, the tensor product $V_\lambda\otimes V_\mu$ of two given irreducible representations V_λ and V_μ is a sum of irreducible representations. We define the *Littlewood-Richardson coefficients* $c^\nu_{\lambda\,\mu}\in\mathbf{N}$ as the corresponding multiplicities:

$$V_\lambda\otimes V_\mu=\sum_\nu c^\nu_{\lambda\,\mu}V_\nu\,. \tag{3}$$

The Knutson-Tao theorem [KT99] was previously known as the

SATURATION CONJECTURE. *If, for some $n>0$, $c^{n\nu}_{n\lambda\,n\mu}\neq 0$ then $c^\nu_{\lambda\mu}\neq 0$.*

This note is about another relation between the Horn conjecture and the sequences of *stretched Littlewood-Richardson coefficients*; that is the sequences $(c^{n\nu}_{n\lambda\,n\mu})_n\in\mathbf{N}$. Namely, we will prove the following

FULTON CONJECTURE. *For any* $n > 0$, $c^{\nu}_{\lambda\,\mu} = 1 \;\Rightarrow\; c^{n\nu}_{n\lambda\,n\mu} = 1$.

This conjecture was first proved by Knutson, Tao and Woodward [KTW04] using the Honeycomb theory. A geometric proof was given by Belkale in [Bel07b]. The aim of this note is to give a short proof of this conjecture based on the geometry of Horn cones.

The proof of the Horn conjecture is much more involved than its statement. In Section 2, we give an idea of the history of this proof and the subjects interplaying with it. Section 3 is concerned with the codimension one faces of the Horn cones. Sections 2 and 3 are mainly expository; we give proofs only when elementary linear algebra allows it. The last section is our proof of Fulton's conjecture.

2. SCHUBERT CALCULUS AND THE HORN CONJECTURE

2.1 SCHUBERT CALCULUS

Let $\mathrm{Gr}(a,b)$ be the *Grassmann variety* of a-dimensional linear subspaces L of $V = \mathbf{C}^{a+b}$. Let $F_\bullet : \{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{a+b} = V$ be a *complete flag* of $\mathbf{C}^{a+b}$ (i.e. F_i is a i-dimensional subspace of V). The relative position of $L \in \mathrm{Gr}(a,b)$ and $F_\bullet$ defines a partition of $\mathrm{Gr}(a,b)$ which is a cellular decomposition and allows one to describe the topology of $\mathrm{Gr}(a,b)$. More precisely, for any subset $I = \{i_1 < \cdots < i_a\}$ of cardinality a in $\{1,\ldots,a+b\}$, we define the *Schubert variety* $\Omega_I(F_\bullet)$ in $\mathrm{Gr}(a,b)$ by

$$\Omega_I(F_\bullet) = \{L \in \mathrm{Gr}(a,b) \,:\, \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq a\}.$$

The open subset of $\Omega_I(F_\bullet)$ defined by $\dim(L \cap F_{i_j}) = j$ for any j is denoted by $\Omega_I^\circ(F_\bullet)$; it is isomorphic to some affine space. The Poincaré dual of the homology class of $\Omega_I(F_\bullet)$ does not depend on $F_\bullet$; it is denoted by σ_I. The σ_I's form a $\mathbf{Z}$-basis for the cohomology group:

$$H^*(\mathrm{Gr}(a,b),\mathbf{Z}) = \bigoplus_{I \in \mathcal{P}(a,a+b)} \mathbf{Z}\sigma_I\,.$$

Now let I, J be in $\mathcal{P}(a,a+b)$. By expanding $\sigma_I.\sigma_J$, we define the *structure-coefficients* c_{IJ}^K of the *cup product* in the Schubert basis:

$$\sigma_I.\sigma_J = \sum_K c_{IJ}^K \sigma_K\,.$$

The class [pt] of the point generates $H^{2ab}(\mathrm{Gr}(a,b),\mathbf{Z})$. For K in $\mathcal{P}(a,a+b)$, we define $K^\vee$ by: $i\in K^\vee$ if and only if $a+b+1-i\in K$. Then, σ_K and $\sigma_{K^\vee}$ are Poincaré dual, that is $\sigma_K.\sigma_{K^\vee}=[\mathrm{pt}]$. So, if the sum of the codimensions of $\Omega_I(F_\bullet)$, $\Omega_J(F_\bullet)$ and $\Omega_I(F_\bullet)$ equals the dimension of $\mathrm{Gr}(a,b)$, we have

$$\sigma_I.\sigma_J.\sigma_K = c_{IJ}^{K^\vee}\,[\mathrm{pt}]\,.$$

The following result gives a simple interpretation of the integers $c_{IJ}^{K^\vee}$ and in particular shows that they are nonnegative:

