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# DIRICHLET'S CALCULATION OF GAUSS SUMS

by Bill CASSELMAN *)

## INTRODUCTION

If we make a list of small odd primes $p$ for which $5$ is a square modulo $p$ we get

$$11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 131, 139, 149, 151, 179, 181, \ldots$$

All the last digits in this list are either $1$ or $9$. So the pattern should be pretty clear: we guess that $5$ is a square modulo $p$ precisely when $p$ is either $1$ or $9$ modulo $10$. No prime is congruent to $4$ or $6$ modulo $10$, so we can reformulate this to say that $5$ is a square modulo $p$ whenever $p$ is congruent to $1$ or $4$ modulo $5$, which is to say a square modulo $5$.

If we make a list of odd primes $p$ for which $3$ is a square modulo $p$, we get

$$11, 13, 23, 37, 47, 59, 61, 71, 73, 83, 97, 107, 109, 131, 157, 167, 179, 181, \ldots$$

but now the pattern is not so clear. It becomes clearer if we enhance this list:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$: | 11 | 13 | 23 | 37 | 47 | 59 | 61 | 71 | 73 | 83 | 97 | 107 | 109 | 131 | 157 | ... |
| $p \bmod 3$: | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | ... |
| $p \bmod 4$: | 3 | 1 | 3 | 1 | 3 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | ... |

Now we can guess that $3$ will be a square modulo $p$ if and only if either (a) $p$ modulo $4$ is $1$ and $p$ modulo $3$ is a square; or (b) neither $p$ modulo $4$ nor $p$ modulo $3$ is a square. We can confirm these guesses by looking at primes other than $3$ and $5$. Thus are we led to the *law of quadratic reciprocity*.

*) I wish to thank students attending a number theory seminar at Humboldt University (Berlin) for listening to a first draft of this, and Henri Darmon for making it possible for me to find time to write a nearly final draft during a visit to McGill (Montreal).

For $p$ an odd prime, $a$ relatively prime to $p$, let

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square modulo } p, \\ -1 & \text{otherwise.} \end{cases}$$

Since for odd $p$ the map $x \mapsto x^2$ is a multiplicative homomorphism with kernel $\{\pm 1\}$, this defines a multiplicative character. The following rule accounts for everything we have guessed at:

THEOREM (Quadratic reciprocity). *If $p$, $q$ are odd primes then*

$$\left(\frac{p}{q}\right) = \varepsilon \left(\frac{q}{p}\right),$$

*where*

$$\varepsilon = \begin{cases} 1 & \textit{if either } p \textit{ or } q \textit{ is } 1 \textit{ modulo } 4, \\ -1 & \textit{if both } p \textit{ and } q \textit{ are } 3 \textit{ modulo } 4. \end{cases}$$

There are many, many proofs of quadratic reciprocity. Some of the most intriguing among them reduce the theorem to a question of the sign of *Gauss sums*. Gauss (in [Gau1], then [Gau2]) was the first to make the connection, and Dirichlet followed him several years later ([Dir2]). In this note I will explain — from a modern perspective — Dirichlet's proof, which was a real *tour de force* in analytic number theory. It is commonly asserted that Dirichlet's proof is just a form of the *Poisson summation formula*. There is some truth to that, but it does not tell the whole story, and more particularly it does not even tell it the way Dirichlet likely saw it. There are a number of modern tools one can use to understand what Dirichlet did, a true *embarras de richesses*. What is to follow will offer an apparently new way to read Dirichlet's paper. But then the real moral, I suppose, is that it is impossible to put ourselves inside Dirichlet's mind and times.

