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A NOTE ON THE CHAS-SULLIVAN PRODUCT

by François LAUDENBACH

ABSTRACT. We give a finite-dimensional approach to the Chas-Sullivan product on the free loop space of a manifold, which is not necessarily orientable or compact.

1. INTRODUCTION

Let M be an n-dimensional manifold with empty boundary; it is not required to be either compact or orientable. Denote by $LM = C^\infty(S^1, M)$ its free loop space. In a famous paper [4], in the case when M is an orientable and closed manifold, M. Chas and D. Sullivan constructed a natural graded algebra structure on the homology $H_*(LM; \mathbf{Z})$, more precisely a product

$$H_i(LM; \mathbf{Z}) \otimes H_j(LM; \mathbf{Z}) \to H_{i+j-n}(LM; \mathbf{Z}),$$

in the same *intersection* spirit as the usual intersection product on $H_*(M; \mathbf{Z})$. But their ideas remained not completely accomplished. A different approach was considered by R. Cohen and J. Jones in [6]. According to their abstract, they describe "a realization of the Chas-Sullivan product in terms of a ring spectrum structure on the Thom spectrum of a certain virtual bundle over the loop space", a difficult technique indeed. Recently in [8], Y. Félix and J.-C. Thomas put the *String Topology* into a broad homotopy theoretical setting; they prove that the *operations* in string topology are preserved by homotopy equivalence, at least in the 1-connected case. On the contrary, in the present note we propose a finite-dimensional approach, very close in spirit to [4], based on suitable transversality arguments. We also treat the case of a non-orientable manifold using local coefficients instead of $\mathbf{Z}$.

We do not think of LM as a topological space but as a simplicial set (except in Section 4). A *k-simplex in LM* is a smooth map $\Sigma\colon \Delta^k \times S^1 \to M$, where Δ^k is the standard k-simplex. Let 0 be the base point of $S^1 = \mathbf{R}/\mathbf{Z}$. The evaluation map $\mathrm{ev}_0\colon LM \to M$ is simplicial: $\mathrm{ev}_0(\Sigma) = \sigma$ where $\sigma\colon \Delta^k \to M$ is the k-simplex of M defined by

$$\sigma(t) = \Sigma(t, 0)\,.$$

It is easy to form a chain complex based on the simplices of LM and a bi-complex based on *bi-simplices* (pairs of simplices). In order to take the non-orientability into account, we limit ourselves to *small* simplices and bi-simplices (see 2.1). To define an *intersection product*, we consider *transverse* bi-simplices. By Thom's transversality theorem, they generate a sub-bicomplex which has the same homology as $LM \times LM$. As we shall see, smallness (resp. transversality property) will refer only to the image of simplices (resp. bi-simplices) through the evaluation map.

Let us introduce the sub-bicomplex $LM \underset{M}{\times} LM$ of $LM \times LM$ made of *composable loops*, namely pairs of loops having a common origin. Performing the composition yields a well-defined map in homology $LM \underset{M}{\times} M \to LM$. The intersection product is not defined on the chain level. But the intersection of two *transverse* cycles in LM produces a "singular" smooth manifold W in $LM \underset{M}{\times} LM$, hence in LM after composing, which becomes a simplicial cycle once W is triangulated. According to Whitehead [13], such a triangulation is unique up to subdivision and isotopy. Therefore the product is well-defined at the homology level.

In the last section it is shown that this definition of the free loop product is not less efficient than the "infinite-dimensional" one when calculations are performed in concrete geometric situations. Results due to several authors (M. Goresky, N. Hingston in [9], D. Chataur, J.-F. Le Borgne in [5]) are rewritten in this setting.

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2. Simplices and bi-simplices at the manifold level

In this section we give a geometric approach to the intersection product in the homology of M. The main point is that this is done in such a way that the intersection product lifts immediately to the homology of LM through the map induced by ev_0. This requires an appeal to Thom's transversality with constraints which is more powerful than the usual transversality which is frequently used (see J. E. McClure [11] in the *PL* case or E. Castillo, R. Diaz [3] in the smooth case).

2.1. The manifold M under consideration is equipped with an atlas $\mathcal{A}$ of charts. A *small* k-simplex is a smooth map

$$\sigma\colon \Delta^k \to M$$

whose image is contained in a chart from $\mathcal{A}$. For each simplex σ a particular chart $U(\sigma) \in \mathcal{A}$ containing its image is chosen once and for all. A *small k-chain* is a linear combination $\xi = \sum n_i\sigma_i$ of finitely many small simplices σ_i with coefficients $n_i \in \mathbf{Z}$. The *orientation twisted boundary* is defined by the formula

$$\partial\sigma = \sum_{i=0}^{k} \varepsilon(-1)^i F_i\sigma\,,$$

where $F_i\sigma$ is the i^{th} face of σ and ε is the sign of the Jacobian of change of coordinates from $U(F_i\sigma)$ to $U(\sigma)$ calculated at any point of the image of $F_i\sigma$. The small chains with this boundary form a chain complex whose homology is $H_*(M;\mathbf{Z}_{or})$, the homology with integer coefficients twisted by the orientation.

In the sequel, *homotopy* will mean smooth homotopy. Given a k-simplex σ, a homotopy $\sigma^t\colon \Delta^k \to M$, $t \in [0,1]$, with $\sigma^0 = \sigma$, induces a homotopy $(F\sigma)^t$ of each face $F\sigma$. So the following definition makes sense.

Definition 2.2. Given a chain $\xi = \sum n_i\sigma_i$, a *boundary-preserving homotopy* of ξ is a one-parameter family $\xi^t, t \in [0,1]$, $\xi^t = \sum n_i\sigma_i^t$, where $t \mapsto \sigma_i^t$ is a homotopy of σ_i into M ($\sigma_i^0 = \sigma_i$), and such that, if σ_i and σ_j have a common face at time $t = 0$, then the corresponding simplices σ_i^t and σ_j^t still have a common face at any time.

For instance, if ξ is a cycle then ξ^t is a cycle for every $t \in [0,1]$. When we consider such a homotopy of small chain, we shall restrict ourselves to the case when the homotopy is small, that is, each summand σ_i^t has values in $U(\sigma_i)$. The following homotopy extension lemma is easy to prove.