THEOREM 1 (Kleiman [Kle74]). *Make the above assumption about the codimensions of Ω_I, Ω_J and Ω_K. Then for flags $F_\bullet$, $G_\bullet$ and $H_\bullet$ in general position,*

$$\Omega_I(F_\bullet)\cap\Omega_J(G_\bullet)\cap\Omega_K(H_\bullet)=\Omega_I^\circ(F_\bullet)\cap\Omega_J^\circ(G_\bullet)\cap\Omega_K^\circ(H_\bullet)$$

consists of $c_{IJ}^{K^\vee}$ points.

2.2 Producing inequalities from Schubert calculus

A spectrum or a partition $(\alpha_1\geq\cdots\geq\alpha_n)$ is said to be *regular* if the α_i's are pairwise distinct. Let A be an $n\times n$ Hermitian matrix with a regular spectrum α. Let $I\in\mathcal{P}(r,n)$, for some positive integer $r<n$. We are going to explain how to express $\sum_{i\in I}\alpha_i$ as an extremum (see inequality (2)).

To A, we associate the complete flag $A_1\subset\cdots\subset A_{n-1}\subset\mathbf{C}^n$, where A_i is the sum of the i eigenlines of A with the i largest eigenvalues (well defined for α regular). We also consider the following Schubert variety of the Grassmannian $\mathrm{Gr}(r,n)$ of r-dimensional subspaces of $\mathbf{C}^n$:

$$\Omega_I(A):=\{V\in\mathrm{Gr}(r,n)\ :\ \dim(V\cap A_i)\geq\#(I\cap\{1,\ldots,i\}),\ 1\leq i\leq n\}.$$

For any linear subspace V of $\mathbf{C}^n$ the *Rayleigh trace* $R_A(V)$ is defined to be the trace of the endomorphism $p_V\circ A_{|V}$, where p_V is the orthogonal projection onto V.

THEOREM 2 ([HZ62]). *If the spectrum of A is regular, we have*

$$\min_{V\in\Omega_I(A)}R_A(V)=\sum_{i\in I}\alpha_i(A)\,.$$

Moreover, the minimum is attained when V is the sum of the eigenlines corresponding to the eigenvalues $\alpha_i(A)$ for $i\in I$.

Let $\Delta^\circ(n)$ denote the set of triples of regular elements in $\Delta(n)$. We now state the first relation between Schubert calculus and the Horn cone.

THEOREM 3 ([Tot94, HR95]). *Let I, J and K be in $\mathcal{P}(r,n)$ such that $c_{IJ}^{K^\vee} \neq 0$. Then, inequality (2) holds for any point in* $\mathrm{Horn}(n)$.

Proof. We admit that $\Delta(n)$ spans $E_0(n)$. This implies that $\Delta^\circ(n)$ is dense in $\Delta(n)$; in particular, it is sufficient to prove the theorem for points in $\Delta^\circ(n)$. Let A, B and C be three Hermitian matrices with regular spectrum such that $A+B+C=0$. Since $\sigma_I.\sigma_J.\sigma_K \neq 0$, Theorem 1 implies that

$$\Omega_I(A) \cap \Omega_J(B) \cap \Omega_K(C)$$

is not empty. Let V_0 belong to this intersection. Theorem 2 implies that

$$\varphi_{IJK}(A,B,C) := \sum_{i\in I} \alpha_i(A) + \sum_{j\in J} \beta_j(B) + \sum_{k\in K} \gamma_k(C) \tag{4}$$

$$\leq \min_{V\in\Omega_I(A)} R_A(V) + \min_{V\in\Omega_J(B)} R_B(V) + \min_{V\in\Omega_K(C)} R_C(V) \tag{5}$$

$$\leq R_A(V_0) + R_B(V_0) + R_C(V_0) \tag{6}$$

$$\leq R_{A+B+C}(V_0) = 0\,. \tag{7}$$

2.3 A COMPLETE SET OF INEQUALITIES FROM SEMISTABILITY

In 1998, Klyachko proved that the inequalities given by Theorem 3 are sufficient to characterize $\Delta(n)$:

THEOREM 4 ([Kly98]). *Let $(\alpha_i,\beta_i,\gamma_i) \in E_0(n) \cap E(n)^+$. Then $(\alpha_i,\beta_i,\gamma_i)$ belongs to $\Delta(n)$ if and only if for every $r=1,\ldots,n-1$, for every $(I,J,K) \in \mathcal{P}(r,n)^3$ such that*

$$c_{IJ}^{K^\vee} \neq 0\,, \tag{8}$$

the following inequality holds:

$$\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k \leq 0\,. \tag{9}$$

We are going to explain one ingredient used by Klyachko. Consider the following basic question:

Given two irreducible representations V_λ and V_μ of GL_n, what are the irreducible subrepresentations of $V_\lambda \otimes V_\mu$?

Let Λ_n^+ be the set of $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbf{Z}^n$. We set

$$\mathrm{LR}(\mathrm{GL}_n) = \{(\lambda, \mu, \nu) \in (\Lambda_n^+)^3 \ : \ (V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n} \neq \{0\}\}.$$

Answering the above question is equivalent to describing the set $\mathrm{LR}(\mathrm{GL}_n)$. Indeed, $(V_\lambda \otimes V_\mu \otimes V_\nu^*)^{\mathrm{GL}_n} \neq \{0\}$ if and only if V_ν is a submodule of $V_\lambda \otimes V_\mu$ if and only if $c^\nu_{\lambda\,\mu} \neq 0$.

Let $\mathcal{F}l(n)$ denote the *variety of complete flags* in $\mathbf{C}^n$ acted upon by GL_n. The Borel-Weil Theorem shows that V_λ can be obtained as the module of regular sections of some GL_n-linearized line bundle $\mathcal{L}_\lambda$. In particular, $(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n} \neq \{0\}$ if and only if some line bundle on $\mathcal{F}l(n)^3$ admits nonzero GL_n-invariant sections. Now, the existence of some positive k such that $(k\lambda, k\mu, k\nu) \in \mathrm{LR}(\mathrm{GL}_n)$ can be interpreted as the existence of semistable points for some action of GL_n. This existence can be verified by linear inequalities using either slopes of vector bundles or the Hilbert-Mumford theorem. In Klyachko's paper, the inequalities (9) are understood as inequalities between slopes of toric vector bundles. The Kempf-Ness theorem [KN79] shows that this existence is equivalent to the fact that 0 belongs to the image of a moment map for some Hamiltonian action of U_n. Making this discussion more precise, we finally obtain the following:

THEOREM 5. *Let (λ, μ, ν) be a triple of nonincreasing sequences of n rational numbers. Then, $(\lambda, \mu, \nu) \in$* Horn(n) *if and only if $(k\lambda, k\mu, k\nu) \in$* $\mathrm{LR}(\mathrm{GL}_n)$ *for some positive integer k.*

2.4 The role of the saturation conjecture

Note that the only difference between the Horn conjecture and Theorem 4 is that condition (1) was replaced by condition (8). The inductive nature of condition (1) is mainly explained by Theorem 5 and the following classical result of Lesieur (see [Les47]):

$$c^K_{IJ} = c^{\lambda_K}_{\lambda_I\,\lambda_J}\,, \tag{10}$$

where λ_I is defined in the introduction. Putting all these remarks together, the missing piece to obtain the Horn conjecture is precisely the saturation conjecture as stated in the introduction. In 1999, Knutson-Tao proved this conjecture using a new model expressing the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ as the number of integral points in some polytope (namely, the *Honeycomb model*). Then, Belkale gave a geometric proof in [Bel06] using mainly the interpretation of the Littlewood-Richardson coefficients in terms of the cohomology of the Grassmannians. Derksen-Weyman gave a proof in [DW00] using an interpretation of the problem in terms of *representations of quivers*. In [KM08], Kapovich-Millson gave a proof using the *Littelmann path model* to translate the problem into geometric terms in some Bruhat-Tits buildings.

