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FROM SPACES OF POLYGONS TO SPACES OF POLYHEDRA FOLLOWING BAVARD, GHYS AND THURSTON

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ABSTRACT. Following work of W. P. Thurston, C. Bavard and É. Ghys constructed particular hyperbolic polyhedra from spaces of deformations of Euclidean polygons. We present this construction as a straightforward consequence of the theory of mixed-volumes.

The gluing of these polyhedra can be isometrically embedded into complex hyperbolic cone-manifolds constructed by Thurston from spaces of deformations of Euclidean polyhedra. It is then possible to deduce the metric structure of the spaces of polygons embedded in complex hyperbolic orbifolds discovered by P. Deligne and G. D. Mostow.

In [Thu98] W. P. Thurston described a natural complex hyperbolic structure on the space of convex polytopes in Euclidean 3-space with fixed cone-angles. Applying this construction to polygons, C. Bavard and É. Ghys pointed out in [BG92] that spaces of convex Euclidean polygons with fixed angles are isometric to particular hyperbolic polyhedra, called *(truncated) orthoschemes*. In Section 1 we obtain the Bavard–Ghys results by using the theory of mixed-area (mixed-volume for polygons). Along the way we obtain Proposition 1.6 which is new. The use of the Alexandrov–Fenchel Theorem might seem artificial at this point (see the discussion after Theorem 1.1), but mixed-area theory sheds light on the relations between convex polygons and hyperbolic orthoschemes via Napier cycles, see Subsection 1.1. Moreover, this is very natural, as mixed–area is the polar form of the quadratic form studied in [Thu98, BG92]. Above all, it indicates a way to generalize the Bavard–Ghys construction from spaces of polygons to spaces of polytopes of any dimension d. In the case $d = 3$, the construction is related to Thurston's, but is different. Further explanations will be given in a forthcoming paper [Fl]. Section 1 ends with a discussion of hyperbolic orthoschemes of Coxeter type, as it appears that the list given by Im Hof in [IH90] is incomplete.

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In Section 2 we glue some of these hyperbolic orthoschemes to get hyperbolic cone-manifolds. This can be seen as hyperbolization of the space of configurations of weighted points on the circle. This has been done several times, especially in lower dimensions, but it seems that the link with orthoschemes has never been clearly established. Proposition 2.3 is new as stated for all dimensions.

Section 3 describes a local parametrization of spaces of polyhedra, equivalent to that in [Thu98], using a "complex mixed-area". We next outline the remainder of the construction of [Thu98] which allows one to recover in a simple way the complex hyperbolic orbifolds listed by Mostow (our interest will be in a sublist first established by Deligne and Mostow [DM86]).

Finally in Section 4 we check that the spaces of polygons embed (isometrically) into the spaces of polyhedra locally as real forms. This is an unsurprising and certainly well-known fact (see for example [KM95]), which is verified here with the parametrizations we defined. We then easily derive Theorem 4.2, which gives the metric structure of those sets of polygons in the Deligne–Mostow orbifolds (are they manifolds, orbifolds, or just cone-manifolds ?).

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1. SPACES OF POLYGONS AND HYPERBOLIC ORTHOSCHEMES

1.1 BASIC FACTS ABOUT HYPERBOLIC POLYHEDRA, NAPIER CYCLES

The *signature* (N,Z,P) of a symmetric bilinear form (or of a Hermitian form) is the triple constituted of its N negative eigenvalues, Z zero eigenvalues and P positive eigenvalues (with multiplicity). We denote by $\mathbf{R}^{n,1}$ the *Minkowski space* of dimension $n+1$, that is $\mathbf{R}^{n+1}$ endowed with the bilinear form of signature $(1,0,n)$

$$\langle x,y\rangle_1 = -x_0y_0 + x_1y_1 + \ldots + x_ny_n\,.$$

A vector x of Minkowski space is said to be *positive* if $\langle x,x\rangle_1 > 0$ and *non-positive* otherwise. The *hyperbolic space* of dimension n is the following submanifold of $\mathbf{R}^{n,1}$ together with the induced metric:

$$\mathbf{H}^n := \{x \in \mathbf{R}^{n,1} \mid \langle x,x\rangle_1 = -1,\ x_0 > 0\}.$$

A *convex polyhedron* P of $\mathbf{H}^n$ is the non-empty intersection of $\mathbf{H}^n$ with a convex polyhedral cone of $\mathbf{R}^{n,1}$ with vertex at the origin. If two *facets* (i.e. codimension-one faces) of P intersect in $\mathbf{H}^n$, their outward normals in $\mathbf{R}^{n,1}$ span a Riemannian plane and the angle between these two vectors is the exterior dihedral angle between the facets. The interior dihedral angle between the facets is π minus the exterior dihedral angle. In this paper, the *dihedral angle* is the interior dihedral angle.

Let us consider the central projection in $\mathbf{R}^{n,1}$ onto the hyperplane $\{x_0 = 1\}$. The image of the hyperbolic space under this projection is the interior of the unit ball of $\mathbf{R}^n$. It is endowed with the metric for which the projection is an isometry. In this model, known as the *Klein projective model* of the hyperbolic space, geodesics are straight lines. The unit sphere in this model is the boundary at infinity of the hyperbolic space. In this paper a *convex generalized polyhedron* of the hyperbolic space is (the hyperbolic part of) a convex polytope of $\mathbf{R}^n$ such that all its edges meet the interior of the unit ball (a *polytope* is a compact polyhedron). A vertex lying outside the interior of the ball is called *hyperideal*. It is *ideal* if it lies on the unit sphere and *strictly hyperideal* otherwise. A vertex in the interior of the unit ball is called *finite*. A strictly hyperideal vertex corresponds to a positive vector of $\mathbf{R}^{n,1}$. The polyhedron is *truncated* if we cut it along the hyperplanes orthogonal to its strictly hyperideal vertices. We get a new hyperbolic polyhedron with new facets, one for each strictly hyperideal vertex v. Such a new facet has the property of being orthogonal to all the facets which had v as a vertex. A hyperbolic convex generalized polyhedron with only finite vertices is compact, and is of finite volume if it has only finite and ideal vertices.

A particularly important class of hyperbolic polyhedra (compact or of finite volume) is that of *Coxeter polyhedra*, whose dihedral angles are integer submultiples of π. This implies in particular that the polyhedron is simple (this means that n facets meet at each finite vertex). For more details about Coxeter polyhedra we refer to [Vin85, Vin93]. Coxeter polyhedra are represented by *Coxeter diagrams*. Each facet is represented by a node. If two facets intersect orthogonally the nodes are not joined. If the two facets intersect at an angle π/k, $k > 2$, the nodes are joined by a line with a k above it. If the facets intersect at infinity we put a ∞ on the line, and if the facets do not intersect the nodes are joined by a dashed line.

Let us consider a set of vectors $e_k \in \mathbf{R}^{n,1}$, $k \in \mathbf{Z}$ modulo $n+3$, such that

- $\langle e_{k-1}, e_k \rangle_1 < 0$ for all k;
- $\langle e_k, e_j \rangle_1 = 0$ for $2 \leq |j-k| \leq n+1$.

Two sets of vectors $\{e_k\}$ and $\{f_k\}$ as above are considered equivalent if, for each k, $e_k = \lambda_k f_k$ with λ_k a positive scalar. The equivalence class is a *Napier cycle*.

This definition comes from [IH90]. As noted in remarks on pp. 526 and 531 of this reference, this definition is suggested by the "Napier pole sequences" introduced in [Deb90]. A *Napier pole sequence* is a sequence of unit vectors in Euclidean space satisfying the same equations as above (with the usual scalar product instead of $\langle . , . \rangle_1$). It is shown in [Deb90, Lemma 5.2] that the sequence is then periodic. In our definition the periodicity is assumed but it is not hard to see that it is implied by the other assumptions, following the lines of the proofs of [Deb90, Lemma 5.2] and [IH90, Proposition 1.2].

In passing, we get that a Napier cycle always contains $n+1$ consecutive positive vectors, which generate the whole cycle, and there are three types of Napier cycles (see [IH90]):

- *type* 1: two adjacent vectors are non-positive;
- *type* 2: one vector is non-positive;
- *type* 3: all vectors are positive.

There are corresponding polyhedra, bounded by the hyperplanes orthogonal to the positive vectors of the Napier cycles. If a Napier cycle has non-positive vectors, then they correspond to vertices of the polyhedron, see [IH90, 2].

- The (ordered) set of outward normals of an *ordinary orthoscheme* generates a Napier cycle of type 1. Its Coxeter diagram (when it is Coxeter) is a linear chain with $n+1$ nodes.
- The set of outward normals of a *simply-truncated orthoscheme* generates a Napier cycle of type 2. Its Coxeter diagram is a linear chain with $n+2$ nodes.
- The set of outward normals of a *doubly-truncated orthoscheme* is a Napier cycle of type 3. Its Coxeter diagram is a cycle with $n+3$ nodes.

By abuse of language we will call a polyhedron of one of the three types above an *orthoscheme*. Usually, the word orthoscheme designates what we called an ordinary orthoscheme. See [IH90] for more details about the terminology.

EXAMPLE. In $\mathbf{H}^2$, an ordinary orthoscheme is a right triangle, a simply-truncated orthoscheme is a quadrilateral with three right angles and a doubly-truncated orthoscheme is a right-angled pentagon; see the figures in [IH90].

REMARK. In [Deb90], corresponding to a Napier pole sequence there is an associated "Napier configuration", which is a sequence of spherical orthoschemes. Geometrically this means that some vertices of a given orthoscheme will be considered as outward normals of another orthoscheme (we refer to [Deb90] for more details). Analogous considerations in our case of Napier cycles would oblige us to consider a larger class of polyhedra than hyperbolic ones. The terminology "Napier pole sequence" comes from the fact that in the sphere of dimension 2, relations in a Napier configuration are Napier's rules (Napier is sometimes written Neper), see [Deb90, 5].