I am going to try to follow Dirichlet very closely, but there will be two significant departures. The first is very simple — Dirichlet did not use complex numbers, but worked rather with real and imaginary parts separately — for example, in the Fourier transforms. I will not follow him in this. The second difference is probably more controversial, and is how my exposition differs from other expositions of Dirichlet's calculation of Gauss sums. Dirichlet works throughout his paper with equations of integrals such as the one

$$\lim_{N\to\infty} \int f(x) \Big( \sum_{-N}^{N} e^{2\pi i n x} \Big)\, dx = f(0)$$

for $f$ a suitably smooth function with period 1. I will replace this with the equation of distributions

$$\sum e^{2\pi inx} = \delta_0 .$$

This is not only convenient, but I think that it gives some insight into the way Dirichlet actually thought. (For a friendly introduction to the modern theory of distributions, I recommend [Schw].)

To give some first-hand flavour of Dirichlet's writing, here is his main result, in a translation of his own words:

*The sum of the finite or infinite series*

$$F(\alpha) = c_0 + c_1 \cos\alpha + c_2 \cos 2\alpha + \cdots$$

*being known, one can always express in terms of the function* $F(\alpha)$ *the new series*

$$c_0 + c_1 \cos 1^2 \frac{2\pi}{n} + c_2 \cos 2^2 \frac{2\pi}{n} + \cdots$$

*and*

$$c_1 \sin 1^2 \frac{2\pi}{n} + c_2 \sin 2^2 \frac{2\pi}{n} + \cdots$$

In addition, he finds an explicit evaluation of these series in terms of some values of $F(\alpha)$.

## 1. Gauss sums

For $n > 1$ and $a$ an integer modulo $n$, define the *normalized Gauss sum* by the formula

$$\gamma_n(a) = \frac{1}{\sqrt{n}} \sum_{x \bmod n} e^{2\pi iax^2/n} = \frac{1}{\sqrt{n}} \sum_{y \bmod n} e^{2\pi iay/n}\, \nu_n(y) ,$$

where $\nu_n(y)$ is the number of $x$ in $\mathbf{Z}/n$ with $x^2 = y$. It is a kind of Fourier transform on $\mathbf{Z}/n$.

It is relatively easy to show (and I shall say more about this later on) that for $p$ an odd prime, $a$ prime to $p$

$$\gamma_p^2(a) = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4 \\ -1 & \text{if } p \equiv 3 \mod 4 , \end{cases}$$

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so that $\gamma_p(a)$ is determined up to sign:

$$\gamma_p(a) = \begin{cases} \pm 1 & \text{if } p \equiv 1 \mod 4 \\ \pm i & \text{if } p \equiv 3 \mod 4 \,. \end{cases}$$

*Can we determine the sign? What does this have to do with quadratic reciprocity?*

As for the first question, the following is one of the most celebrated results of Gauss:

THEOREM (Gauss, 1805). *For* $n > 1$,

$$\gamma_n(1) = \frac{1 + i^{-n}}{1 + i^{-1}} = \begin{cases} 1 + i & \textit{if } n \equiv 0 \mod 4 \\ 1 & \textit{if } n \equiv 1 \mod 4 \\ 0 & \textit{if } n \equiv 2 \mod 4 \\ i & \textit{if } n \equiv 3 \mod 4 \,. \end{cases}$$

I will first say something about the elementary mathematical background to this result, explain why it is interesting, say something about its history, and finally explain Dirichlet's approach to it.

Gauss sums

$$G(\chi) = \sum_{x \bmod n} \chi(x)\, e^{2\pi i x/n}$$

can be attached to any primitive character of $(\mathbf{Z}/n)^\times$. An argument about Fourier transforms on $\mathbf{Z}/n$ tells us that its absolute value is $\sqrt{n}$. But there is no general result about these higher-order sums analogous to Gauss' determination of the sign of quadratic sums (see [B-E]).