3. Faces of $\Delta(n)$

3.1 Deleting inequalities

For $n = 3, 4, 5$ and 6, the Horn conjecture describes $\Delta(n)$ by respectively $18 (= 6+12)$, $50 (= 9+41)$, $154 (= 12+142)$ and $537 (= 15+522)$ inequalities. In the sums, the first term corresponds to the inequalities of $E(n)^+$ and the second one to the inequalities (2). Using a computer software on convex geometry, one can verify that for $n = 3, 4, 5$ and 6, the cone $\Delta(n)$ has respectively 18, 50, 154 and 536 faces of codimension one. So the Horn conjecture gives one redundant inequality for $n = 6$. This is

$$\alpha_2 + \alpha_4 + \alpha_6 + \beta_2 + \beta_4 + \beta_6 + \gamma_2 + \gamma_4 + \gamma_6 \leq 0 .$$

This inequality corresponds to the coefficient $c_{II}^{I^\vee} = 2$ with $I = \{2, 4, 6\}$. In 2000, Belkale improved Theorem 4 as follows:

THEOREM 6 ([Bel01]). *The point $(\alpha_i, \beta_i, \gamma_i) \in E_0(n) \cap E(n)^+$ belongs to $\Delta(n)$ if and only if for every $r = 1, \ldots, n-1$, for every $(I, J, K) \in \mathcal{P}(r, n)^3$ such that*

$$c_{IJ}^{K^\vee} = 1 , \tag{11}$$

the following inequality holds:

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k \leq 0 . \tag{12}$$

We are now going to explain with the material already introduced why Theorem 6 should be true. Let I, J and K be such that $c_{IJ}^{K^\vee} \neq 0$. In Theorem 4, we can forget inequality (9) if when you saturate it you obtain no point in $\Delta^\circ(n)$. So, let us assume that there exist three Hermitian matrices A, B and C with regular spectrum such that $A + B + C = 0$ and

$$\sum_{i \in I} \alpha_i(A) + \sum_{j \in J} \beta_j(B) + \sum_{k \in K} \gamma_k(C) = 0 . \tag{13}$$

Arguing as in the proof of Theorem 3, we obtain that any point V in the intersection $\Omega_I(A) \cap \Omega_J(B) \cap \Omega_K(C)$ satisfies $\sum_{i \in I} \alpha_i(A) = R_A(V)$. Now, Theorem 2 implies that V is the sum of the eigenlines corresponding to I. This proves that $\Omega_I(A) \cap \Omega_J(B) \cap \Omega_K(C)$ is reduced to one point. To obtain Theorem 6, it remains to prove that the intersection is transverse.

3.2 The Knutson-Tao-Woodward Theorem

In 2004, Knutson-Tao-Woodward proved that Theorem 6 is optimal, in the sense that no inequality can be deleted.

THEOREM 7. *The hyperplanes* $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$ *and* $\gamma_i = \gamma_{i+1}$ *spanned by the codimension one faces of* $E(n)^+$ *intersect* $\Delta(n)$ *along faces of codimension one.*

For any I, J *and* K *in* $\mathcal{P}(r,n)$ *(for some* $1 \leq r \leq n-1$*) such that* $c_{IJ}^{K^\vee} = 1$, *the hyperplane* $\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k = 0$ *intersects* $\Delta(n)$ *along a face* $\mathcal{F}_{IJK}$ *of codimension one intersecting* $\Delta^\circ(n)$.

The Knutson-Tao-Woodward proof uses their Honeycomb model. In [Res10], we give an alternative proof using the Geometric Invariant Theory viewpoint. To prove this result, we have to produce points in $\Delta(n)$ which satisfy equality (13). In [Res10], these points are interpreted as line bundles on some product of manifolds that have nonzero invariant sections (see Section 2.3). We produce such line bundles by methods of algebraic geometry.