1.2 EUCLIDEAN POLYGONS AND NAPIER CYCLES

Let P be a convex polygon of $\mathbf{R}^2$ with $n+3$ vertices such that the origin is contained in its interior. We denote by F_k the edges of P, labeled in cyclic order, α_k is the exterior angle between F_{k-1} and F_k, and h_k is the distance of F_k to the origin. The angles α_k satisfy

$$\text{(A)} \qquad 0 < \alpha_k < \pi\,, \qquad \sum_{k=1}^{n+3} \alpha_k = 2\pi\,.$$

We denote by ℓ_k the length of F_k, and we have (see Figure 1)

$$\text{(1.1)} \qquad \ell_k = \ell_k^r + \ell_k^l = \frac{h_{k-1} - h_k \cos(\alpha_k)}{\sin(\alpha_k)} + \frac{h_{k+1} - h_k \cos(\alpha_{k+1})}{\sin(\alpha_{k+1})}\,.$$

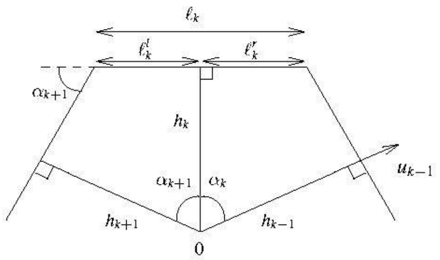

FIGURE 1
Notation for a convex polygon

We identify the set of *heights* $h_1, \dots, h_{n+3}$ (also called *support numbers*) with $\mathbf{R}^{n+3}$ so that the set of outward unit normals $u_1, \dots, u_{n+3}$ of the convex polygon P corresponds to the canonical basis. For a vector $P \in \mathbf{R}^{n+3}$, $h_k(P)$ is the kth coefficient of P for the basis $u_1, \dots, u_{n+3}$. In particular $h_k(u_i) = \delta_i^k$. We define $\ell_k(P)$ as the right-hand side of (1.1), after replacing the entries h_i by $h_i(P)$.

A TRIVIAL EXAMPLE. An element of $\mathbf{R}^{n+3}$ describes a polygon (not necessarily convex) with $n+3$ edges (maybe of length 0), which is such that the kth edge has outward normal u_k and is on a line at distance h_k from the origin. Let us consider the element u_k of $\mathbf{R}^{n+3}$. There is one edge on a line l with normal u_k and at distance 1 from the origin. The edge with normal u_{k-1} (resp. u_{k+1}) is at distance 0 from the origin, hence it is on the unique line from the origin making an angle α_k (resp. α_{k+1}) with l. The other edges are reduced to the origin because they link the origin to itself. So u_k describes a triangle, see Figure 2.

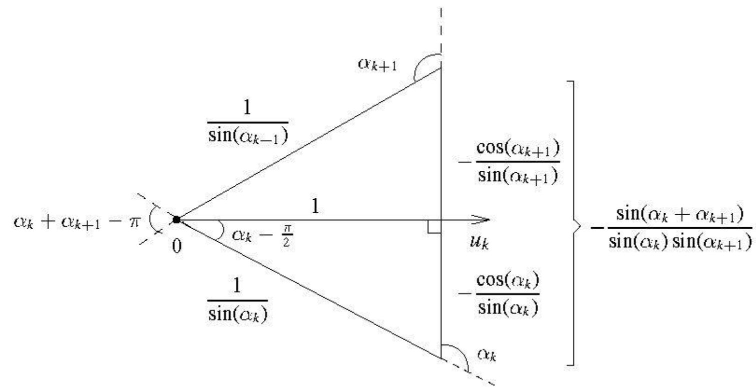

FIGURE 2
The geometric meaning of (1.2): it is (minus) the signed area of the triangle described by u_k

We define the following bilinear form on $\mathbf{R}^{n+3}$:

$$m(P, Q) := -\frac{1}{2} \sum_{k=1}^{n+3} h_k(P)\ell_k(Q).$$

If P and Q are two convex polygons with outward unit normals $u_1, \dots, u_{n+3}$, then $m(P, Q)$ is known as (minus) the mixed-area of P and Q, and $m(P,P)$ is minus the area of P, which is, up to the sign, the quadratic form used in [BG92] (in the present paper the minus sign serves only to get a more usual signature below). We immediately get

$$m(u_k,u_k) = -\frac{1}{2}\ell_k(u_k) = \frac{1}{2}\frac{\sin(\alpha_k+\alpha_{k+1})}{\sin(\alpha_k)\sin(\alpha_{k+1})}. \tag{1.2}$$

This formula has a geometric meaning: it is (minus) the signed area of the triangle described by u_k (see Figure 2 and the example above).

It is also straightforward that

$$m(u_k,u_j) = \begin{cases} 0 & \text{if } 2 \le |j-k| \le n+1, \\ -\frac{1}{2}\frac{1}{\sin(\alpha_k)} & \text{if } j = k-1, \\ -\frac{1}{2}\frac{1}{\sin(\alpha_{k+1})} & \text{if } j = k+1. \end{cases} \tag{1.3}$$

It follows from (1.3) that m is symmetric, that is:

$$m(P,Q) = -\frac{1}{2}\sum_{k=1}^{n+3} h_k(Q)\ell_k(P).$$

This also follows from general properties of the mixed-volume [Sch93, Ale05].

THEOREM 1.1. *The symmetric bilinear form m has signature* $(1,2,n)$.

Theorem 1.1 is a straightforward adaptation of the analogous result proved in [Thu98] dealing with convex polytopes of $\mathbf{R}^3$ (see Section 3). A more embracing statement is obtained in [BG92] with the same method (see the remark after the proof of Proposition 1.6). Theorem 1.1 is also proved with greater effort in [KNY99] in the case where all the α_k are equal. This statement is generalized to some cases of "convex generalized polygons" in [BI08, Lemma 3.15] (where "generalized" has a meaning different from ours).

Theorem 1.1 is also a particular case of classical results about mixed-volumes, even if this is far from the simplest way of proving it. It appears in Alexandrov's proof of the so-called *Alexandrov–Fenchel Theorem* (or *Alexandrov–Fenchel Inequality*) for convex polytopes of $\mathbf{R}^d$ (here $d=2$) [Ale37, Sch93, Ale96]. Actually Theorem 1.1 can be derived from the *Minkowski Inequality* for convex polygons [Sch93, Note 1, p. 321], [Kla04]: if P and Q are convex polygons then

$$m(P,Q)^2 \ge m(P,P)\,m(Q,Q), \tag{1.4}$$

and equality occurs if and only if P and Q are homothetic. The way of going from Minkowski's Inequality to Theorem 1.1 is part of Alexandrov's proof of the Alexandrov–Fenchel Theorem. This is also done in a wider context in [Izm08, Appendix A]. Note that the Minkowski Inequality (1.4) can be thought of as the reversed Cauchy–Schwarz inequality in the case where the time-like vectors correspond to convex polygons. One advantage in considering the Bavard–Ghys construction as a consequence of the Alexandrov–Fenchel Theorem in the particular case when $d = 2$, rather than as a consequence of Thurston's construction, is that the former generalizes immediately to any dimension d. This generalization will be the subject of [FI].

The height of the sum of two polygons is the sum of their heights, see e.g. [Ale96, Chapter IV]. It follows that a translation of a polygon P is the same as adding to the heights of P the heights of a point. Hence the kernel of m consists of heights spanning a point, since the area is invariant under translations (in the general case this is a step in the proof of the Alexandrov–Fenchel Theorem, see [Ale96, Lemma III, p. 71], [Sch93, Proposition 3, p. 329]).

A TRIVIAL EXAMPLE. Let us consider $\mathbf{R}^3$, with basis $\{u_1, u_2, u_3\}$. From (1.2), (1.3) and (A) we easily deduce that the matrix of m is

$$-\frac{1}{2}\begin{pmatrix} \frac{\sin(\alpha_3)}{\sin(\alpha_1)\sin(\alpha_2)} & \frac{1}{\sin(\alpha_2)} & \frac{1}{\sin(\alpha_1)} \\ \frac{1}{\sin(\alpha_2)} & \frac{\sin(\alpha_1)}{\sin(\alpha_2)\sin(\alpha_3)} & \frac{1}{\sin(\alpha_3)} \\ \frac{1}{\sin(\alpha_1)} & \frac{1}{\sin(\alpha_3)} & \frac{\sin(\alpha_2)}{\sin(\alpha_3)\sin(\alpha_1)} \end{pmatrix}$$

and that the vectors

$$v := \begin{pmatrix} 1 \\ 0 \\ -\frac{\sin(\alpha_3)}{\sin(\alpha_2)} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{\sin(\alpha_1)}{\sin(\alpha_2)} \end{pmatrix}$$

span the kernel of m. In Figure 3 we check that the polygon corresponding to the vector v is a single point. Another eigenvector of m is

$$\begin{pmatrix} 1 \\ \frac{\sin(\alpha_1)}{\sin(\alpha_3)} \\ \frac{\sin(\alpha_2)}{\sin(\alpha_3)} \end{pmatrix}$$

with negative eigenvalue

$$-\frac{1}{2}\frac{\sin(\alpha_1)^2 + \sin(\alpha_2)^2 + \sin(\alpha_3)^2}{\sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)},$$

so the signature of m is $(1, 2, 0)$.