## 2. FINITE FOURIER TRANSFORMS

Assign $\mathbf{Z}/n$ the measure according to which every point has measure $1/\sqrt{n}$. Thus the 'integral' of $f(x)$ over $\mathbf{Z}/n$ becomes the sum

$$\frac{1}{\sqrt{n}} \sum_{x \bmod n} f(x) \,.$$

Let $C(\mathbf{Z}/n)$ be the vector space of complex-valued functions on $\mathbf{Z}/n$. For $f$ in $C(\mathbf{Z}/n)$, its *Fourier transform* is the function

$$\widehat{f}(y) = \frac{1}{\sqrt{n}} \sum_{x \bmod n} f(x)\, e^{-2\pi ixy/n} .$$

Since

$$\sum_{x=0}^{n-1} e^{2\pi ixy/n} = \begin{cases} 0 & \text{if } y \not\equiv 0 \mod n \\ n & \text{if } y \equiv 0 \mod n\,, \end{cases}$$

it is easy to see here that, conversely,

$$f(x) = \frac{1}{\sqrt{n}} \sum_{y \bmod n} \widehat{f}(y)\, e^{2\pi ixy/n} ,$$

thereby defining the inverse Fourier transform, and that

$$\langle \widehat{F}, f \rangle = \frac{1}{\sqrt{n}} \sum \widehat{F}(x) f(x) = \langle F, \widehat{f} \rangle .$$

This implies in turn that the $L^2$-norms of $f$ and $\widehat{f}$ are the same (*Plancherel formula*).

As we have seen, there is one very useful connection between the Fourier transform and Gauss sums:

$$\gamma_n(a) = \frac{1}{\sqrt{n}} \sum_{y \bmod n} e^{2\pi iay/n}\, \nu_n(y) ,$$

where *$\nu_n(y)$ is the number of $x$ in $\mathbf{Z}/n$ with $x^2 = y$*. In other words, $\gamma_n$ is the inverse Fourier transform of $\nu_n$. But there is another connection between Gauss sums and the Fourier transform, one that will turn out to be particularly relevant to Dirichlet's paper. If $f(x) = e^{2\pi ix^2/n}$, then its Fourier transform is a function I define to be

$$\begin{aligned} \eta_n(y) &= \widehat{f}(y) \\ &= \frac{1}{\sqrt{n}} \sum_{x \bmod n} e^{2\pi ix^2/n}\, e^{-2\pi ixy/n} \\ &= \frac{1}{\sqrt{n}} \sum_{x \bmod n} e^{2\pi i(x^2-xy)/n} , \end{aligned}$$

which for $y = 0$ gives

$$\gamma_n(1) = \eta_n(0) = \widehat{f}(0) .$$

Some other of these Fourier coefficients are variants of the original Gauss sums. I recall that, if $p$ is an odd prime, the *Legendre symbol* is the function on $\mathbf{Z}/p$ defined by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if } x \text{ is a square unit modulo } p; \\ -1 & \text{if } x \text{ is a unit but not a square;} \\ 0 & \text{if } x \text{ is not a unit.} \end{cases}$$

Then

$$\nu_p(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \text{ is not a square} \\ 2 & \text{if } x \text{ is a unit that is a square,} \end{cases}$$

so $\nu_p(x) = 1 + \left(\frac{x}{p}\right)$.

Thus, for $a \not\equiv 0 \mod p$,

$$\begin{aligned} \gamma_p(a) &= \frac{1}{\sqrt{p}} \sum_{x \bmod p} e^{2\pi i a x/p} + \frac{1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{x}{p}\right) e^{2\pi i a x/p} \\ &= \frac{1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{x}{p}\right) e^{2\pi i a x/p} = \left(\frac{a}{p}\right) \frac{1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{ax}{p}\right) e^{2\pi i a x/p}, \end{aligned}$$

from which it follows that

$$\gamma_p(a) = \left(\frac{a}{p}\right) \gamma_p(1), \tag{1}$$

since $a$ is a unit modulo $p$.

An easy calculation then shows, because the Fourier transform preserves $L^2$-norms, that $|\gamma_p(a)|^2 = \gamma_p(a)\,\gamma_p(-a) = 1$ and hence $\gamma_p(1)^2 = \left(\frac{-1}{p}\right)$. More explicitly,

$$\gamma_p(1)^2 = \begin{cases} 1 & \text{if } p \text{ modulo 4 is 1}, \\ -1 & \text{if } p \text{ modulo 4 is 3}. \end{cases}$$

The problem raised by Gauss is, *which square root?* The surprising answer is, *the simplest guess*.