3.3 Description of the faces of $\Delta(n)$

Let I, J and K be in $\mathcal{P}(r,n)$. Define the linear isomorphism ρ_{IJK} by:

$$\begin{aligned} E(n) &\longrightarrow E(r) \oplus E(n-r) \\ (\alpha_i, \beta_i, \gamma_i) &\mapsto ((\alpha_i)_{i\in I}, (\beta_i)_{i\in J}, (\gamma_i)_{i\in K}) + ((\alpha_i)_{i\notin I}, (\beta_i)_{i\notin J}, (\gamma_i)_{i\notin K}) \,. \end{aligned}$$

This isomorphism puts together the eigenvalues $(\alpha_i)_{i\in I}$. We assume that $c_{IJ}^{K^\vee} \neq 0$. Then, by Theorem 3 inequality (2) holds for any point in $\Delta(n)$. Consider the associated face (perhaps of small dimension):

$$\mathcal{F}_{IJK} = \{(\alpha, \beta, \gamma) \in \Delta(n) : \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k = 0\}. \tag{14}$$

We can now describe $\mathcal{F}_{IJK}$ in terms of smaller Horn cones. Indeed, we will prove that the points of $\mathcal{F}_{IJK}$ correspond to simultaneously block diagonal matrices as in equation (15). Let $E(n)^{++}$ denote the open convex cone in $E(n)$ consisting of regular triples in $E(n)^+$.

PROPOSITION 1. *Recall that* $c_{IJ}^{K^\vee} \neq 0$. *Let* $(\alpha, \beta, \gamma) \in E(n)^+$. *Then,*
(i) *If* $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$ *then* $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK}$.
(ii) *Conversely, if* $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK} \cap E(n)^{++}$ *then* $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$.

Proof. Assume that $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$. Let $A', B', C' \in H(r)$ and $A'', B'', C'' \in H(n-r)$ such that $A' + B' + C' = 0$ and $A'' + B'' + C'' = 0$ whose spectrum correspond to $\rho_{IJK}(\alpha, \beta, \gamma)$. Consider the following three matrices of $H(n)$:

$$A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}, \quad B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \quad C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix}. \tag{15}$$

By construction, α is the spectrum of A and $\sum_I \alpha_i = \mathrm{tr}(A')$, and similarly for B and C. We deduce that $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK}$.

Conversely, let $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK} \cap E(n)^{++}$. It remains to prove that $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$.

Let us now choose three Hermitian matrices A, B and C with spectrum α, β and γ and such that $A + B + C = 0$. We use the notation of the proof of Theorem 3. By assumption, $\varphi_{IJK}(A, B, C) = 0$ and inequality (6) becomes an equality. Thus, $R_A(V_0) = \min_{V \in \Omega_I(A)} R_A(V)$. Now, since α is regular by assumption, Theorem 2 implies that V_0 is the sum of the eigenlines of A corresponding to I. Similarly, V_0 is stable by B and C and the spectrum of the restrictions is respectively $(\beta_j)_{j \in J}$ and $(\gamma_k)_{k \in K}$. We deduce that $((\alpha_i)_{i \in I}, (\beta_i)_{i \in J}, (\gamma_i)_{i \in K})$ belongs to $\Delta(r)$. By considering the restrictions of A, B and C to the orthogonal subspace of V_0, we obtain similarly that $((\alpha_i)_{i \notin I}, (\beta_i)_{i \notin J}, (\gamma_i)_{i \notin K})$ belongs to $\Delta(n-r)$.

REMARK. The second assertion of Proposition 1 holds without the assumption of regularity on α, β and γ. In other words, the first assertion is an equivalence. This fact is more difficult to prove and is not useful here.

COROLLARY 1. *Let* I, J *and* K *be as in the proposition. Then, if* $\mathcal{F}_{IJK}$ *contains regular triples, it has codimension one. In particular,* $c_{IJ}^{K^\vee} = 1$.

Proof. By Proposition 1, $\mathcal{F}_{IJK} \cap E(n)^{++}$ is isomorphic to an open subset of $\Delta(r) \times \Delta(n-r)$. So, $\mathcal{F}_{IJK}$ has codimension 2 in $E(n)$ and so codimension one in $\Delta(n)$. Now, Theorem 6 implies that $c_{IJ}^{K^\vee} = 1$.

REMARK. Corollary 1 is proved in [Res10, Theorem 8] by purely geometric invariant theoretic methods.

Achevé de composer le 28 juin 2011 à 15 :18

4. Proof of the Fulton conjecture

Let λ, μ and ν be three partitions (with at most r parts) such that $c^{\nu}_{\lambda\,\mu} = 1$ and let N be a positive integer. We have to show that $c^{N\nu}_{N\lambda\,N\mu} = 1$.