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LEMMA 2.3. *Let τ be a face of one of the summands of ξ. Any (small) homotopy of τ extends as a (small) boundary-preserving homotopy of ξ. The same statement holds for a family of homotopies with an extension depending continuously on the parameter.*

DEFINITION 2.4. A (p,q)-bi-simplex of $M \times M$ is a pair (u,v), where u is a p-simplex and v is a q-simplex of M. It is said to be *small* when both factors are small simplices of M.

It is convenient to use the notation $u \times v$ for a bi-simplex, considering the map $u \times v \colon \Delta^p \times \Delta^q \to M \times M$, $(x,y) \mapsto (u(x), v(y))$. The (small) bi-simplices generate a bi-complex $C_{**}(M \times M)$ whose elements are bi-chains, with two boundary operators twisted by the orientation,

$$\partial_1(u \times v) = (\partial u) \times v\,, \qquad \partial_2(u \times v) = u \times \partial v\,,$$

and a *total boundary operator*

$$D(u \times v) = \partial u \times v + (-1)^p u \times \partial v\,.$$

DEFINITION 2.5. A (small) bi-simplex $u \times v$ is said to be *transverse* when the map $u \times v$ and all its faces (they are bi-simplices) are transverse to the diagonal Δ_M.

In what follows, we tacitly assume all transverse bi-simplices to be small. The advantage of this is that $W = (u \times v)^{-1}(\Delta_M)$ is a proper orientable submanifold of codimension n (with corners) in $\Delta^p \times \Delta^q$. Moreover, if we use the charts $U(u)$ and $U(v)$, whose product contains the image of $u \times v$, W receives a canonical orientation.

Notice that, when $u \times v$ is a transverse bi-simplex, a small homotopy of its factors keeps it transverse. The transverse bi-simplices generate a sub-bicomplex $C^{tr}_{**}(M \times M)$ of $C_{**}(M \times M)$. A bi-chain is said to be *transverse* when each of its bi-simplices is transverse. It is said to be a *product* bi-chain if it has the form $\xi \times \eta$ where both factors are chains in M.

LEMMA 2.6. *Let $\xi \times \eta$ be a product bi-chain. There exists an arbitrarily C^∞-small boundary-preserving homotopy ξ^t of $\xi^0 = \xi$ such that $\xi^1 \times \eta$ is transverse. Moreover, when $\partial\xi \times \eta$ is transverse, the homotopy can be chosen stationary on $\partial\xi \times \eta$. The same statement holds for the second factor.*

Considering the composition of loops we have in mind, it is very important to have such approximation of bi-chains.

Proof. We first consider the case when $\xi \times \eta = u \times v$ is a bi-simplex. The conclusion follows from Thom's transversality theorem, not in its elementary form but in the form known as transversality with constraints [12]. Indeed, only the first factor is moved to guarantee that bi-simplices remain bi-simplices during the deformation. We argue as follows. Let S be a small n-ball in the vector space containing the chart $U(u)$, so small that for any $s \in S$, the image of $u^s := u + s$ is still contained in $U(u)$. We introduce the family $(u^s \times v)$ parametrized by S. It is transverse to Δ_M as well as its restriction to any face of $\Delta^p \times \Delta^q$. Therefore, according to Sard's lemma (used *à la Thom*), for almost every $s \in S$, $u^s \times v$ is a transverse bi-simplex.

For the general case, it is useful to observe that, in the above argument, the space S of parameters can be chosen arbitrarily small. When considering a product bi-chain $\xi \times \eta$, its k-skeleton is the collection of all the k-faces of the bi-simplices appearing in $\xi \times \eta$. Arguing recursively, we may assume that all bi-simplices in the k-skeleton are transverse. Let $(u \times v)$ be a $(k+1)$-bi-simplex, which we endow with a family S of translations in $U(u)$. According to Lemma 2.3, the translation by any $s \in S$ extends as a boundary-preserving homotopy of ξ (ending at ξ^s) and it can be chosen small enough so that each bi-simplex of the k-skeleton remains transverse. Therefore, for almost every $s \in S$, ξ^s is transverse along its k-skeleton and $u^s \times v$. Repeating this process, we successively make all $(k+1)$-bi-simplices transverse.

For the relative version, we notice that if Fu is a face of u and $Fu \times v$ is transverse, then $u \times v$ is automatically transverse to Δ_M on a neighborhood of the domain of $Fu \times v$. In that case we can moderate the translation by s when approaching $Fu \times v$, so that it becomes stationary along $Fu \times v$ (here, a moderate translation has the form $u + \rho s$ where ρ is a non-negative function on the domain of u). The general relative version follows easily.

Let $[\xi]$ and $[\eta]$ be two classes of respective degree p and q in $H_*(M; \mathbf{Z}_{or})$. According to Lemma 2.6, they can be represented by small cycles so that the bi-cycle $\xi \times \eta$ is transverse. This bi-cycle is unique up to *transverse homology*; more precisely we claim the following uniqueness lemma:

LEMMA 2.7. *Let $\xi \times \eta$ and $\xi' \times \eta'$ be two transverse bi-cycles in the same bi-homology class ($[\xi] = [\xi']$ and $[\eta] = [\eta']$). There exists a transverse bi-chain Ω whose total boundary is $D\Omega = \xi' \times \eta' - \xi \times \eta$.*

Proof. There exists a cycle ξ'', homotopic to ξ, such that $\xi'' \times \eta$ and $\xi'' \times \eta'$ are both transverse. Indeed, $\xi \times \eta$ is already transverse and this property is preserved under approximations of ξ. According to Lemma 2.6, such an approximation exists, making $\xi'' \times \eta'$ transverse.

If ω is a $(p+1)$-chain such that $\partial\omega = \xi - \xi''$, there is a boundary-preserving homotopy $\omega^t, t \in [0,1]$, of ω relative to its boundary such that $\omega^1 \times \eta$ is transverse. By the same argument, there is a $(p+1)$-chain ω' with $\partial\omega' = \eta - \eta'$ such that $\xi'' \times \omega'$ is transverse. Finally, there exists ω'' with $\partial\omega'' = \xi' - \xi''$ such that $\omega'' \times \eta'$ is transverse. By concatenating the three transverse homologies $\omega^1 \times \eta$, $\xi'' \times \omega'$ and $\omega'' \times \eta'$, we obtain a transverse homology [1]) joining the two given bi-cycles.