## 3. Gauss sums and quadratic reciprocity

This matter of signs is related to quadratic reciprocity.

LEMMA 3.1. *If $p \neq q$ are two odd primes then*

$$\gamma_p(q)\,\gamma_q(p) = \gamma_{pq}(1)\,.$$

*Proof.* By the Chinese Remainder Theorem we have

$$\begin{aligned}\gamma_{pq}(1) &= \sum_{x \bmod pq} e^{2\pi i x^2/pq} = \sum_{\substack{a \bmod q\\ b \bmod p}} e^{2\pi i (pa+qb)^2/pq} \\ &= \sum_{a,\,b} e^{2\pi i p a^2/q} e^{2\pi i q b^2/p} = \gamma_q(p)\,\gamma_p(q)\,. \qquad \square\end{aligned}$$

From this lemma we can continue:

$$\begin{aligned}\gamma_{pq}(1) &= \left(\frac{p}{q}\right)\left(\frac{q}{p}\right)\gamma_p(1)\,\gamma_q(1) \\ \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) &= \frac{\gamma_{pq}(1)}{\gamma_p(1)\gamma_q(1)}\,.\end{aligned}$$

Finally, from equation (1) and from Gauss' theorem:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{cases} \dfrac{1}{1\cdot 1} = 1 & \text{if } p \equiv 1,\ q \equiv 1 \mod 4\,, \\ \dfrac{i}{1\cdot i} = 1 & \text{if } p \equiv 1,\ q \equiv 3 \mod 4\,, \\ \dfrac{i}{i\cdot 1} = 1 & \text{if } p \equiv 3,\ q \equiv 1 \mod 4\,, \\ \dfrac{1}{i\cdot i} = -1 & \text{if } p \equiv 3,\ q \equiv 3 \mod 4\,. \end{cases}$$

## 4. Historical remarks

Gauss raised the question of the sign of these sums in *Disquisitiones Arithmeticae*, asserting without proof the correct determination — not making it quite clear, as Weil [We] mentions, whether he had proved this or was merely conjecturing it. In the English translation, from §356 of [Gau1]:

> *We observe that the upper signs always hold ... These theorems ... are on a higher level of investigation, and we will reserve their consideration for another occasion.*

It turns out that he did not have a proof at that time, but he did come up with one in 1805, mentioning it at the time in a letter to his friend Olbers, and published a complete account in 1811. His proof relied on some relatively simple algebraic formulas (in the domain of what is now often known as *q-calculus*), and used only algebra in their derivation, albeit algebra perhaps suggested by the *theory of $\vartheta$-functions*. For $n = p$ an odd prime, Gauss' principal identity is

$$1 + \omega + \omega^4 + \cdots + \omega^{(p-1)^2} = (\omega - \omega^{-1})(\omega^3 - \omega^{-3}) \ldots (\omega^{p-2} - \omega^{-(p-2)}),$$

where $\omega$ is any primitive $p$-th root of $1$. This is essentially a product of sines, and it is easy to calculate its sign. Gauss' proof must have seemed rather odd to his early readers, but later on one could see that it was in some sense quite natural. The expressions involved suggest theta functions, which arise in the discussion of functional equations of $\zeta$- and $L$-functions, as do Gauss sums. The book [Dav] is a great reference for this subject.

A very readable account of this history is contained in [Pat]. Gauss' formula was reproved many times in the nineteenth century by mathematicians applying an astonishing range of techniques. Perhaps the most elegant account is that of Schur [Schur], who starts with the observation that the Gauss sum is essentially the trace of the Fourier transform.

Patterson observed that Gauss' formula can be condensed to

$$\gamma_n(1) = (1 + i^{-n}) \int_{-\infty}^{\infty} e^{2\pi i x^2} \, dx ,$$

and in a very short argument, following Dirichlet somewhat loosely in spirit, he proves this directly. But instead of following Patterson, I want to look more closely at what Dirichlet wrote.