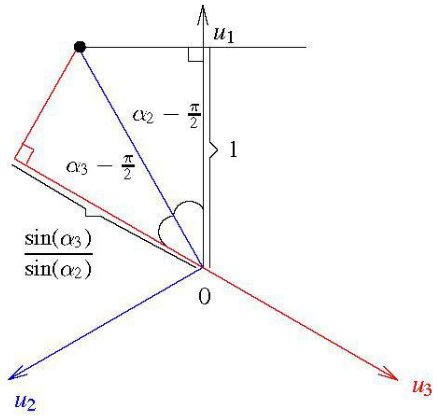

FIGURE 3
The polygon with heights 1, 0 and $-\frac{\sin(\alpha_3)}{\sin(\alpha_2)}$ is a point

By Theorem 1.1 the quotient of $\mathbf{R}^{n+3}$ by the kernel of m is isometric to the Minkowski space $\mathbf{R}^{n,1}$. We denote the quotient map by Π. By definition, (1.3) and Theorem 1.1 imply

COROLLARY 1.2. *The set of vectors* $(\Pi(u_1), \ldots, \Pi(u_{n+3}))$ *is a Napier cycle.*

Up to global isometries, this Napier cycle depends only on the angles α_k between the u_k. Let $(\alpha_1, \ldots, \alpha_{n+3})$ be an ordered list of real numbers, up to cyclic permutations, satisfying (A). We denote by $\mathcal{H}(\alpha_1, \ldots, \alpha_{n+3})$ the hyperbolic orthoscheme corresponding to the Napier cycle given by the corollary above. We now verify that, as announced in the title, $\mathcal{H}(\alpha_1, \ldots, \alpha_{n+3})$ is a space of convex polygons.

LEMMA 1.3. *The orthoscheme* $\mathcal{H}(\alpha_1, \ldots, \alpha_{n+3})$ *is in bijection with the set of convex polygons with the* k*th exterior angle equal to* α_k*, up to direct isometries and homotheties.*

Proof. The interior of the hyperbolic polyhedron $\mathcal{H}(\alpha_1, \ldots, \alpha_{n+3})$ is the set of vectors P satisfying $m(P, u_k) < 0$ for all positive u_k, i.e. the set of P such that $\ell_k(P) > 0$ (it is not hard to see that this is true even if one or two u_k are non-positive). So P belongs to the set of convex polygons with the u_k as outward unit normals, which is the set of convex polygons with kth exterior angle equal to α_k, up to rotations as the u_k are arbitrarily placed in the plane. We saw that to quotient by the kernel of m is equivalent to consider the polygons up to translation. Finally $\mathcal{H}(\alpha_1,\dots,\alpha_{n+3})$ is a subset of the hyperbolic space, hence it contains only polygons of unit area, that is the same as polygons up to homotheties.

In the sequel, we identify the space of polygons described in the lemma with $\mathcal{H}(\alpha_1,\dots,\alpha_{n+3})$.

PROPOSITION 1.4. *Let $n \geq 2$. The hyperbolic orthoscheme $\mathcal{H}(\alpha_1,\dots,\alpha_{n+3})$ has the following properties.*

i) *Its type is equal to 3 minus the number of k such that $\alpha_k+\alpha_{k+1}\geq \pi$.*

ii) *It has non-obtuse dihedral angles, and if $\Pi(u_{k-1})$ and $\Pi(u_k)$ are space-like, the dihedral angle Θ between the corresponding facets is acute and satisfies*

$$\cos^2(\Theta) = \frac{\sin(\alpha_{k-1})\sin(\alpha_{k+1})}{\sin(\alpha_{k-1}+\alpha_k)\sin(\alpha_k+\alpha_{k+1})}. \tag{1.5}$$

iii) *It is of finite volume. Moreover it is compact if and only if there is no couple k,k' for which $\alpha_k+\cdots+\alpha_{k'}=\pi$.*

The last condition about compactness can be rephrased by saying that the polygons of $\mathcal{H}(\alpha_1,\dots,\alpha_{n+3})$ have no parallel edges. The cases $n=0$ and $n=1$ are geometrically meaningless. Note that if $\alpha,\beta,\gamma,\delta$ are the angles of a parallelogram, then $\mathcal{H}(\alpha,\beta,\gamma,\delta)$ is the whole "hyperbolic line" $\mathbf{H}^1$.

Proof. First note that with (1.2) the character of the vector $\Pi(u_k)$ is easily determined: it is

- space-like if $\alpha_k+\alpha_{k+1}<\pi$,
- light-like if $\alpha_k+\alpha_{k+1}=\pi$,
- time-like if $\alpha_k+\alpha_{k+1}>\pi$,

and i) follows. The dihedral angles are either $\pi/2$ or else minus the cosine of the angle is given by

$$\frac{m(u_{k-1},u_k)}{\sqrt{m(u_{k-1},u_{k-1})}\sqrt{m(u_k,u_k)}} = -\sqrt{\frac{\sin(\alpha_{k-1})\sin(\alpha_{k+1})}{\sin(\alpha_{k-1}+\alpha_k)\sin(\alpha_k+\alpha_{k+1})}}, \tag{1.6}$$

which is a real negative number if $\alpha_{k-1}+\alpha_k<\pi$ and $\alpha_k+\alpha_{k+1}<\pi$; this proves ii). In fact, orthoschemes always have non-obtuse dihedral angles and finite volume [IH90, Proposition 2.1]. This gives the first part of iii). It remains to prove the assertion about compactness. The polyhedron is not compact if and only if one of its faces contains an ideal point. This can be the case if a vector u_k is light-like, that is if $\alpha_k + \alpha_{k+1} = \pi$. Suppose now that u_k is space-like. A facet of the polyhedron is given by the polygons P which satisfy $\ell_k(P) = 0$. This facet is itself a polyhedron of one dimension lower. More precisely it is isometric to $\mathcal{H}' := \mathcal{H}(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{n+3})$. The hyperbolic polyhedron $\mathcal{H}'$ corresponds to a cone in an ambient Minkowski space, in which (the vector corresponding to) u_{k+1} is light-like if and only if $\alpha_k + \alpha_{k+1} + \alpha_{k+2} = \pi$. This is easy to check from the definition of the bilinear form, see also Figure 4. Now if u_{k+1} is space-like in $\mathcal{H}'$, we repeat the reasoning for a facet of $\mathcal{H}'$, and so on.

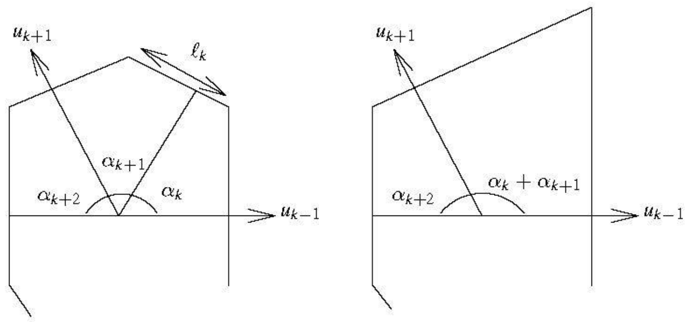

FIGURE 4

On the left u_{k+1} is a space-like vector and it becomes a light-like vector on the right because $\alpha_k + \alpha_{k+1} + \alpha_{k+2} = \pi$

We easily check below that equation (1.5) is another form of the main formula of [BG92] (written in the present paper as equation (1.7)). It also appears in the form of (1.5) in [KNY99, MN00] for $n = 2$ or 3. We denote by U_k the line spanned by the vector u_k in $\mathbf{R}^2$, and we define the *cross-ratio* by

$$[a, b, c, d] = \frac{d-a}{a-b}\,\frac{b-c}{c-d}\,.$$

COROLLARY 1.5. *Let* Θ *be as in* ii) *of Proposition* 1.4. *Then*

$$\tan^2(\Theta) = -[U_{k-1}, U_k, U_{k+1}, U_{k+2}]\,. \tag{1.7}$$

Proof. This follows from (1.5) and from the well-known fact that

$$\frac{\sin(\alpha_{k-1})\sin(\alpha_{k+1})}{\sin(\alpha_{k-1}+\alpha_k)\sin(\alpha_k+\alpha_{k+1})} = \frac{1}{1-[U_{k-1},U_k,U_{k+1},U_{k+2}]}\,. \tag{1.8}$$

This formula is stated in [BG92] for the directions of the edges of the polygon, and not for the directions of the normals of the polygon as above, but the cross-ratios are the same.

We proved, following [BG92], that from any convex polygon one can construct a hyperbolic orthoscheme. One can prove the converse. This result is not given in [BG92], but this reference contains the main point for the proof. Let us be more formal.

Let $A(n)$ be the space of all the ordered lists $(\alpha_1,\dots,\alpha_{n+3})$ up to the action of the dihedral group $\mathbf{D}_{n+3}$, with α_k real numbers satisfying (A). Note that if $\sigma\in\mathbf{D}_{n+3}$ then $\mathcal{H}(\alpha_1,\dots,\alpha_{n+3})$ and $\mathcal{H}(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n+3)})$ are isometric. We say that two elements $(\alpha_1,\dots,\alpha_{n+3})$ and $(\alpha'_1,\dots,\alpha'_{n+3})$ of $A(n)$ are *equivalent* if, given a set of planar vectors $\sqcap=\{u_1,\dots,u_{n+3}\}$ with α_k the angle between u_{k-1} and u_k and a set of planar vectors $\sqcap'=\{u'_1,\dots,u'_{n+3}\}$ with α'_k the angle between u'_{k-1} and u'_k, then there exists a projective map φ of $\mathbf{R}^2$ fixing the origin and sending $\sqcap$ to $\sqcap'$, i.e. such that $\varphi(u_k)=u'_k$ for all k. We denote by $\overline{A}(n)$ the quotient of $A(n)$ by this equivalence relation, and by $H(n)$ the space of orthoschemes of $\mathbf{H}^n$ up to global isometries.

PROPOSITION 1.6. *There is a bijection between $\overline{A}(n)$ and $H(n)$.*

Proof. Let $(\alpha_1,\dots,\alpha_{n+3})\in A(n)$. We know that these numbers define a Napier cycle. The corresponding orthoscheme is defined by the hyperplanes orthogonal to the positive vectors of the Napier cycle. Hence it is defined by the Gram matrix of the normalized positive vectors of the Napier cycle, whose coefficients are either 1 or given by (1.6), which is a projectively invariant formula (see (1.8)). Hence there is a well-defined map from $\overline{A}(n)$ to $H(n)$.