2.8. Intersection of cycles

We are now ready to define the *intersection of cycles*. To begin with, we consider a transverse product bi-chain $\xi \times \eta$ of degree (p,q), which is a sum of transverse bi-simplices

$$\xi \times \eta = \sum n_{ij} u_i \times v_j .$$

Let W_{ij} be the preimage of Δ_M by $u_i \times v_j$. As mentioned earlier, W_{ij} is an oriented manifold with corners of codimension n. If $F(u_i \times v_j)$ is a face of the bi-simplex, FW_{ij} denotes the corresponding face of W_{ij}.

According to Whitehead [13], W_{ij} can be smoothly triangulated by a PL-triangulation T_{ij}. Moreover, if some faces have been already triangulated, then, by using the relative version of Whitehead's theorem, T_{ij} can be chosen so that the triangulated faces are subcomplexes. If two bi-simplices have common faces $F(u_i \times v_j) = F(u_k \times v_\ell)$, then we have a canonical diffeomorphism $FW_{ij} \to FW_{k\ell}$ which we think of as an identification. The triangulations of these faces are chosen accordingly. We consider the chain of $\Delta_M \cong M$,

$$\xi \cdot \eta := (-1)^{n(n-q)} \sum n_{ij}(u_i \times v_j)|(W_{ij}, T_{ij}) ,$$

which is called the *intersection product*. Of course, as a chain, this depends on the chosen triangulations. But, since these triangulations are unique up to subdivision and boundary-preserving isotopy (that is, smooth isotopy of each simplex in W_{ij} preserving the triangulation property), the ambiguity is not severe. The sign, which we call the *Dold sign*, will be commented upon later.

[1]) A *homology from a cycle c to a cycle c'* is a chain whose boundary is $c' - c$.

LEMMA 2.9. *When $\xi \times \eta$ is a transverse bi-cycle, the intersection product $\xi \cdot \eta$ is a cycle of degree $p+q-n$ (with orientation twisted coefficients). If $\xi \times \eta$ is changed by a transverse homology (in the sense of Lemma 2.7), $\xi \cdot \eta$ is changed by a homology. Finally, $[\xi] \cdot [\eta]$ is well-defined in $H_{p+q-n}(M; \mathbf{Z}_{or})$.*

Note that a change of triangulation of the W_{ij}'s may be thought of as a special case of a change by a transverse homology.

Proof. As W_{ij} is an oriented proper submanifold, (W_{ij}, T_{ij}) is a relative cycle in $\Delta^p \times \Delta^q$. Thus, the total boundary of $(u_i \times v_j)|(W_{ij}, T_{ij})$ is

$$\big((\partial u_i) \times v_j\big)|(W_{ij}, T_{ij}) + (-1)^p\big(u_i \times (\partial v_j)\big)|(W_{ij}, T_{ij}).$$

By summing over ij we obtain the boundary of $\xi \cdot \eta$. As ξ and η are cycles, each hyperface in the latter sum appears twice with opposite sign. The rest of the statement is easy to prove.

REMARK 2.10. In his book [7] (Chap. VIII, §13.3), A. Dold explains that the chosen sign makes the intersection product on homology and the (unsigned) cup-product on cohomology fit together via the Poincaré duality. Another advantage of this sign is the following. Set $\mathbf{H}_*(M; \mathbf{Z}_{or}) = H_{*+n}(M; \mathbf{Z}_{or})$. This regraded homology endowed with the above intersection product becomes a commutative ring in the graded sense.

3. SIMPLICES AND BI-SIMPLICES AT THE FREE LOOP SPACE LEVEL

3.1. Recall the evaluation map $\mathrm{ev}_0 \colon LM \to M$. A simplex $\Sigma \colon \Delta^k \times S^1 \to M$ is said to be *small* when $\sigma = \mathrm{ev}_0(\Sigma)$ is so. The i^{th} face of Σ is obtained by restricting Σ to $F_i\Delta^k \times S^1$. We have $F_i(\mathrm{ev}_0(\Sigma)) = \mathrm{ev}_0(F_i\Sigma)$. The *orientation twisted boundary* of Σ is

$$\partial\Sigma = \sum_{i=0}^{k} \varepsilon(-1)^i F_i\Sigma,$$

where ε is the sign of the Jacobian of change of coordinates from $U(\mathrm{ev}_0(F_i\Sigma))$ to $U(\mathrm{ev}_0(\Sigma))$. The small chains with this boundary form a sub-complex of $C_*(LM)$, the singular chain complex of LM, whose homology is $H_*(LM; \mathbf{Z}_{or})$. Indeed, this sub-complex is obtained from $C_*(LM)$ by two operations which induce homotopy equivalences: subdivision and smoothing. The notion of boundary-preserving homotopy is similar to that given in 2.1.

LEMMA 3.2. *The evaluation map has the lifting property for boundary-preserving homotopy of chains with any initial chain. Moreover, if the lifting of the homotopy is given along some faces this partial lifting can be extended to a global lifting.*

Proof. It is clear for $\Delta^k \times S^1 \times [0,1]$ retracts onto
$$\Delta^k \times S^1 \times \{0\} \cup \Delta^k \times \{0\} \times [0,1] \cup F \times S^1 \times [0,1]$$
where F is any union of faces in Δ^k.

A (p,q)-bi-simplex $u \times v$ in $LM \times LM$ is a map
$$\Delta^p \times \Delta^q \times S^1 \to M \times M\,,$$
$$(x,y,\theta) \mapsto \big(u(x,\theta), v(y,\theta)\big)\,.$$
It is said to be *transverse* when $\big(\mathrm{ev}_0(u), \mathrm{ev}_0(v)\big)$ is a transverse bi-simplex of $M \times M$. In that case, the preimage of the diagonal Δ_M yields a submanifold with corners $W \subset \Delta^p \times \Delta^q$. For each $(x,y) \in W$, the loops $u(x,-)$ and $v(y,-)$ are composable since they have common base points $u(x,0) = v(y,0)$. Therefore, taking a triangulation of W and the Dold sign as in 2.8, we get a $(p+q-n)$-chain of composable loops, which we call the *loop intersection product*:
$$u * v := (-1)^{n(n-q)}(u \times v)\big|W \times S^1\,.$$
Performing the composition (in the prescribed order, u before v) yields a $(p+q-n)$-chain in LM, called the *Chas-Sullivan product* or *loop product* $u \underset{CS}{\cdot} v$. Of course, it depends on the choice of the triangulation of W. Here, we see that the product structure of $u \times v$ is very important; without it we lose the entries of the composition. Thus, when making a bi-simplex transverse, it is crucial to do so by homotopy through bi-simplices. This is why we used transversality with constraints.