In spite of much attention paid by others to Gauss' number-theoretical investigations, his calculation of the sign of $\gamma$ seems to have escaped much attention until 1835, when Dirichlet gave a new proof by very different methods. He says of Gauss' result, perhaps expressing some degree of ambivalence:

> *Parmi les conséquences nombreuses et inattendues que Mr. Gauss a tirées de sa belle théorie des équations binômes, il y en a une qui présente une singularité très remarquable.* [1])

[1]) Among the numerous and unexpected consequences that Mr. Gauss has drawn from his beautiful theory of binomial equations, there is one that presents a remarkable singular quality.

He also complains mildly about the lack of motivation in Gauss' later derivation. His own proof might equally well be said to present a remarkable singularity. It is a clever application of Fourier series, perhaps not then used in theoretical mathematics by anyone except him. It was Dirichlet, in fact, who made the theory of Fourier series a rigorous subject and made as well the first important applications to something outside mathematical physics.

Gauss' evaluation of his sums went case by case, depending on how $n$ factored. It has the virtue of being elementary, if somewhat mysterious. Like many of Gauss' proofs, it seems to have descended directly from Heaven, without having passed through the normal genesis. Dirichlet's proof, on the other hand, amounts to an explicit formula valid for all $n$ at once, but is not at all elementary. I cannot say without reservation that it is well motivated, but I can say that it offers intriguing connections between things not often seen, especially in the early nineteenth century, to be connected.

## 5. Fresnel integrals

In order to state Dirichlet's formula, I need to recall the *Fresnel integral*

$$I_c = \int_{-\infty}^{\infty} e^{i\pi c x^2}\, dx = 2\int_0^{\infty} e^{i\pi c x^2}\, dx$$

for $c \neq 0$ and real. It is best thought of as an analogue of the Gauss sum for $\mathbf{R}$.

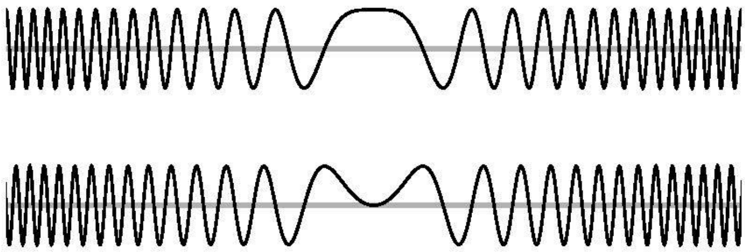

Figure 1
Graphs of the real and imaginary parts of $e^{ix^2}$

The integral is conditionally convergent, as can be seen by a change of variables $x^2 = y$. It then becomes

$$\int_0^\infty \frac{e^{i\pi cy}}{\sqrt{y}}\, dy \,,$$

which is an alternating sum of decreasing terms, as Figure 1 shows more directly.

The first reference to this integral (or rather to its real and imaginary parts separately) that I know of is in a paper of Cauchy ([Cau]), who calculates it along with several other definite integrals by a technique I must confess I do not follow. For him, this is just an exercise in calculus, without any apparent link to other problems. The integral next occurs in the paper [Fre] by the physicist Augustin Fresnel, who founded the *theory of diffraction* and for whom this integral is crucial.

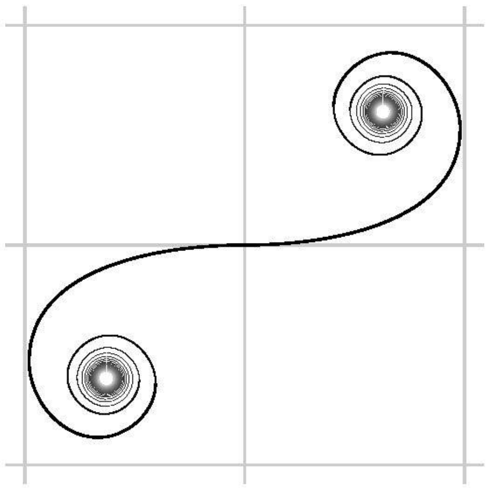

FIGURE 2

*Cornu's spiral*, deservedly famous, shows the track of the path $t \to \int_0^t e^{i\pi x^2}\, dx$, as $t \to \pm\infty$