Let $H\in H(n)$ and let $(e_1,\dots,e_{n+3})$ be the Napier cycle generated by the outward unit normals of H. We know that there at most two non-positive vectors. Moreover if there are two non-positive vectors, they are consecutive. In this case, up to a change of labeling, we suppose that the non-positive vectors are e_{n+2} and e_{n+3}. If there is only one non-positive vector, we suppose that it is e_{n+3}. Suppose now that the two elements $(\alpha_1,\dots,\alpha_{n+3})$ and $(\alpha'_1,\dots,\alpha'_{n+3})$ of $A(n)$ lead to H. Up to a projective transformation we may consider that $\alpha_1=\alpha'_1$ and $\alpha_2=\alpha'_2$. We denote by $s(.,.,.)$ the function defined by the right side of (1.6). As $s(\alpha_1,\alpha_2,\alpha_3)$ and $s(\alpha'_1,\alpha'_2,\alpha'_3)$ are both equal to $m(e_1,e_2)$ it easily follows that $\alpha_3 = \alpha'_3$ (this is straightforward with the cross-ratio). Next, as $s(\alpha_2, \alpha_3, \alpha_4)$ and $s(\alpha'_2, \alpha'_3, \alpha'_4)$ are both equal to $m(e_2, e_3)$ it follows that $\alpha_4 = \alpha'_4$, and so on until $s(\alpha_n, \alpha_{n+1}, \alpha_{n+2}) = m(e_n, e_{n+1}) = s(\alpha_n, \alpha_{n+1}, \alpha'_{n+2})$, which gives $\alpha_{n+2} = \alpha'_{n+2}$. The last equality $\alpha_{n+3} = \alpha'_{n+3}$ follows as the sum of the angles is equal to 2π. Hence the map from $\overline{A}(n)$ to $H(n)$ is injective.

For an arbitrary $H \in H(n)$, we can find angles $(\alpha_1, \ldots, \alpha_{n+3})$ in a similar way: α_1, α_2 and α_3 are chosen between 0 and π and such that $s(\alpha_1, \alpha_2, \alpha_3) = m(e_1, e_2)$ (it is clear that such a triple always exists). It is easy to see that the angle α_4 (assumed to be between 0 and π) is determined by $s(\alpha_2, \alpha_3, \alpha_4) = m(e_2, e_3)$ and so on until $s(\alpha_n, \alpha_{n+1}, \alpha_{n+2}) = m(e_n, e_{n+1})$ which determines $0 < \alpha_{n+2} < \pi$. We must now examine different cases, according to the character of e_{n+2}:

- if e_{n+2} is space-like, $0 < \alpha_{n+3} < \pi$ is given as above;
- if e_{n+2} is light-like, we define $\alpha_{n+3} = \pi - \alpha_{n+2}$. It is between 0 and π;
- if e_{n+2} is time-like, we again define α_{n+3} with the help of (1.6). The squared norm of e_{n+1} is positive and that of e_{n+2} is negative. As they are elements of a Napier cycle, $m(e_{n+1}, e_{n+2})$ is negative, hence the right-hand side of (1.6) must belong to $i\mathbf{R}_-$. As $\sin(\alpha_{n+1})$ and $\sin(\alpha_{n+1} + \alpha_{n+2})$ are positive and $\sin(\alpha_{n+2} + \alpha_{n+3})$ is negative (because e_{n+2} is time-like), $\sin(\alpha_{n+3})$ must be positive, so that $0 < \alpha_{n+3} < \pi$.

We now have $n+3$ angles between 0 and π. For each such set of angles, (1.2) and (1.3) allow one to define a bilinear form (geometrically the signed area of the polygons constructed with the angles). In our case this form has signature $(1, 2, n)$, because it is the Gram matrix of a Napier cycle. But it is known that such forms have this signature only if the α_i sum up to 2π [BG92, Proposition p. 209]. Hence we have constructed an element of $A(n)$.

REMARK. It would be interesting to know whether some of the many results about hyperbolic orthoschemes can be translated in terms of Euclidean polygons (for such results, see [Deb90] and the references therein). Moreover [BG92] also contains a computation of the signature of the (signed) area form for spaces of non-convex polygons. In particular one can construct Euclidean and spherical polyhedra.

1.3 COXETER ORTHOSCHEMES

The aim of [IH90] (previously announced in [IH85]) is to find all Coxeter orthoschemes. Some subfamilies were already known, especially in dimensions 2 and 3 (see [IH85, IH90] and [Vin75, Vin85] for more details).

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For these dimensions there exist infinite families of Coxeter orthoschemes. For dimension ≥ 4, Im Hof found a list of 75 Coxeter orthoschemes up to dimension 9 (he proved that they cannot exist for higher dimension). In [BG92] the existence of Coxeter orthoschemes is verified, by giving for each one a list of real numbers representing the slopes of the lines parallel to the edges of a suitable convex polygon.

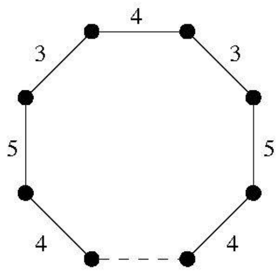


FIGURE 5
The Tumarkin polyhedron is a compact Coxeter orthoscheme of dimension 5 and type 3

Figure 5 represents another example of a Coxeter orthoscheme, given in [Tum07]. We name it the *Tumarkin polyhedron*. It is not difficult to find a convex polygon P of $\mathbf{R}^2$ whose space of angle-preserving deformations is isometric to the Tumarkin polyhedron. The following list gives the slopes of the lines containing the normals u_k of P:

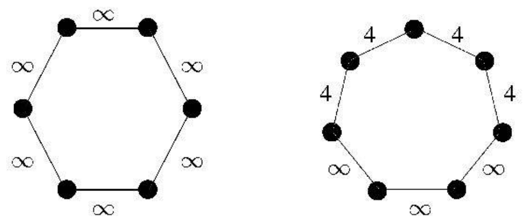
$$\left(\sqrt{5}, -2, -1, 0, 1, \infty, -3, \frac{\sqrt{5}-3}{2}\right).$$

This list confirms the existence of the Tumarkin polyhedron (dihedral angles can easily be computed with (1.7)). The fact that all the slopes are different indicates that the polyhedron is compact. This polyhedron does not appear in [IH90] (nor in [BG92]). It seems to have been overlooked by Im Hof, and it is now natural to ask the following question:

QUESTION 1. *Is Im Hof's list together with the Tumarkin polyhedron complete?*

ADDED IN PROOF. Hans-Christoph Im Hof informed the author that the list of Coxeter orthoschemes (including Tumarkin polyhedron) was checked by Bettina Kistler in her master thesis (winter 2010–2011). A corrigendum to [IH90] should follow, asserting that the list is now complete.

We call an angle of the form $q\pi$, $q \in \mathbf{Q}$ a *rational angle*. If H is an orthoscheme, we cannot expect that there exists a set of rational angles $(\alpha_1, \dots, \alpha_n)$ such that $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is isometric to H. A natural question is whether H is a Coxeter orthoscheme, but I was unable to find such a set of angles for the Tumarkin polyhedron. Conversely two different sets of rational angles can lead to the same Coxeter orthoscheme; examples are shown in Figure 6.



$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$ $\qquad$ $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$

$\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ $\qquad$ $\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$

FIGURE 6

The Coxeter diagram on the left represents a Coxeter orthoscheme of dimension 3 and type 3 (the one shown in [Thu98, Figure 3]). The Coxeter diagram on the right represents a Coxeter orthoscheme of dimension 4 and type 3. Below each diagram we give two different lists of rational angles $(\alpha_1, \dots, \alpha_{n+3})$ such that $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is isometric to the orthoscheme.

2. SPACES OF POLYGONS AND HYPERBOLIC CONE-MANIFOLDS

In this section we assume that $n \geq 2$. Let $\alpha_1, \dots, \alpha_{n+3}$ be $n+3$ real numbers satisfying (A). We denote by $R(\alpha_1, \dots, \alpha_{n+3})$ the set of all permutations of the set $\{\alpha_1, \dots, \alpha_{n+3}\}$ up to the action of the dihedral group. Hence $R(\alpha_1, \dots, \alpha_{n+3})$ has $(n+2)!/2$ elements. Each element $\sigma \in R(\alpha_1, \dots, \alpha_{n+3})$ will be identified with the orthoscheme $\mathcal{H}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n+3)})$. There is a natural way of gluing all these polyhedra together. Let α_k, α_j be such that $\alpha_k + \alpha_j < \pi$ (such a pair always exists as the α_k satisfy (A)). It is easy to see that both orthoschemes $\mathcal{H}(\alpha_1, \dots, \alpha_k, \alpha_j, \dots, \alpha_n)$ and $\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_k, \dots, \alpha_n)$ have a facet isometric to $\mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_j, \dots, \alpha_n)$, see Figure 7. We glue them isometrically along this facet. We do so for all facets of all orthoschemes in $R(\alpha_1, \dots, \alpha_{n+3})$. At the end we obtain a space which is by construction a hyperbolic cone-manifold of dimension n. We again denote this space by $R(\alpha_1, \dots, \alpha_{n+3})$.