This loop product extends linearly to the transverse bi-chains. When performing it on a product bi-cycle, the result is a cycle in LM whose homology class in $H_{p+q-n}(LM; \mathbf{Z}_{or})$ is well-defined. Strictly speaking, the system of coefficients is $\mathrm{ev}_0^*(\mathbf{Z}_{or})$, which we write as $\mathbf{Z}_{or}$ for brevity, and we shall do so each time a loop space is in question. More precisely, we have the following proposition.

PROPOSITION 3.3. *Let $[\xi] \in H_p(LM; \mathbf{Z}_{or})$ and $[\eta] \in H_q(LM; \mathbf{Z}_{or})$. These classes may be represented by cycles in LM such that $\xi \times \eta$ is a transverse bi-cycle. The class of $\xi \underset{CS}{\cdot} \eta$ is uniquely defined in $H_{p+q-n}(LM; \mathbf{Z}_{or})$.*

Proof. Starting with arbitrary representatives of the given homology classes in LM, we apply Lemma 2.6. This produces a homotopy of their images by the evaluation map, making them a transverse bi-cycle in $M \times M$. The lifting homotopy property (Lemma 3.2) allows one to make $\xi \times \eta$ a transverse bi-cycle in $LM \times LM$. For this representative, the loop product $\xi \underset{CS}{\cdot} \eta$ is well-defined. If another representative $\xi' \times \eta'$ is used, one can prove that $\xi' \times \eta'$ and $\xi \times \eta$ are joined by a transverse homology, that is, a transverse bi-chain in $LM \times LM$. This is simply a loop version of the uniqueness Lemma 2.7, which can be deduced from the latter by applying the lifting homotopy property. As a consequence the homology class of $\xi \underset{CS}{\cdot} \eta$ is well-defined.

REMARK 3.4. Of course, the composition of smooth loops produces a piecewise smooth loop only. There are two ways of correcting this problem. One consists in doing a smoothing (boundary-preserving) homotopy. The other consists in using $LM = C^0(S^1, M)$ equipped with a mixed simplicial structure: a k-simplex will be a continuous map $u\colon \Delta^k \times S^1 \to M$ whose restriction to $\Delta^k \times 0$ is smooth.

In what follows, we use the following simplified notation: $\xi \cdot \eta := \xi \underset{CS}{\cdot} \eta$, which is defined when the bi-cycle $\xi \times \eta$ is transverse, and $[\xi] \cdot [\eta] := [\xi \underset{CS}{\cdot} \eta]$ which is well-defined. Actually, there is a one-parameter family of compositions $- \cdot_s -$, $s \in [0,1]$, defined as follows. Two loops u and v are said to be *s-composable* when $u(s) = v(0)$; in that case the composed loop $u \cdot_s v$ is made of $u|[0, 1-s] * v * u|[1-s, 1]$. When $s = 0$, it is the usual composition and when $s = 1$ we have $u \cdot_1 v = v \cdot u$. Notice that if two loops are 0-composable, they are also 1-composable.

PROPOSITION 3.5. *At the level of homology, the loop product is commutative up to sign. Precisely, if ξ and η are respectively a p-cycle and a q-cycle of LM, then $[\xi] \cdot [\eta] = (-1)^{(p-n)(q-n)}[\eta] \cdot [\xi]$. The loop product is also associative.*

If the regrading $\mathbf{H}_* = H_{*+n}$ is applied, then $\mathbf{H}_*(LM, \mathbf{Z}_{or})$ endowed with the loop product becomes a graded commutative and associative ring.

Proof. We assume that the bi-cycle $\xi \times \eta$ in $LM \times LM$ is transverse. We first prove:

$$[\xi] \cdot [\eta] = [\xi \cdot_1 \eta] .$$

Writing $\xi \times \eta = \sum n_{ij} u_i \times v_j$, one can make a boundary-preserving homotopy of ξ so that for every ij, the map

$$(x, y, s) \in \Delta^p \times \Delta^q \times [0,1] \mapsto (u_i(x,s), v_j(y,0), s) \in M \times M \times [0,1]$$

is transverse to $\Delta_M \times [0,1]$. This transversality yields a $(p+q-n+1)$-chain ω in LM whose boundary is $\partial\omega = \xi \cdot_1 \eta - \xi \cdot \eta$. Note that the loops in ω have $u(x,0)$ as base points; therefore, even when the loops in u are orientation reversing, still no sign appears in the formula for $\partial\omega$.

Now, we are reduced to proving:

$$[\xi \cdot_1 \eta] = (-1)^{(p-n)(q-n)}[\eta] \cdot [\xi] \,.$$

On both sides of this equality, the composition is the same. The difference comes from the orientation of the manifold W_{ij} associated to each transverse bi-simplex $u_i \times v_j$ appearing in $\xi \times \eta$. The permutation of the two factors in this bi-simplex induces a sign $(-1)^n$ in the co-orientation of the diagonal Δ_M, and a sign $(-1)^{pq}$ due to the order of the factors Δ^p and Δ^q in the source. Moreover, the Dold sign $(-1)^{n(n-q)}$ is changed to $(-1)^{n(n-p)}$. Altogether, the sign rule follows.

For associativity, consider three chains ξ, η, ζ in LM of respective degrees p, q, r. It is easy to define the transversality of the triple $\xi \times \eta \times \zeta$. If they are cycles and if the triple is transverse, the triple composition $[\xi] \cdot [\eta] \cdot [\zeta]$ is well-defined in $H_{p+q+r-2n}(LM; \mathbf{Z}_{or})$. Moreover, the following facts are easily checked:

- $\xi \times \eta$ is transverse;
- $(\xi * \eta) \times \zeta$ is transverse (where $*$ stands for the loop intersection product);
- $(\xi \cdot \eta) \cdot \zeta$ coincides with $\xi \cdot \eta \cdot \zeta$ up to a canonical reparametrization of the circle.