The Fresnel integral can be explicitly evaluated by a contour integral, comparing it to the better known integral

$$\int_{-\infty}^\infty e^{-\pi x^2}\, dx = 1 \,.$$

We get

$$\int_{-\infty}^{\infty} e^{i\pi cx^2}\,dx = \frac{1}{\sqrt{|c|}}\,\frac{1 \pm i}{\sqrt{2}}$$

with $\pm$ chosen to be the sign of $c$. The only difficulty is to show that the integral over the arc from 0 to $\pi/4$ has limit 0 as $R \to \infty$.

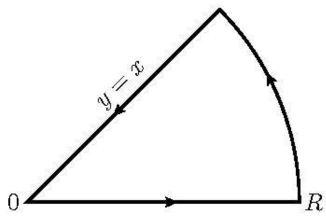


FIGURE 3
The contours for evaluating Fresnel's integral

Fresnel shows in his note of 1818 that he knew an exact formula for the integral now named after him, but he does not indicate how he knew it. He might have been aware of the slightly earlier work of Cauchy. One can also speculate that he simply adapted without rigour the formula

$$\int_{0}^{\infty} e^{\alpha x^2}\,dx = \sqrt{\frac{-\pi}{4\alpha}}\,,$$

well known at that time for $\alpha < 0$, to the case where $\alpha$ is imaginary. Fresnel's note also included a table of values of the integral from 0 to $x$. Later, Cauchy used Taylor series to reproduce Fresnel's table with slightly greater accuracy.

A consequence of this explicit formula is a closely related one, essentially for the Fourier transform of $\varphi_c(x) = e^{i\pi cx^2}$, considered as a *tempered distribution* on $\mathbf{R}$:

$$\begin{aligned}
\widehat{\varphi}_c(y) &= \int_{-\infty}^{\infty} e^{i\pi cx^2} e^{-2\pi ixy}\,dx = \int_{-\infty}^{\infty} e^{i\pi cx^2 - 2\pi ixy}\,dx \\
&= \int_{-\infty}^{\infty} e^{i\pi c(x^2 - 2xy/c)}\,dx = e^{-i\pi^2 y^2/c} \int_{-\infty}^{\infty} e^{i\pi c(x^2 - 2xy/c + y^2/c^2)}\,dx \\
&= e^{-i(\pi/c)y^2} \int_{-\infty}^{\infty} e^{i\pi cx^2}\,dx = I_c\, e^{-i(\pi/c)y^2}\,.
\end{aligned}$$

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This argument is still valid even if we do not know $I_c$ explicitly. This is worthwhile keeping in mind, because the argument of Dirichlet's that we are going to follow will evaluate Gauss sums and the Fresnel integral at the same time. So for now I just write:

$$\widehat{\varphi}_c(y) = \int_{-\infty}^{\infty} e^{i\pi cx^2} e^{-2\pi ixy}\, dx = I_c\, e^{-i(\pi/c)y^2}\,, \tag{2}$$

where

$$I_c = \int_{-\infty}^{\infty} e^{i\pi cx^2}\, dx\,,$$

without specifying $I_c$ explicitly.

The underlying idea of Dirichlet, roughly speaking, is to show that the calculation of Gauss sums and the calculation of the Fresnel integral are in fact intimately related — in fact, fall out almost simultaneously. This, certainly, is very satisfying.

## 6. ... AS PERIODIC DISTRIBUTIONS

The function $\varphi_c(x) = e^{i\pi cx^2}$ is bounded, hence determines a *tempered distribution* on $\mathbf{R}$, i.e.

$$\int_{-\infty}^{\infty} \varphi_c(x) f(x)\, dx$$

makes sense if $f$ decreases sufficiently rapidly at $\pm\infty$. But Dirichlet does something more interesting, and explains how to interpret this integral when $f$ is periodic. In modern terminology, he constructs from the function $\varphi_c(x)$ a *periodic distribution*. This is not quite a straightforward matter.