Strategy of the proof. The fact that $c^{N\nu}_{N\lambda\,N\mu} \neq 0$ is a direct consequence of the Borel-Weil Theorem. By the Lesieur Theorem (see equation (10)) and Theorem 7, the coefficient $c^{\nu}_{\lambda\,\mu}$ equal to one corresponds to some face $\mathcal{F}$ of some Horn cone. By interpreting the conclusion $c^{N\nu}_{N\lambda\,N\mu} = 1$ in similar terms, we have to prove that a certain face of some Horn cone also has codimension one. Producing points on this face becomes a game with block diagonal matrices.

In the paragraph just before Theorem 5, we already mentioned that by the Borel-Weil Theorem, if $c^{\nu}_{\lambda\,\mu} = 1$ then there exists some GL_r-invariant section σ of some line bundle $\mathcal{L}$ on $\mathcal{F}l(r)^3$. The fact that $\sigma^{\otimes N}$ is a nonzero GL_r-invariant section of $\mathcal{L}^{\otimes N}$ implies that $c^{N\nu}_{N\lambda\,N\mu} \neq 0$.

We draw the three partitions λ, μ and ν in a same rectangle: we fix an integer n such that $n-r$ is greater or equal to λ_1, μ_1 and ν_1. Set $I = \{n-r+i-\lambda_i \,:\, i=1,\dots,r\} \in \mathcal{P}(r,n)$ in such a way $\lambda_I = \lambda$ with the notation of the introduction. Similarly, we associate J and K to μ and ν. By equality (10), we have $c^K_{IJ} = 1$.

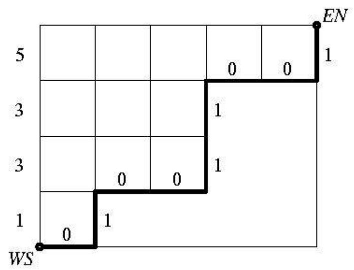
By Theorem 7, $\mathcal{F}_{IJK^\vee}$ is a face of codimension one of $\Delta(n)$. Let (A,B,C) (resp. (A',B',C')) be three Hermitian matrices of size r (resp. $n-r$) such that $A+B+C=0$ and $A'+B'+C'=0$. We assume that their spectra belong to the relative interior of $\rho_{IJK^\vee}(\mathcal{F}_{IJK^\vee})$.

Now let I'', J'' and K'' be the three subsets of $r+N(n-r)$ of cardinality r corresponding to the three partitions $N\lambda$, $N\mu$ and $N\nu$ whose Young diagram is contained in the rectangle with r lines and $N(n-r)$ columns. Since $c^{N\nu}_{N\lambda\,N\mu} \neq 0$, we can consider the face $\mathcal{F}_{I''J''K''^\vee}$ of $\Delta(r+N(n-r))$ as in Proposition 1. By Corollary 1, it remains to prove that $\mathcal{F}_{I''J''K''^\vee}$ intersects $E(r+N(n-r))^{++}$.

Consider N generic perturbations (A'_i, B'_i, C'_i) of (A',B',C') satisfying $A'_i + B'_i + C'_i = 0$. Consider now the block diagonal matrix A'' of size $r+N(n-r)$ with blocks $A, A'_1, \dots, A'_N$; and similarly B'' and C''. We have $A''+B''+C''=0$.

It remains to prove that the point of $\Delta(r+N(n-r))$ corresponding to (A'', B'', C'') belongs to $\mathcal{F}_{I''J''K''^\vee}$. By Proposition 1, it is sufficient to prove that the spectrum of A (resp. B and C) consists of the eigenvalues of A'' (resp. B'' and C'') indexed by I'' (resp. J'' and $K''^{\vee}$).

Let us explain how to recover I from λ_I. First, draw the Young diagram of λ_I. Consider the path from WS to EN; it has length n. Mark each horizontal step by 0 and each vertical step by 1. We have just obtained a word of length n containing r 1's: it is the characteristic function χ_I of I. We illustrate this remark by Figure 1. This description of the map $\lambda_I \mapsto I$ implies that $\chi_{I''}$ is obtained from χ_I by replacing each 0 by N ones.