The last item yields the associativity in homology once it is observed that the same is true for the other bracketing.

Now the only question is how to make $\xi \times \eta \times \zeta$ transverse when it is not. It is not sufficient to move one factor, as one factor in $M \times M \times M$ is not transverse to the small diagonal. It is necessary to move two factors, say $\xi \times \eta$, while keeping the product structure, that is, moving through product chains $\xi^t \times \eta^t$. Again transversality with constraints is used.

REMARK 3.6. Each free loop γ, $\theta \mapsto \gamma(\theta)$, gives rise to a 1-cycle $\overline{\Delta}(\gamma)$ of loops by rotating the source:

$$\overline{\Delta}(\gamma)(t)(\theta) = \gamma(t+\theta)\,, \quad t \in S^1 \,.$$

This map induces $\Delta\colon H_*(LM) \to H_{*+1}(LM)$, with twisted coefficients when M is not orientable. Arguing similarly as in Proposition 3.5, one could probably prove the theorem of Chas-Sullivan that $\mathbf{H}_*(LM)$, endowed with the loop product and Δ, is a Batalin-Vilkovisky algebra.

4. A MULTIPLICATIVE SPECTRAL SEQUENCE

In this section, using our definition of the loop product, we discuss multiplicative properties which were stated and proved by Mark Goresky and Nancy Hingston in [9], §12 (up to the system of coefficients[2])). The setting is the one that R. Bott first considered in his seminal paper [1], where he studied the standard n-sphere, and that he extended in [2]. We summarize his results as follows. Let M be an n-dimensional closed Riemannian manifold whose primitive geodesics are all simple loops with the same length ($= 1$, say). Denote by Λ the space of loops parametrized proportionally to arc length. The class of regularity is not very important here; a good class is the Sobolev class $H^1(S^1, M)$. Let $\ell^2\colon \Lambda \to \mathbf{R}$ be the length squared. Bott proved that this is a nondegenerate function (now called a *Morse-Bott function*) and he calculated the index of the critical points, which are the closed geodesics. For $p \in \mathbf{N}$, let Λ_p be the subspace of loops of length $\leq p$ and Σ_p be the space of geodesics of length p. A geodesic in Σ_p is just a primitive geodesic which is traversed p times. As a manifold, $\Sigma_0 \cong M$ and, for $p \geq 1$, $\Sigma_p \cong UM$, where UM stands for the unit tangent space to M. Let α_p be the index of the Hessian of ℓ^2 at any point of Σ_p; obviously $\alpha_0 = 0$. Bott proved the iteration formula:

$$\alpha_p = p\alpha_1 + (p-1)(n-1).$$

Moreover he calculated (with $\mathbf{Z}_2$ coefficients) the spectral sequence derived from the filtration $\Lambda_0 \subset \Lambda_1 \subset \dots$ of Λ. We are going to consider the same spectral sequence, up to some regrading.

DEFINITION 4.1. A spectral sequence $\{E^r_{p,q}, d^r\}_{r\geq 1}$ is said to be *multiplicative* when it is endowed with a product $E^r_{p,q} \otimes E^r_{p',q'} \to E^r_{p+p',q+q'}$ such that:
1) d^r is a derivation in the graded sense:
$$d^r(x\cdot y) = d^r(x)\cdot y + (-1)^{|x|} x \cdot d^r(y),$$
where $|\cdot|$ stands for the total degree $|x| = p+q$ when $x \in E^r_{p,q}$;
2) the product on $E^{r+1}_{*,*} \cong H_*(E^r_{*,*}, d^r)$ is induced by that of the ideal $\ker d^r$.

[2]) The authors use $\mathbf{Z}_2$-coefficients when M is non-orientable.

Following Chataur-Le Borgne in [5], we set

$$\mathbf{E}^1_{p,q} = E^1_{p,q+n} = H_{p+q+n}\big(\Lambda_p, \Lambda_{p-1}; \operatorname{ev}_0^*(\mathbf{Z}_{or})\big)\,.$$

The differential d^1 is the connecting homomorphism in the long exact sequence of the triple $(\Lambda_p, \Lambda_{p-1}, \Lambda_{p-2})$. More generally, $d^r\colon E^r_{*,*} \to E^r_{*,*}$ is defined by the usual algorithm associated to the filtration $\cdots \Lambda_{p-1} \subset \Lambda_p \subset \Lambda_{p+1} \subset \cdots \Lambda$ (see [10], Chap. 2). Notice that page 0 exists in this spectral sequence, but it has no multiplicative structure.

PROPOSITION 4.2. *The loop product endows $\{\mathbf{E}^r_{*,*}\}_{r\geq 1}$ with a multiplicative structure.*

Proof. We first look at the desired properties on page $r=1$. When ξ is a relative i-cycle of the pair $(\Lambda_p, \Lambda_{p-1})$, η is a relative j-cycle of the pair $(\Lambda_{p'}, \Lambda_{p'-1})$ and $\xi \times \eta$ is transverse, then the loop intersection product $\xi * \eta$ is a $(i+j-n)$-chain of composable loops. By performing the composition we get a chain c of loops of length $\leq p+p'$, since the length of the composed loop is just the sum of the lengths of the entries. The boundary of c is a cycle in $\Lambda_{p+p'-1}$. The class of $[\xi]\cdot[\eta]$ is well-defined in $H_{i+j-n}(\Lambda_{p+p'}, \Lambda_{p+p'-1})$. After regrading, this product behaves well with respect to the bi-degree. It remains to check that d^1 is a derivation.

We look first at a transverse bi-simplex $u \times v$ of bi-degree (i,j) in $\Lambda \times \Lambda$. Let $W \subset \Delta^i \times \Delta^j$ be the preimage of the diagonal Δ_M by $\operatorname{ev}_0(u) \times \operatorname{ev}_0(v)$. Its boundary consists of two parts:

- $\partial_1 W := W \cap (\partial\Delta^i \times \Delta^j)$,
- $\partial_2 W := W \cap (\Delta^i \times \partial\Delta^j)$.