If $\varphi_c$ were itself of sufficiently rapid decay on $\mathbf{R}$, we would just calculate (with $f$ periodic):

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_c(x) f(x)\, dx &= \sum_n \int_n^{n+1} \varphi_c(x) f(x)\, dx \\ &= \int_0^1 \overline{\varphi}_c(x) f(x)\, dx\,,\end{aligned}$$

where

$$\overline{\varphi}_c(x) = \sum_{\mathbf{Z}} \varphi_c(x+n) = \sum_{\mathbf{Z}} e^{\pi ic(x+n)^2}\,.$$

But of course $e^{i\pi cx^2}$ is not at all rapidly decreasing, so this does not seem to work. What Dirichlet does comes in two stages: (1) he shows that if $\varphi_c = e^{i\pi cx^2}$ then (expressing things in modern terms) the sum

$$\overline{\varphi}_c(x) = \sum_{\mathbf{Z}} e^{\pi ic(x+n)^2}$$

defines a distribution on $\mathbf{R}/\mathbf{Z}$; (2) he shows how to evaluate it in elementary terms for certain values of $c$.

To get a feel for how this works, I will look next at an analogous but more familiar example.

## 7. Convergence of Fourier series

A few years before, Dirichlet had given the first rigorous proof of the fundamental result on Fourier series. The modern formulation is only slightly different from his. Suppose $f$ to be a smooth function on the closed interval $[0,1]$, i.e. one obtained by restriction to $[0,1]$ of a smooth function on some neighbourhood of $[0,1]$. Its Fourier series will converge to $f$ in the interior $(0,1)$, and at the end points it will converge to the average value at those end points, so that

$$\begin{aligned}\lim_{N\to\infty} \sum_{m=-N}^{N} f_m &= \lim \sum_m \int_0^1 f(x)\, e^{-2\pi imx}\, dx \\ &= \lim \int_0^1 f(x)\Big( \sum_{m=-N}^{N} e^{-2\pi imx}\Big)\, dx = \tfrac{1}{2}\big(f(0)+f(1)\big)\,.\end{aligned}$$

Here $f_m$ is the $m$-th Fourier coefficient of $f$. The important point is that the sum is over a set of consecutive integers that in the limit covers $\mathbf{Z}$.

Not too outrageously, one can formulate this nicely by writing (as I shall do frequently without hesitation)

$$\sum_m e^{2\pi imx} = \tfrac{1}{2}(\delta_0 + \delta_1)\,.$$

Rigorously, this is to be taken as an equation involving a limit of distributions, applied to smooth functions on the closed interval $[0,1]$.

I will not prove Dirichlet's result here, since it is (or used to be) standard fare in calculus courses, but I will do something the texts rarely do — I will try to make it plausible.

Let $q = e^{-2\pi i x}$. Then since

$$\sum_{-n}^{n} e^{-2\pi i m x} = \sum_{-n}^{n} q^m = \frac{q^{n+\frac{1}{2}} - q^{-(n+\frac{1}{2})}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sin(2n+1)\pi x}{\sin \pi x},$$

Dirichlet's claim means to us that the sequence of functions $\dfrac{\sin(2n+1)\pi x}{\sin \pi x}$ converges as a distribution to $\delta_0$. This is reasonable, since this function is rapidly oscillating in the interior of $[0,1]$ with spikes of height $(2n+1)$ at $0$ and $1$. Integrated against a smooth function, the rapidly oscillating part vanishes. As the spikes concentrate, they contribute the values of $f$ at $0$ and $1$.