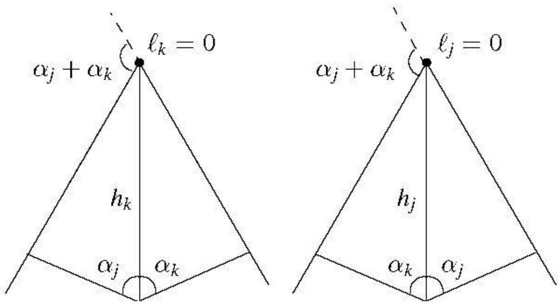


FIGURE 7

Let $\alpha_k + \alpha_j < \pi$. A facet of $\mathcal{H}(\alpha_1, \dots, \alpha_k, \alpha_j, \dots, \alpha_n)$ is defined by $m(u_k, P) = 0$, that is $\ell_k(P) = 0$, and this space is isometric to $\mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_j, \dots, \alpha_n)$. This last space is also isometric to the facet of $\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_k, \dots, \alpha_n)$ defined by $m(u_j, P) = 0$.

LEMMA 2.1. *The cone-manifold $R(\alpha_1, \dots, \alpha_{n+3})$ is connected. It has finite volume, and it is compact if and only if there is no couple k, k' for which $\alpha_k + \dots + \alpha_{k'} = \pi$.*

Finiteness of volume and the description of compactness are straightforward consequences of Proposition 1.4. In order to prove connectedness we will prove the following lemma, which implies Lemma 2.1. We denote by $\widetilde{R}(\alpha_1, \dots, \alpha_{n+3})$ the double-covering of $R(\alpha_1, \dots, \alpha_{n+3})$ obtained by distinguishing a list from the one obtained by reversing the order.

LEMMA 2.2. *The cone-manifold $\widetilde{R}(\alpha_1, \dots, \alpha_{n+3})$ is not connected if and only if there exist $\alpha_i, \alpha_j, \alpha_k$ $(i \neq j \neq k \neq i)$ such that*

$$\alpha_i + \alpha_j \geq \pi\,, \quad \alpha_j + \alpha_k \geq \pi\,, \quad \alpha_k + \alpha_i \geq \pi\,.$$

In this case $\widetilde{R}(\alpha_1, \dots, \alpha_{n+3})$ has two connected components, which are identified by reversing the order of the angles.

Note that this last case can occur only if $\widetilde{R}(\alpha_1, \dots, \alpha_{n+3})$ contains orthoschemes of type 1.

Proof. To prove the lemma it suffices to know when there is no sequence of glued polyhedra from

$$\mathcal{H}(\alpha_1, \ldots, \alpha_i, \alpha_j, \ldots, \alpha_{n+3})$$

to

$$\mathcal{H}(\alpha_1, \ldots, \alpha_j, \alpha_i, \ldots, \alpha_{n+3}).$$

Below are all the possible configurations.

- If all the pairs α_i, α_j satisfy $\alpha_i + \alpha_j < \pi$, the polyhedra are glued along $\mathcal{H}(\alpha_1, \ldots, \alpha_i + \alpha_j, \ldots, \alpha_{n+3})$.
- If there is only one pair α_i, α_j such that $\alpha_i + \alpha_j \geq \pi$, we can glue

  $$\mathcal{H}(\alpha_1, \ldots, \alpha_i, \alpha_j, \alpha_{j+1}, \ldots, \alpha_{n+3})$$

  to

  $$\mathcal{H}(\alpha_1, \ldots, \alpha_i, \alpha_{j+1}, \alpha_j, \ldots, \alpha_{n+3})$$

  and so on, i.e. we can always glue a polyhedron to the one obtained by permuting α_j with the angle to its right. As the list of angles is determined up to cyclic order, we arrive at $\mathcal{H}(\alpha_1, \ldots, \alpha_j, \alpha_i, \ldots, \alpha_{n+3})$, hence $\widetilde{R}(\alpha_1, \ldots, \alpha_{n+3})$ is connected.
- By adapting the above argument with suitable permutations, it is easy to show that $\widetilde{R}(\alpha_1, \ldots, \alpha_{n+3})$ is still connected if there are $\alpha_i, \alpha_j, \alpha_k$ such that $\alpha_i + \alpha_j \geq \pi$, $\alpha_i + \alpha_k \geq \pi$ and $\alpha_j + \alpha_k < \pi$.
- In the same way it is easy to show that if $\alpha_i, \alpha_j, \alpha_k$ are as in the statement of the lemma, then

  $$\mathcal{H}(\alpha_1, \ldots, \alpha_i, \alpha_j, \alpha_k, \ldots, \alpha_{n+3})$$

  and

  $$\mathcal{H}(\alpha_1, \ldots, \alpha_j, \alpha_i, \alpha_k, \ldots, \alpha_{n+3})$$

  cannot be joined and hence $\widetilde{R}(\alpha_1, \ldots, \alpha_{n+3})$ is not connected. Moreover every other polyhedron of $\widetilde{R}(\alpha_1, \ldots, \alpha_{n+3})$ can be joined to one of these two polyhedra, and reversing the order of the angles is a bijection between these two components.
- As the sum of the α_k is equal to 2π (and $n \geq 2$) there cannot be a fourth angle α_l such that $\alpha_i, \alpha_j, \alpha_k$ are as in the statement of the lemma and $\alpha_l + \alpha_x \geq \pi$ for $x \in \{1, \ldots, n+3\}$.

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We know by Poincaré's Theorem [Thu98, Thm 4.1] that $R(\alpha_1,\ldots,\alpha_{n+3})$ is isometric to a hyperbolic orbifold if and only if the angle around each singular set of codimension 2 is $2\pi/k$ with integer $k>0$. If all these angles are 2π then the orbifold is a manifold. The singular sets of codimension 2 in $R(\alpha_1,\ldots,\alpha_{n+3})$ correspond to codimension 2 faces of the polyhedra around which facets are glued. There are two possibilities:

1. if $\alpha_k+\alpha_{k+1}<\pi$ and $\alpha_j+\alpha_{j+1}<\pi$ with $2\leq |j-k|\leq n+1$, then around the codimension 2 face isometric to
$$N := \mathcal{H}(\alpha_1,\ldots,\alpha_k+\alpha_{k+1},\ldots,\alpha_j+\alpha_{j+1},\ldots,\alpha_{n+3})$$
are glued four orthoschemes, corresponding to the four ways of ordering (α_k,α_{k+1}) and (α_j,α_{j+1}). As we know that the dihedral angle of each orthoscheme at such a codimension 2 face is $\pi/2$, the total angle around N in $R(\alpha_1,\ldots,\alpha_{n+3})$ is 2π. Hence metrically N is actually not a singular set;
2. if $\alpha_k+\alpha_{k+1}+\alpha_{k+2}<\pi$, then around the codimension 2 face isometric to
$$S := \mathcal{H}(\alpha_1,\ldots,\alpha_k+\alpha_{k+1}+\alpha_{k+2},\ldots,\alpha_{n+3})$$
are glued six orthoschemes corresponding to the six ways of ordering
$$(\alpha_k,\alpha_{k+1},\alpha_{k+2})\,.$$

Let us examine the angle θ around S. It is a sum of six dihedral angles. Formula (1.5) gives the cosine of each dihedral angle. It is symmetric in two variables, hence θ is twice the sum of three different dihedral angles. Moreover

PROPOSITION 2.3. *We have that* $\cos(\theta/2)$ *is equal to*
$$\frac{\sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)-\sin(\alpha_1+\alpha_2+\alpha_3)\big(\sin(\alpha_1)\sin(\alpha_2)+\sin(\alpha_2)\sin(\alpha_3)+\sin(\alpha_3)\sin(\alpha_1)\big)}{\sin(\alpha_1+\alpha_2)\sin(\alpha_2+\alpha_3)\sin(\alpha_3+\alpha_1)}\,.$$

Proof. This formula is proved in [KNY99], hence we only outline the proof and refer to this paper for more details about the computation. Note that in this reference the result is stated only for $n=2$ or 3 as the authors get (1.5) only for these dimensions. The idea is the following one. We have to consider the sum of three dihedral angles, and as we know that they are acute, this sum is less than 2π. Hence a gluing of three orthoschemes having those dihedral angles can be isometrically embedded in the hyperbolic space (or at least a neighborhood of S). The gluing involves four facets glued around S, that gives four outward unit normals e_1, e_2, e_3, e_4 spanning a space-like plane in the Minkowski space. Hence we can consider these vectors as unit vectors in the Euclidean plane, and the problem is now reduced to finding the angle between e_1 and e_4 knowing the angles between e_1 and e_2, e_2 and e_3 and e_3 and e_4 (the exterior dihedral angles of the orthoschemes).

Here is a natural question.

QUESTION 2. *For which values of the parameters $(\alpha_1, \dots, \alpha_{n+3})$ is the cone-manifold $R(\alpha_1, \dots, \alpha_{n+3})$ isometric to an orbifold?*

An intermediate step could be to determine whether there can exist such orbifolds for all dimensions n. This is motivated by the fact that neither Coxeter orthoschemes nor Mostow orbifolds (see Section 3) exist for $n > 9$. Another analogy is that there exist (at least) 98 Coxeter orthoschemes (counting one for each infinite family in dimension ≤ 3) and 94 Mostow orbifolds (but for dimension ≥ 4 there are at least 76 Coxeter orthoschemes and only 28 Mostow orbifolds). But obviously there is no relation between the fact that $R(\alpha_1, \dots, \alpha_{n+3})$ is an orbifold and the fact that the orthoschemes constituting it are Coxeter. For example it is easy to check that $R(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ is a cone-manifold, made of Coxeter orthoschemes (some of them are isometric to the one on the left in Figure 6).

Another intermediate step could be to determine whether there exist non-rational angles α_i such that $R(\alpha_1, \dots, \alpha_{n+3})$ is an orbifold (in the case of Mostow orbifolds, the angles have to be rational [DM86, 3.12]).

I tried the formula of Proposition 2.3 with a computer program, with $\alpha_i = p\pi/q$, p and q integers, $p < q < 100$, as data. A value of the form $\cos(\pi/k)$ (actually $\cos(\pi/2)$) was obtained only for:

$$\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{12}\right). \tag{2.1}$$

The program may have missed some values and more involved computations would lead too far from the scope of this paper. It is easy to check that if $R(\alpha_1, \dots, \alpha_{n+3})$ is an orbifold with only (2.1) leading to a singular stratum then $n = 2$. Two examples of such orbifolds appear in Table 1, namely $R(\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ and $R(\frac{2\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4})$, and using (1.5) we note that the orthoschemes involved in the gluings are not of Coxeter type.