Each part gives rise to an $(i+j+n-1)$-chain in Λ which is a part of $\partial(u \cdot v)$. As in the proof of Lemma 2.9, the second chain is endowed with the sign $(-1)^i$ according to the formula for the total boundary given after Definition 2.4. Taking the Dold sign into account yields:

$$\partial(u \cdot v) = (\partial u)\cdot v + (-1)^{n-i} u \cdot (\partial v)\,.$$

By summing such a formula over all bi-chains forming $\xi \times \eta$, we get:

$$\partial(\xi \cdot \eta) = (\partial \xi)\cdot \eta + (-1)^{n-i} \xi \cdot (\partial \eta)\,.$$

After regrading, it becomes a derivation formula. When ξ and η are relative cycles as above, $d^1([\xi]\cdot[\eta]) = (d^1[\xi])\cdot[\eta] + (-1)^{|\xi|}[\xi]\cdot d^1[\eta]$; hence, property 1 from Definition 4.1 holds for $r=1$.

The product on $\mathbf{E}^2_{*,*}$ is defined by taking ξ, a relative cycle of the pair $(\Lambda_p, \Lambda_{p-2})$, and η, a relative cycle in the pair $(\Lambda_{p'}, \Lambda_{p'-2})$, with $\xi \times \eta$ transverse. Thus the chain of composed loops is a relative cycle of the pair $(\Lambda_{p+p'}, \Lambda_{p+p'-2})$. It can be checked that elements in $\ker d^1$ are represesentable by such cycles. Hence property 2 from Definition 4.1 holds for $r = 1$.

The same argument applies for all $r \geq 1$ once one remembers the definition of $E^r_{p,*}$ associated to the filtration of Λ. The product on $\mathbf{E}^r_{*,*}$ is defined by taking ξ, a relative cycle of the pair $(\Lambda_p, \Lambda_{p-r})$, and η, a relative cycle of the pair $(\Lambda_{p'}, \Lambda_{p'-r})$, where $\xi \times \eta$ is transverse. The boundary operator d^r is induced by ∂ and is not affected by the value of r. After regrading, d^r, like d^1, becomes a derivation and the product which is induced on its homology is the one given on $E^{r+1}_{*,*}$.

4.3. The Thom isomorphism

There is a Morse-Bott version of the famous Morse lemma. It yields a “normal” form for a Morse-Bott function near a critical manifold. We are going to apply it to the function $\ell^2 \colon \Lambda \to \mathbf{R}$ near the critical manifold Σ_p whose index is α_p. If one feels uncomfortable about applying this lemma in infinite dimension, one can take a finite-dimensional approximation of Λ near Σ_p by considering the space of geodesics polygons (still parametrized proportionally to arc length) with Np edges of equal length, where $1/N$ is less than the injectivity radius. Let E^-_p be the vector bundle of rank α_p over Σ_p generated by the eigenvectors of the Hessian of ℓ^2 (with respect to some Riemannian metric on Λ) whose eigenvalues are negative; this is a sub-bundle of $T\Lambda|\Sigma_p$. With notation borrowed from [9], let $\Sigma^-_p = exp(E^-_p)$ denote the unstable manifold of Σ_p with respect to the gradient of ℓ^2; we are mainly interested in its germ along the critical manifold. Finally Λ^-_p denotes the open set of loops whose length is less than p. We have $\Sigma^-_p \smallsetminus \Sigma_p \subset \Lambda^-_p$ as a consequence of the Taylor expansion.

The *Morse-Bott lemma* states that:

1. Λ^-_p retracts by deformation to Λ_{p-1};
2. Λ_p retracts by deformation to $\Lambda_{p-1} \cup \Sigma^-_p$.

As a consequence, the inclusions of pairs induce the following isomorphisms:

$$H_*(\Lambda_p, \Lambda_{p-1}) \cong H_*(\Lambda_p, \Lambda^-_p) \cong H_*(\Sigma^-_p, \Sigma^-_p \smallsetminus \Sigma_p)\,.$$

Here the system of coefficients is $\mathrm{ev}_0^*(\mathbf{Z}_{or})$. Since the orientation of the fibre bundle E^-_p is twisted exactly as $\mathrm{ev}_0^*(\mathbf{Z})$ is (see [9], Prop. 12.2),we are ready to apply the Thom isomorphism:

$$h_p\colon H_*(\Sigma_p;\mathbf{Z}) \cong H_{*+\alpha_p}\big(\Sigma_p^-, \Sigma_p^- \smallsetminus \Sigma_p; \mathrm{ev}_0^*(\mathbf{Z}_{or})\big)\,.$$

For $p>0$, this may be written as

$$h_p\colon H_*(UM;\mathbf{Z}) \cong H_{*+\alpha_p}\big(\Lambda_p, \Lambda_{p-1}; \mathrm{ev}_0^*(\mathbf{Z}_{or})\big)\,.$$

Also note that the Gysin morphism

$$H_*(M;\mathbf{Z}_{or}) \to H_{*+n-1}(UM;\mathbf{Z})$$

makes $H_*(UM;\mathbf{Z})$ a $H_*(M;\mathbf{Z}_{or})$-module for the intersection product. In the next proposition the coefficients are omitted; they are meant as we just specified them.

PROPOSITION 4.4 (Goresky-Hingston [9], Theorem 12.5). *The Thom isomorphisms carry the intersection product of $H_*(UM)$ into the loop product of $\oplus_{p>0}H_{*+\alpha_p}(\Lambda_p,\Lambda_{p-1})$. Moreover, they carry the $H_*(M)$-module structure of $H_*(UM)$ to the $H_*(\Lambda_0)$-module structure of $H_*(\Lambda_p,\Lambda_{p-1})$.*

The proof below mainly follows the same line as [9]. But it is based on the notion of loop product that we introduced in the previous section.

Proof. There are several steps.

A) One can factorize the intersection product *int* of $H_*(UM)$ in the following way:

$$\begin{array}{ccc}
H_i(UM)\otimes H_j(UM) & \xrightarrow{int_0} & H_{i+j-n}(UM \underset{M}{\times} UM)\\
\Big\downarrow{\scriptstyle int} & & \Big\downarrow{\scriptstyle int_1}\\
H_{i+j-(2n-1)}(UM) & \xrightarrow{=} & H_{i+j-(2n-1)}(UM \underset{UM}{\times} UM)\,.
\end{array}$$

Indeed, starting with a transverse bi-cycle $\xi\times\eta$ we may first intersect it with the fibered product $UM \underset{M}{\times} UM \subset UM\times UM$ yielding a $(i+j-n)$-cycle ζ (this induces the morphism int_0), which, in turn, is transverse to the diagonal $\Delta_{UM} = UM \underset{UM}{\times} UM$ (this induces the morphism int_1).