$r = 4,\ n = 9$
$\lambda_I = (5 \geq 3 \geq 3 \geq 1)$
$\chi_I = 010011001$
$I = \{2, 5, 6, 9\}$

FIGURE 1
From λ_I to I

Now, the spectrum of A'' is obtained from the spectrum of A by replacing each eigenvalue between two ones indexed by I by N closed eigenvalues. We deduce that $(\alpha(A'')_i)_{i \in I''} = (\alpha(A)_i)_{i \in I}$. This implies that $(\alpha(A''), \alpha(B''), \alpha(C''))$ belongs to $\mathcal{F}_{I''J''K''^{\vee}}$, ending the proof of Fulton's conjecture.

REMARK. As pointed out by P. Belkale the construction of A'' is close to the construction of $\mathcal{W}(N)$ in [Bel07a, p. 11].

REFERENCES

[Bel01] BELKALE, P. Local systems on $\mathbf{P}^1 - S$ for S a finite set. *Compositio Math.* *129* (2001), 67–86.

[Bel06] —— Geometric proofs of Horn and saturation conjectures. *J. Algebraic Geom.* *15* (2006), 133–173.

Achevé de composer le 28 juin 2011 à 15 :18

[Bel07a] —— Extremal unitary local systems on $\mathbf{P}^1 - \{p_1, \ldots, p_s\}$. In: *Algebraic Groups and Homogeneous Spaces*, 37–64. Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007.

[Bel07b] —— Geometric proof of a conjecture of Fulton. *Adv. Math. 216* (2007), 346–357.

[DW00] DERKSEN, H. and J. WEYMAN. Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. *J. Amer. Math. Soc. 13* (2000), 467–479.

[FH91] FULTON, W. and J. HARRIS. *Representation Theory. A First Course*. Graduate Texts in Mathematics *129*. Readings in Mathematics. Springer-Verlag, New York, 1991.

[HR95] HELMKE, U. and J. ROSENTHAL. Eigenvalue inequalities and Schubert calculus. *Math. Nachr. 171* (1995), 207–225.

[HZ62] HERSCH, J. et B. ZWAHLEN. Évaluations par défaut pour une somme quelconque de valeurs propres γ_k d'un opérateur $C = A + B$ à l'aide de valeurs propres α_i de A et β_j de B. *C.R. Acad. Sci. Paris 254* (1962), 1559–1561.

[Hor62] HORN, A. Eigenvalues of sums of Hermitian matrices. *Pacific J. Math. 12* (1962), 225–241.

[KM08] KAPOVICH, M. and J. J. MILLSON. A path model for geodesics in Euclidean buildings and its applications to representation theory. *Groups Geom. Dyn. 2* (2008), 405–480.

[KN79] KEMPF, G. and L. NESS. The length of vectors in representation spaces. In: *Algebraic Geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, 233–243. Lecture Notes in Mathematics *732*. Springer, Berlin, 1979.

[Kir84] KIRWAN, F. Convexity properties of the moment mapping. III. *Invent. Math. 77* (1984), 547–552.

[Kle74] KLEIMAN, S. L. The transversality of a general translate. *Compositio Math. 28* (1974), 287–297.

[Kly98] KLYACHKO, A. A. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.) 4* (1998), 419–445.

[KT99] KNUTSON, A. and T. TAO. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc. 12* (1999), 1055–1090.

[KTW04] KNUTSON, A., T. TAO and C. WOODWARD. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc. 17* (2004), 19–48.

[Les47] LESIEUR, L. Les problèmes d'intersection sur une variété de Grassmann. *C.R. Acad. Sci. Paris 225* (1947), 916–917.

[Res10] RESSAYRE, N. Geometric invariant theory and the generalized eigenvalue problem. *Invent. Math. 180* (2010), 389–441.

[Tot94] TOTARO, B. Tensor products of semistables are semistable. In: *Geometry and Analysis on Complex Manifolds*, 242–250. World Sci. Publ., River Edge, NJ, 1994.

[Wey12] WEYL, H. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann. 71* (1912), 441–479.

(Reçu le 12 juin 2009; version révisée reçue le 3 mai 2010)

Nicolas Ressayre

Département de Mathématiques
Université Montpellier II
CC 51–Place Eugène Bataillon
34095 Montpellier Cedex 5
France
e-mail: ressayre@math.univ-montp2.fr

Achevé de composer le 28 juin 2011 à 15 :18