There is an easy case when $R(\alpha_1,\dots,\alpha_{n+3})$ is a manifold: this is when there does not exist any singular set of codimension 2, i.e. when the sum of each triple of angles is at least equal to π. It is easy to check that this can happen only for $n=2$ or 3. If $n=3$ the sum of each triple of angles must be π, which implies that all the angles are equal. Hence there is only one case, obtained by gluing 60 times the orthoscheme shown on the left in Figure 6. For $n=2$ there are infinitely many $R(\alpha_1,\dots,\alpha_5)$ which are manifolds, for example those obtained by slightly deforming the angles in the list $\left(\frac{2\pi}{5},\frac{2\pi}{5},\frac{2\pi}{5},\frac{2\pi}{5},\frac{2\pi}{5}\right)$ (they are obtained by gluing right-angled pentagons).

It is proved in [KNY99, YNK02] that if $n=2$ or 3 and $R(\alpha_1,\dots,\alpha_{n+3})$ contains only orthoschemes of type 3, then its metric structure is uniquely determined by the angles $(\alpha_1,\dots,\alpha_{n+3})$. This contrasts with the fact that an orthoscheme can be constructed from an infinite number of lists of angles. The reason is that permutation of angles and projective transformations do not commute in general. This leads to the following question:

QUESTION 3. *Does $R(\alpha_1,\dots,\alpha_{n+3})=R(\alpha'_1,\dots,\alpha'_{n+3})$ imply that, up to the action of the dihedral group, $(\alpha_1,\dots,\alpha_{n+3})=(\alpha'_1,\dots,\alpha'_{n+3})$?*

The study of the deformation spaces of polygons with fixed angles appears in [Thu98] as a particular case of the study of the deformation spaces of polyhedra (see the next section). It also appears as an exercise in [Thu97, Problem 2.3.12]. Detailed studies can be found in [KY93, Yos96, AY98] for $n=2$ and in [AY99, MN00] for $n=3$. Both cases are treated in [KNY99, YNK02]. Note that these references deal mainly with orthoschemes of type 3.

Due to the so-called *Schwarz–Christoffel map*, $R(\alpha_1,\dots,\alpha_{n+3})$ can be thought of as a real hyperbolization of the space of configurations of points on the circle, depending on weights $(\alpha_1,\dots,\alpha_{n+3})$ [Thu98, KNY99, IP99, MN00]. For this reason this construction can be related to many others, see e.g. [KM95, IP99] and references therein. In [KM95] it is proved that the spaces of polygons with fixed angles are homeomorphic to the spaces of polygons with fixed edge lengths (so Lemma 2.2 is the analog of [KM95, Theorem 1, Lemma 6]). Natural metrics on the moduli spaces of convex polygons in Riemannian and Lorentzian spaces of constant curvature are also introduced in [Sch07].

3. Spaces of polyhedra

3.1 Configurations of points on the sphere

In [DM86], the space of configurations of points on the sphere with suitable weights is endowed with a complex hyperbolic structure, depending on the weights. It is then possible to find a list of (compact or finite-volume) complex hyperbolic orbifolds. The list was enlarged in [Mos86] (some of them had been known for a long time; see [DM86] for more details). Due to a generalization of the Schwarz–Christoffel map [Thu98, 8], the space of configurations of weighted points on the sphere is homeomorphic to the space of Euclidean cone-metrics on the sphere with prescribed cone-angles (see below). (This also follows from general theorems about the determination of metrics on surfaces by the curvatures [Tro86, Tro91].) In [Thu98], the results of [DM86, Mos86] are recovered by studying such spaces of metrics on the sphere. The two constructions are outlined in parallel in [Koj01]. A bridge between the two constructions is clearly exposed in [Tro07]. Moreover this last reference also concerns surfaces of higher genus. We won't review further the extensive work based on [DM86] and [Thu98].

A *Euclidean metric with N cone singularities of positive curvature* on the sphere $\mathbf{S}^2$ is a flat metric on $\mathbf{S}^2$ minus N points $x_1, \dots, x_N$, $N \geq 3$, such that a neighborhood of each x_k is isometric to the neighborhood of the apex of a Euclidean cone of angle $0 < \theta_k < 2\pi$ (the curvature is $2\pi - \theta_k$). Such a metric is uniquely determined up to homotheties by the conformal class of the N-punctured sphere and by the numbers α_k (satisfying the Gauss–Bonnet condition). Let $(\alpha_1, \dots, \alpha_{n+3})$ satisfy (A). We denote by $C(\alpha_1, \dots, \alpha_{n+3})$ the set of Euclidean metrics on the sphere with $n+3$ cone singularities of positive curvature $2\alpha_i$, up to direct isometries and homotheties. Hence $C(\alpha_1, \dots, \alpha_{n+3})$ is a connected topological manifold of dimension $2n$. We suppose that the cone-singularities are labeled: if x_k and x_j have the same cone-angle, then exchanging them leads to another metric. (Our $C(\alpha_1, \dots, \alpha_{n+3})$ should be written $P(A; 2\alpha_1, \dots, 2\alpha_{n+3})$ in the notation of [Thu98, p. 524]. The notation $C(\alpha_1, \dots, \alpha_{n+3})$ defined in [Thu98, p. 524] concerns non-labeled cone-singularities, see the remarks after Theorem 4.2.)

Because of the following famous theorem, $C(\alpha_1, \dots, \alpha_{n+3})$ can also be defined as the space of convex polytopes of $\mathbf{R}^3$ with $n+3$ labeled vertices x_k such that the sum of the angles on the faces around x_k is $2\pi - 2\alpha_k$, up to Euclidean direct isometries and homotheties. In the sequel we will identify the metric and the polytope.

THEOREM 3.1 (Alexandrov's Theorem, [Ale42, Ale06]). *Let g be a Euclidean metric on the sphere with cone singularities of positive curvature. There exists a convex polytope P in $\mathbf{R}^3$ such that the induced metric on the boundary of P is isometric to g. Moreover P is unique up to ambient isometries.*

It is proved in [KM96] that the spaces of configurations of points on the sphere are also homeomorphic to the spaces of polygons in $\mathbf{R}^3$ with fixed edge lengths. Using a theorem of Minkowski, it is not hard to see that such spaces of polygons are homeomorphic to the spaces of convex polytopes of $\mathbf{R}^3$ with fixed face areas. A duality between the spaces of polygons (in the plane) with fixed angles and the spaces of polygons with fixed edge lengths is proved in [KM95]. In dimension 3 this duality is expressed between the spaces of polytopes with fixed cone-angles and the spaces of polytopes with fixed face areas. A proof of Minkowski's Theorem is given in [Kla04], together with a proof of the Minkowski Inequality, which is the basic result for proving the Alexandrov–Fenchel Theorem. A quaternionic structure on the spaces of polygons of $\mathbf{R}^5$ with fixed edge lengths is described in [FL04].

To emphasize the analogy between the case of convex polygons in $\mathbf{R}^2$ and the case of convex polytopes in $\mathbf{R}^3$, we describe in the next subsection a local parametrization of $C(\alpha_1, \dots, \alpha_{n+3})$ close to Thurston's. A difference to his approach is in the choice of unfolding a polytope on the plane. The use of the Alexandrov Theorem is never really necessary, but it simplifies some arguments, and the construction uses elements from the original proof by A. D. Alexandrov.

3.2 POLYTOPES AS COMPLEX POLYGONS

A *N-gon* of the Euclidean plane is an ordered N-tuple of points $(a_1, \dots, a_N)$ (the vertices) with line segments joining a_{k-1} to a_k (with $a_{N+1} = a_1$). Considering the Euclidean plane as the complex plane, the set of N-gons is identified with $\mathbf{C}^N$. Let P be a convex polytope representing an element of $C(\alpha_1, \dots, \alpha_{n+3})$. We will associate P with a $(2n+6)$-gon $A(P)$.

Let us choose a point s on (the boundary of) P. The point s is a *source point*. We suppose that s is *generic*: it is not a vertex and for each vertex x_k there is a unique shortest geodesic from s to x_k. If we cut P along these shortest geodesics, then it can be unfolded into the plane as a $2(n+3)$-gon: $n+3$ vertices are the images of the vertices x_k, which alternate with the

$n+3$ images of s. These will be denoted by $(s_1,\ldots,s_{n+3})$. By an abuse of notation the images of the vertices x_k will still be denoted by letters x_k, but the vertices of the $2(n+3)$-gon are labeled so that s_k is between x_k and x_{k+1} for the direct order. An example is shown in Figure 8.

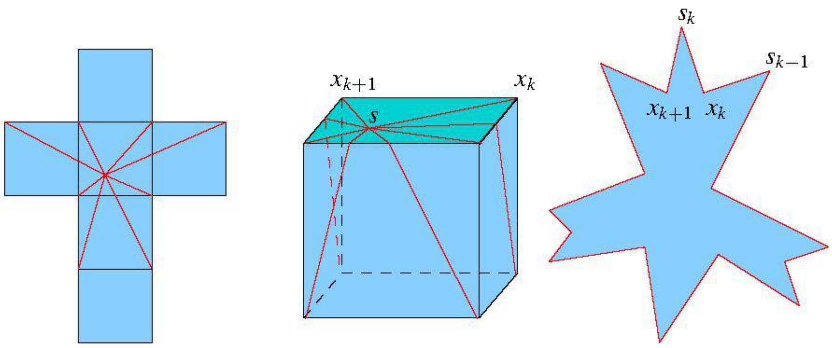

FIGURE 8

On the right is an Alexandrov unfolding of the cube. On the left is an edge unfolding of the cube, used to determine the shortest geodesics from the source point to the cone points.