B) Since $\mathrm{ev}_0|\Sigma_p$ is a smooth submersion, the fibered product $\Sigma_{p,p'} := \Sigma_p \underset{M}{\times} \Sigma_{p'}$ is a smooth manifold. A point of it is a pair of closed geodesics of respective lengths p and p' which are composable as loops. In general the composed loop is not a geodesic.

Similarly, since Σ_p^- is tangent to E_p^-, which is a fibre bundle over Σ_p, then $\mathrm{ev}_0|\Sigma_p^-$ is also a submersion onto M near Σ_p. Thus $\Sigma_{p,p'}^- := \Sigma_p^- \underset{M}{\times} \Sigma_{p'}^-$ is smooth near $\Sigma_{p,p'}$. The normal space to $\Sigma_{p,p'}$ in $\Sigma_{p,p'}^-$ is the restriction to $\Sigma_{p,p'}$ of $E_p^- \times E_{p'}^-$, which is a vector bundle of rank $\alpha_p + \alpha_{p'}$. So we have a Thom isomorphism:

$$h_{p,p'}\colon H_*(\Sigma_{p,p'}) \cong H_{*+\alpha_p+\alpha_{p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p,p'}) .$$

For small simplices, the Thom isomorphism at the chain level is generated by the following Thom "extension": take a small simplex in the base of a disk bundle (which hence is trivial over the considered small simplex) and take a direct product with the fiber. If $\xi \times \eta$ is a bi-cycle in $\Sigma_p \times \Sigma_{p'}$ transverse to $\Sigma_{p,p'}$, then the Thom extension $\tilde{\xi} \times \tilde{\eta}$ is transverse to $\Sigma_{p,p'}^-$ and its intersection with $\Sigma_{p,p'}^-$ is the Thom extension of the intersection cycle in the base $\Sigma_{p,p'}$. This proves that the Thom isomorphism carries the loop intersection product $H_i(\Sigma_p) \otimes H_j(\Sigma_{p'}) \to H_{i+j-n}(\Sigma_{p,p'})$ to the suitable relative version of the loop intersection product $H_{i+\alpha_p}(\Sigma_p^-) \otimes H_{j+\alpha_{p'}}(\Sigma_{p'}^-) \to H_{i+j+\alpha_p+\alpha_{p'}-n}(\Sigma_{p,p'}^-)$. Note that, after identification, the first morphism is nothing but int_0 from A).

C) We observe that $\Sigma_{p+p'}$ lifts (by a section of the composition map) as a submanifold of codimension $n-1$ in $\Sigma_{p,p'}$. Indeed, any smooth geodesic of length $p+p'$ splits uniquely into two geodesics of respective lengths p and p'. Conversely, $(\gamma, \gamma') \in \Sigma_{p,p'}$ belongs to this lifting of $\Sigma_{p+p'}$ if and only if the initial velocities $\dot\gamma(0)$ and $\dot\gamma'(0)$ are positively proportional, which is a condition of codimension $n-1$. Thus, the composition is a map of pairs:

$$comp\colon (\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p+p'}) \to (\Lambda_{p+p'}, \Lambda_{p+p'} \smallsetminus \Sigma_{p+p'}) .$$

The normal bundle to $\Sigma_{p+p'}$ in $\Sigma_{p,p'}^-$, denoted by ν, is the direct sum $E_p^- \oplus E_{p'}^- \ominus \nu_{p,p'}$, where $\nu_{p,p'}$ denotes the normal bundle to $\Sigma_{p+p'}$ in $\Sigma_{p,p'}$. The rank of ν is $\alpha_p + \alpha_{p'} + n - 1$. According to Bott's iteration formula, it equals $\alpha_{p+p'}$, which is the rank of $E_{p+p'}^-$.

If ξ is a relative cycle in $(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p,p'})$ transverse to $\Sigma_{p+p'}$, its trace in the pair $(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p+p'})$ is the Thom extension of its intersection with $\Sigma_{p+p'}$. In other words we have the following commutative diagram:

$$\begin{tikzcd}
H_k(\Sigma_{p,p'}) \arrow[r, "int_1"] \arrow[d] & H_{k-n+1}(\Sigma_{p+p'}) \arrow[d] \\
H_{k+\alpha_p+\alpha_{p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p,p'}) \arrow[r] & H_{k-n+1+\alpha_{p+p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \smallsetminus \Sigma_{p+p'}),
\end{tikzcd}$$

where the vertical arrows are the respective Thom isomorphisms. In order to finish the proof we need to apply $comp_*$ and see that it commutes with the Thom isomorphisms.

D) Some difficulty comes here from the fact that the composition map $comp\colon \Sigma^-_{p,p'} \to \Lambda_{p+p'}$ could be singular along $\Sigma_{p+p'}$ in the direction of $\nu_{p,p'}$ since $comp$ maps $\Sigma_{p,p'}$ into the critical level set of ℓ whose value is $p+p'$. We are going to construct the following two objects:

a) a linear embedding $\varphi\colon \nu \to T\Lambda|\Sigma_{p+p'}$, over the identity of $\Sigma_{p+p'}$, such that $Hess(\ell^2)\circ\varphi$ is negative definite;

b) a homotopy from $comp$ to $exp\circ\varphi\circ exp^{-1}$ among the maps of pairs

$$(\Sigma^-_{p,p'}, \Sigma^-_{p,p'} \smallsetminus \Sigma_{p+p'}) \to (\Lambda_{p+p'}, \Lambda_{p+p'} \smallsetminus \Sigma_{p+p'})\,.$$

If these two objects exist,

$$comp_*\colon H_*(\Sigma^-_{p,p'}, \Sigma^-_{p,p'} \smallsetminus \Sigma_{p+p'}) \to H_*(\Lambda_{p+p'}, \Lambda_{p+p'} \smallsetminus \Sigma_{p+p'})$$

commutes with the respective Thom isomorphisms, as it is true for φ_*. Indeed, by a), φ induces an isomorphism of fibre bundles $\nu \cong E^-_{p+p'}$ over $\Sigma_{p+p'}$. Hence, the proof of Proposition 4.4 is complete.