This procedure is known as *Alexandrov unfolding* or *star unfolding* (even if the resulting polygon is not necessarily star-shaped, hence we will avoid this last terminology), and is apparently due to Alexandrov ([Ale05, 4.1.2], [Ale06, VI,1]). Actually in those references the source point s is a vertex. With this restriction it is proved that the Alexandrov unfolding is non-overlapping in [Thu98, Proposition 7.1] (but in Figure 16 of [Thu98] the source point is generic). In our case with s a generic point, the Alexandrov unfolding is also non-overlapping [AO92]. See also [MP08, Pak08]. Hence $A(P)$ is a simple polygon. Alexandrov unfolding is used in [Web93] to parametrize the spaces of cone-metrics on the sphere with four cone-singularities of positive curvature and one cone-singularity of negative curvature (i.e. the angle around the singularity is $> 2\pi$). Another way of unfolding cone-metrics with five cone-points in the complex plane is described in [Par06].

By knowing only $(s_1,\ldots,s_{n+3})$ we can recover $A(P)$ and hence P, because the x_k are determined by

$$s_k - x_k = e^{i2\alpha_k}(s_{k-1}-x_k)\,. \tag{3.1}$$

It follows that $A(P)$ lives in a complex vector space of (complex) dimension $n+3$. We identify this space with $\mathbf{C}^{n+3}$. On $\mathbf{C}^{n+3}$ we define the following

Hermitian form:

$$(3.2)\quad M(P,Q) = -\frac{1}{4i}\sum_{k=1}^{n+3} s_k(P)\overline{x_k(Q)} - x_k(P)\overline{s_k(Q)} + x_{k+1}(P)\overline{s_k(Q)} - s_k(P)\overline{x_{k+1}(Q)}$$

(the signed area of the triangle $0ab$, $(a,b)\in \mathbf{C}^2$, is $\frac{1}{4i}(b\overline{a}-a\overline{b})$). If P is an element of $C(\alpha_1,\dots,\alpha_{n+3})$, then $M(P,P)$ is minus the area of $A(P)$ (the face-area of P). Here is the analog of Theorem 1.1:

THEOREM 3.2. *The Hermitian form M has signature $(1,2,n)$.*

This is proved by induction on n. For $n=0$, polytopes are doubled triangles and the result follows from Theorem 1.1 (see the proof of Lemma 4.1). This can also be deduced directly from the facts that the triangles have positive area and that the area is invariant under translations.

For any n, we go back to $n-1$ by the process called "cutting and gluing", apparently due to Alexandrov ([Ale06, Lemma 1, p. 226], see also [Bus58, 17.5] and [Thu98, Proposition 3.3]). Cutting and gluing proceeds as follows: if two cone points, with curvatures $2\alpha_k$ and $2\alpha_j$ such that $\alpha_k+\alpha_j<\pi$, are sufficiently close (such points always exist), then we cut the geodesic joining them. One can glue a Euclidean cone of curvature $2(\alpha_k+\alpha_j)$ to the two resulting geodesics in such a way that the singularities at x_k and x_j disappear. The area of the old metric is the area of the new metric minus the area of the cone. A similar procedure applied to polygons instead of polytopes is used in [BG92] to prove Theorem 1.1 of the current paper.

The Hermitian form (3.2) can be considered as the complex mixed-area. It is a classical form on the space of the N-gons, see e.g. [FRS85] and the references therein. In [FRS85] the form is related to a larger family of Hermitian forms. Moreover polygons are considered as finite Fourier series. It is possible to prove the Alexandrov–Fenchel Theorem (actually the Minkowski Inequality) for convex curves using Fourier series. Perhaps it is possible to compute the signature of M using this point of view. For more details we refer to [Gro96], especially Formula (4.3.3) and Remarks and References of 4.1.

The quotient of $\mathbf{C}^{n+3}$ by the kernel of M describes the unfoldings up to translations. This quotient is a complex vector space of (complex) dimension $n+1$. An Alexandrov unfolding is clearly a well-defined and injective map from a (sufficiently small) neighborhood U of P in $C(\alpha_1,\dots,\alpha_{n+3})$ if the unfoldings are moreover considered up to rotations and homotheties. Hence U is mapped homeomorphically to a subset of the set of negative vectors (for a Hermitian form of signature $(1,n)$) of the quotient of a complex vector space of dimension $n+1$ by complex conjugation: Alexandrov unfolding provides charts from $C(\alpha_1,\dots,\alpha_{n+3})$ to $\mathbf{CH}^n$, the *complex hyperbolic space* of (complex) dimension n — we refer to [Eps87, Gol99] for details about $\mathbf{CH}^n$.

Let P be an element of $C(\alpha_1,\dots,\alpha_{n+3})$. Let T be a geodesic triangulation of P such that the cone-points are exactly the vertices of T (we will call such a triangulation a *cone-triangulation*). The fact that cone-triangulations exist is obvious if we use the Alexandrov Theorem (it suffices to triangulate the faces of the corresponding convex polytope), but precisely the existence of cone-triangulations is a step in the proof of this theorem, see [Bus58, p. 130], [Thu98, Proposition 3.1]. If we cut along some edges of T it is possible to unfold P to the complex plane: this is an *edge unfolding* (it is not necessarily non-overlapping). To each edge of T is associated a complex number, and $n+1$ complex numbers suffice to recover P [Thu98, Proposition 3.2]: edge unfolding provides another chart from $C(\alpha_1,\dots,\alpha_{n+3})$ to $\mathbf{CH}^n$. We check that these two kinds of local coordinates are compatible.

LEMMA 3.3. *Let P be in $C(\alpha_1,\dots,\alpha_{n+3})$ and let A be an Alexandrov unfolding of P. Let U be a neighborhood of P. If U is sufficiently small, then there exists an edge unfolding E of P such that A and E are homeomorphisms on U and such that there is an isometric linear bijection taking $A(U)$ to $E(U)$.*

Proof. The Alexandrov unfolding $A(P)$ contains a simple polygon $SA(P)$ with vertices $x_1,\dots,x_{n+3}$ (it is simple because $A(P)$ is). To go from $SA(P)$ to $A(P)$ one must add to each edge x_kx_{k+1} of $SA(P)$ the triangle $T_k := x_ks_kx_{k+1}$. We will "roll" all the triangles T_k around $SA(P)$. More precisely, we perform on T_2 a rotation of angle $2\alpha_3$ and center x_3. T_2 is now glued on the edge x_3s_3. We rotate the union of T_2 and T_3 around x_4 by an angle $2\alpha_4$, and so on. Finally all the T_k are glued around T_1 (all the s_k go to s_1); see Figure 9 for an example with the cube. The gluing of all the T_k around s_k gives a simple polygon (because the sum of the angles around s is 2π). So we get two simple polygons glued along the edge x_1x_2. A triangulation of each of them is exactly an edge unfolding of P (note that there is no reason for the union to be simple). Let us denote this union by Q.

The coordinates of $E(U)$ are vectors associated to the diagonals of Q, namely the differences of the coordinates of their endpoints. These endpoints are the vertices of Q. By the preceding paragraph, these vertices are obtained from $x_1,\dots,x_{n+3}$ by composition of rotations, and by (3.1), the x_i are linear functions of the s_i, which are the coordinates of $A(U)$. This describes a linear map sending $A(U)$ to $E(U)$. Obviously this operation preserves the area and is bijective.

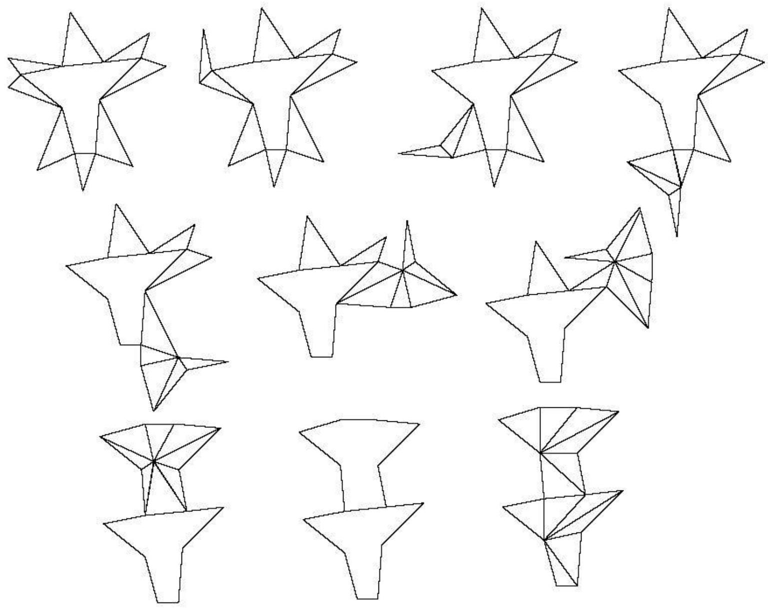

FIGURE 9
From an Alexandrov unfolding to an edge unfolding of the cube
(example for the proof of Lemma 3.3)

From now on we summarize the results of [Thu98]. For the coordinates given by the edge unfoldings on $C(\alpha_1, \dots, \alpha_{n+3})$, the changes of charts correspond to flipping the edges of T (when two triangles form a quadrilateral with a diagonal, to *flip* is to delete the diagonal and to choose the other one). In terms of the complex coordinates, the changes of charts are linear maps, and obviously isometries. This gives a structure of complex hyperbolic manifold of complex dimension n on $C(\alpha_1, \dots, \alpha_{n+3})$. This manifold is not complete (as cone-points can collide). We denote its metric completion by $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$. Then $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ has a structure of complex hyperbolic cone manifold. (A *cone manifold structure* for non constant curvature is less obvious to define than in the constant curvature case. We refer to [Thu98] for a precise definition.)