First, we choose $E^-_p \subset T\Lambda|\Sigma_p$ such that every vector V in $\big(E^-_p\big)_\gamma$ corresponds to a vector field along γ which vanishes at $\gamma(0)$. This is easy as the Jacobi fields in $T_\gamma\Sigma_p$ generate $T_{\gamma(0)}M$. Thus, $\big(E^-_p\big)_\gamma$ embeds canonically into $\big(T\Lambda_{p+p'}\big)_{\gamma*\gamma'}$, just by extending V by 0 along γ'. And similarly for $E^-_{p'}$. This allows us to choose φ to be the identity of the factor $E^-_p \oplus E^-_{p'}$.

Let $(\gamma,\gamma') \in \Sigma_{p+p'} \subset \Sigma_{p,p'}$ and X be a vector in $\big(\nu_{p,p'}\big)_{(\gamma,\gamma')}$. This vector indicates an infinitesimal deformation of (γ,γ') among the pairs of composable closed smooth geodesics, a deformation which separates the directions of their initial velocities. More precisely, there is a one-parameter family (γ_u,γ'_u), $u \in [0,\varepsilon)$, of pairs of closed geodesics such that $\gamma_u(0) = \gamma(0) = \gamma'(0) = \gamma'_u(0)$ and $\frac{d}{du}\big(\dot\gamma_u(0) - \dot\gamma'_u(0)\big)_{|u=0} = X$ (up to a positive scalar); here we identify $\big(\nu_{p,p'}\big)_{(\gamma,\gamma')}$ with the fibre $UM_{\gamma(0)}$. Instead of taking this deformation which leaves the length unchanged, we consider the following shortening deformation made of geodesic triangles T_u: the first edge in T_u is $\gamma_u(t), t\in[0,1-\varepsilon]$, the second edge joins geodesically the point $\gamma_u(1-\varepsilon)$ to the nearby point $\gamma'_u(\varepsilon)$ and the third edge is $\gamma'_u(t), t\in[\varepsilon,1]$ (the triangle is parametrized proportionally to arc length). We define $\varphi(X)$ to be the infinitesimal generator of this family. By estimating $\ell(T_u)$ it is easily seen that $Hess(\ell^2)\big(\varphi(X)\big) < 0$. Moreover $\varphi(X)$ is independent of $E^-_p \oplus E^-_{p'}$ since, as a tangent field to M along the composed loop $\gamma * \gamma'$, it does not vanish at the junction point $\gamma(1) = \gamma'(0)$. This proves item a). In order to obtain item b) it is sufficient to make ε tend to 0.

When $p > 0$, taking the different regradings into account, the Thom isomorphism yields

$$\mathbf{E}^1_{p,q} \cong H_{p+q+n-\alpha_p}(\Sigma_p, \mathbf{Z}) \cong H_{p+q+n-\alpha_p}(UM, \mathbf{Z}) \cong \mathbf{H}_{p+q+n-\alpha_p-(2n-1)}(UM, \mathbf{Z}),$$

where $\mathbf{H}_*(UM; \mathbf{Z})$ denotes the regraded intersection ring of the $(2n-1)$-dimensional manifold UM. Similarly, after regrading the Gysin morphism

$$H_*(M; \mathbf{Z}_{or}) \to H_{*+n-1}(UM; \mathbf{Z})$$

becomes a morphism of degree 0, $\mathbf{H}_*(M; \mathbf{Z}_{or}) \to \mathbf{H}_*(UM; \mathbf{Z})$, giving $\mathbf{H}_*(UM; \mathbf{Z})$ the structure of an $\mathbf{H}_*(M; \mathbf{Z}_{or})$-module for the intersection product.

The multiplicative structure on $\mathbf{E}^1_{*,*}$ may be interpreted in terms of $\mathbf{H}_*(UM)$. The following statement is essentially due to Chataur-Le Borgne in [5] (up to orientability assumption).

PROPOSITION 4.5. 1) *There is an isomorphism of bigraded rings:*

$$\mathbf{E}^1_{*,*} \cong \mathbf{H}_*(M; \mathbf{Z}_{or}) \oplus \mathbf{H}_*(UM; \mathbf{Z})[T]_{\geq 1}.$$

Here $\mathbf{E}^1_{*,*}$ is endowed with the bigraded ring structure yielded by Proposition 4.2. The intersection rings $\mathbf{H}_*(M)$ and $\mathbf{H}_*(UM)$ have bi-degree $(0, *)$. The new variable T has bi-degree $(1, \alpha_1 + n - 2)$ and appears with positive powers. Regarding $\mathbf{H}_*(UM)$ as an $\mathbf{H}_*(M)$-module, the right hand side has a well-defined ring structure.

2) *The differential d^1 on page 1 vanishes at every place.*

3) *The page ∞ inherits the same isomorphism of bigraded rings as page 1:*

$$\mathbf{E}^\infty_{*,*} \cong \mathbf{H}_*(M; \mathbf{Z}_{or}) \oplus \mathbf{H}_*(UM; \mathbf{Z})[T]_{\geq 1}.$$

Proof. 1) We read the first page of the spectral sequence, $\mathbf{E}^1_{*,*}$, *via* the Thom isomorphism, taking Proposition 4.4 into account. For instance, we have (without writing the coefficients):

$$\mathbf{H}_0(UM) = H_{2n-1}(UM) = H_{2n-1+\alpha_1}(UM) \cong E^1_{1,2n-2+\alpha_1} = \mathbf{E}^1_{1,n-2+\alpha_1}$$

and T is simply the image of the unit $\mu \in \mathbf{H}_0(UM)$ (that is, the fundamental class of UM) through the Thom isomorphism. Once the desired ring isomorphism is specified on T it extends globally using the multiplicative property of the Thom isomorphism.

2) As $\mathbf{H}_*(\Lambda_0)$ is a direct factor in $\mathbf{H}_*(\Lambda)$, $d^1\colon \mathbf{E}^1_{1,*} \to \mathbf{E}^1_{0,*}$ has to vanish. In particular, $d^1(T) = 0$. Since d^1 is a derivation (Prop. 4.2), it vanishes everywhere.

3) As a consequence, page 2 of the spectral sequence is isomorphic to page 1 as a bi-graded ring. Therefore the differential d^2 vanishes for the same reason as d^1. Proceeding recursively through the successive pages yields the conclusion.

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