The collision of two cone-points x_k and x_j describes a singular stratum of (complex) codimension 1 in $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ (the collision is possible only if $\alpha_k + \alpha_j < \pi$). The main point is that the singular curvature around the stratum is $2\alpha_k + 2\alpha_j$ [Thu98, Proposition 3.5]. Hence it is easy to know for which $(\alpha_1, \dots, \alpha_{n+3})$ the cone-angles around the (real) codimension 2 strata are of the form $2\pi/k$. By the Poincaré Theorem [Thu98, Theorem 4.1] this means that $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ is a complex hyperbolic orbifold. There exist 36 of these orbifolds, listed in Table 1.

Moreover $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ has finite volume, and it is compact if and only if there is no subset of $(\alpha_1, \dots, \alpha_{n+3})$ summing to π [Thu98, proof of Theorem 0.2].

4. Spaces of polygons into spaces of polyhedra

The set of convex polytopes contains *degenerated (convex) polytopes* which are obtained by "doubling" a convex polygon. Doubling is gluing isometrically along the edges a polygon to its image by a reflection in a line. Such a reflection reverses the labeling of the angles, and hence there is a canonical injection f from $R(\alpha_1, \dots, \alpha_{n+3})$ to $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$. The metric structure on each space is given by the face-area, so it is not surprising that f is an isometry.

LEMMA 4.1. *For each choice of an order on $(\alpha_1, \dots, \alpha_{n+3})$, the composition of f with an Alexandrov unfolding extends to an isometric linear map from $(\mathbf{R}^{n+1}, 2m)$ to $(\mathbf{C}^{n+1}, M)$.*

Proof. Let P be a convex polygon with exterior angles $(\alpha_1, \dots, \alpha_{n+3})$ and let $(u_1, \dots, u_{n+3})$ be its set of outward unit normals. We choose a point s in the interior of P. Without loss of generality, suppose that s is the origin. We denote by s_k the image of s by a reflection in the edge $x_k x_{k+1}$. The polygon $x_1 s_1 x_2 \dots x_{n+3} s_{n+3}$ is an Alexandrov unfolding of the doubling of P, and then the linear extension of f is the following map from $\mathbf{R}^{n+1}$ to $\mathbf{C}^{n+1}$:

$$(h_1, \dots, h_{n+3}) \mapsto (2h_1 u_1, \dots, 2h_{n+3} u_{n+3}) .$$

It follows that $M(f(u_k), f(u_j)) = 0$ if $2 \leq |j-k| \leq n+1$, and writing (3.1) as

$$x_k = \frac{1}{2i} \frac{1}{\sin(\alpha_k)} \left(e^{i\alpha_k} s_{k-1} - e^{-i\alpha_k} s_k \right)$$

we easily compute that M is twice m (compare with (1.2) and (1.3)):

$$\begin{aligned}
M(f(u_k),f(u_{k+1})) &= 4M(u_k,u_{k+1}) \\
&= -\frac{1}{i}\left(-u_k\overline{x_{k+1}(u_{k+1})} - x_{k+1}(u_k)\overline{u_{k+1}}\right) \\
&= \frac{1}{i}\left(\frac{1}{2i}\frac{1}{\sin(\alpha_{k+1})} + \frac{1}{2i}\frac{1}{\sin(\alpha_{k+1})}\right) \\
&= -\frac{1}{\sin(\alpha_{k+1})} = 2m(u_k,u_{k+1}),
\end{aligned}$$

$$\begin{aligned}
M(f(u_k),f(u_k)) &= 4M(u_k,u_k) \\
&= -\frac{1}{i}\left(u_k\overline{x_k(u_k)} - \overline{u_k}x_k(u_k) + \overline{u_k}x_{k+1}(u_k) - u_k\overline{x_{k+1}(u_k)}\right) \\
&= -\frac{1}{i}\left(\frac{1}{i}\frac{\cos(\alpha_k)}{\sin(\alpha_k)} + \frac{1}{i}\frac{\cos(\alpha_{k+1})}{\sin(\alpha_{k+1})}\right) \\
&= \frac{\sin(\alpha_k+\alpha_{k+1})}{\sin(\alpha_k)\sin(\alpha_{k+1})} = 2m(u_k,u_k).
\end{aligned}$$

It follows that on the image of $\mathbf{R}^{n+1}$ in $\mathbf{C}^{n+1}$, the Hermitian form M has real values. Moreover this image has maximal real dimension, hence it is a real form of $\mathbf{C}^{n+1}$. To each real form is associated a unique *real structure* (= anti-linear involution) compatible with the Hermitian structure, whose fixed-point set is the real form. Here the real structure corresponds exactly to the complex conjugation. We follow [Gol99] for the definitions and refer to it for more details.

The real structure on charts given by polygons comes from a global isometric involution reversing the orientation on $\overline{C}(\alpha_1,\dots,\alpha_{n+3})$, denoted by ρ. The involution ρ can be described as the reflection of polytopes in a plane. As the vertices are labeled, the fixed-point set of ρ is exactly $R(\alpha_1,\dots,\alpha_{n+3})$. We easily get the metric structure of this set for the orbifolds discovered by Deligne and Mostow.

THEOREM 4.2. *Table 1 gives the metric structure of $R(\alpha_1,\dots,\alpha_{n+3})$ for Deligne–Mostow orbifolds.*

Proof. If $R(\alpha_1,\dots,\alpha_{n+3})$ has no singular set of codimension 2, then it is a manifold. This occurs when $\alpha_i+\alpha_j+\alpha_k \geq \pi$ for each triple $(\alpha_i,\alpha_j,\alpha_k)$. The two cases marked as orbifolds in Table 1 have only one singular stratum, which is represented by the triple $(\frac{\pi}{4},\frac{\pi}{4},\frac{5\pi}{12})$ (see (2.1)).

From the discussion after Proposition 2.3, we know that the other examples are neither manifolds nor orbifolds. We check this fact. If there exist three angles $\alpha_i, \alpha_j, \alpha_k$ such that the dihedral angles given by $(\alpha_i, \alpha_j, \alpha_k)$ and $(\alpha_j, \alpha_i, \alpha_k)$ are $\pi/4$, then the total angle around the singular set defined by these angles is at least four times $\pi/4$, plus something smaller than π (we know that these dihedral angles of orthoschemes are $< \pi/2$), hence it cannot be $2\pi/k$. It is easy to see that such a situation occurs for the triples $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6})$, $(\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{8})$, $(\frac{\pi}{10}, \frac{2\pi}{5}, \frac{2\pi}{5})$, $(\frac{\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12})$ and $(\frac{\pi}{18}, \frac{4\pi}{9}, \frac{4\pi}{9})$, which cover all the cases indicated in Table 1.

One can also study the spaces of cone-metrics without labeling the cone-points. In this case there are more orbifolds: if $\alpha_k = \alpha_j$ and $\alpha_k + \alpha_j < \pi$, then the angle around the stratum is half that obtained with labeling. Such orbifolds were introduced in [Mos86, Mos88]. The complete list was given in [Thu98] (it is known to be complete due to [DM86, 3.12] and [Fel97]). This list contains the list introduced in [DM86] and given in Table 1. In this case of non-labeling of the cone-points, the fixed-point set of ρ contains the spaces of polygons and some polytopes obtained by doubling convex caps (which can be considered as convex isometric embeddings of Euclidean metrics with conical singularities on the closed disc). Answering the following question should be a step in the study of the fixed-point set of ρ for Mostow orbifolds.

QUESTION 4. *Is it possible to describe a (real) hyperbolic structure on the space of convex caps with fixed cone-angles ?*

The following work concerns real forms of complex hyperbolic orbifolds, with approaches different from ours: [AY98] for $n = 2$, [ACT07b, ACT06] for $n = 3$, [Yos01] and [ACT07a] (announced in [ACT03]) for $n = 4$, [Chu07] for $n = 5$, and the references therein. The following question arises from the fixed-point sets appearing in this work:

QUESTION 5. *Is it possible to describe a (real) hyperbolic structure on the space of centrally symmetric convex polytopes with fixed cone-angles ?*

TABLE 1

The angles $(\alpha_1, \ldots, \alpha_{n+3})$ are those for which $\overline{C}(\alpha_1, \ldots, \alpha_{n+3})$ is a complex hyperbolic orbifold, given by the list in [DM86]. Column **T** gives the number of the orbifold in the list of [Thu98]. Column **S** gives the structure of $R(\alpha_1, \ldots, \alpha_{n+3})$: M means that it is a manifold, O that it is an orbifold, and C that it is a cone-manifold.

T	**Angles**	**S**	**T**	**Angles**	**S**
	Dimension 5		43	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M
3	$\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	45	$\frac{3\pi}{4}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	C
	Dimension 4		46	$\frac{5\pi}{8}, \frac{5\pi}{8}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
4	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	47	$\frac{\pi}{2}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	M
	Dimension 3		48	$\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}$	M
1	$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M	49	$\frac{\pi}{10}, \frac{7\pi}{10}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	C
5	$\frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	57	$\frac{2\pi}{3}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	C
6	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	65	$\frac{7\pi}{12}, \frac{7\pi}{12}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}$	C
39	$\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	C	68	$\frac{5\pi}{6}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
44	$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	C	69	$\frac{2\pi}{3}, \frac{7\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
66	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	70	$\frac{2\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}$	O
67	$\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	71	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$	O
	Dimension 2		72	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
2	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M	73	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M
7	$\frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	74	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
8	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$	M	75	$\frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
9	$\frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	M	78	$\frac{4\pi}{15}, \frac{8\pi}{15}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	M
40	$\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	C	79	$\frac{\pi}{18}, \frac{11\pi}{18}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}$	C
41	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}$	C	85	$\frac{7\pi}{10}, \frac{11\pi}{20}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
42	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$	M	89	$\frac{7\pi}{12}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	M

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