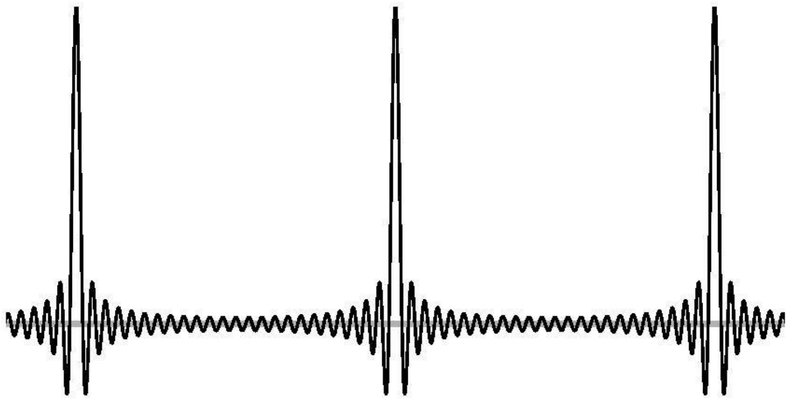

FIGURE 4

One of Dirichlet's approximations to the Dirac distribution ($n = 24$), scaled down to have maximum value 1 instead of $(2 \cdot 24 + 1)$

If we apply this to functions of period $T$, set $T = 1/n$, and sum, we get what might be called *Dirichlet's basic equation*:

$$\sum_m e^{2\pi i n m x} = \frac{1}{n}\left(\tfrac{1}{2}\delta_0 + \delta_{1/n} + \cdots + \delta_{(n-1)/n} + \tfrac{1}{2}\delta_1\right),$$

still as an equation of distributions on $C^\infty[0,1]$. Equivalently:

$$\lim_{N\to\infty} \int_0^1 f(x)\Big(\sum_{m=-N}^{N} e^{2\pi i n m x}\Big)\,dx = \frac{1}{n}\Big(\tfrac{1}{2}f(0) + f\big(\tfrac{1}{n}\big) + \cdots + f\big(\tfrac{n-1}{n}\big) + \tfrac{1}{2}f(1)\Big).$$

So now we know that

$$\sum_m e^{2\pi inmx} = \frac{1}{n}\left(\tfrac{1}{2}\delta_0 + \delta_{1/n} + \cdots + \delta_{(n-1)/n} + \tfrac{1}{2}\delta_1\right),$$

and we want to find a similar formula for

$$\sum_n e^{\pi i\alpha(x+n)^2}$$

as a distribution on $\mathbf{R}/\mathbf{Z}$.

## 8. Computing the Fresnel integral

In a second interlude before explaining Dirichlet's main theorem, we can see how he calculated the Fresnel integral:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\pi x^2}\,dx &= \lim_{N\to\infty}\int_{-N}^{N+1} e^{i\pi x^2}\,dx \\ &= \lim_{N\to\infty}\sum_{k=-N}^{N}\int_k^{k+1} e^{i\pi x^2}\,dx \\ &= \lim_{N\to\infty}\sum_{k=-N}^{N}\int_0^1 e^{i\pi(x+k)^2}\,dx \\ &= \lim_{N\to\infty}\int_0^1 e^{i\pi x^2}\left(\sum_{k=-N}^{N} e^{i\pi k^2}e^{2\pi ixk}\right)dx\,.\end{aligned}$$

In less formal terms:

$$\int_{-\infty}^{\infty} e^{i\pi x^2}\,dx = \int_0^1 e^{i\pi x^2}\left(\sum_k e^{i\pi k^2}e^{2\pi ixk}\right)dx\,.$$

The factor $e^{i\pi k^2}$ is 1 if $k$ is even, and $-1$ if it is odd, so we separate the sum into two parts, setting $k = 2\ell$ or $k = 2\ell + 1$. We now get

$$\int_{-\infty}^{\infty} e^{i\pi x^2}\,dx = \int_0^1 e^{i\pi x^2}\sum_\ell e^{4\pi i\ell x}\,dx + \int_0^1 e^{i\pi(x^2-2x)}\sum_\ell e^{4\pi i\ell x}\,dx\,.$$

From the basic formula of Dirichlet with $n = 2$, letting $f(x)$ be first $e^{i\pi x^2}$ and then $e^{i\pi(x^2-2x)}$ we see that this is half of

$$\tfrac{1}{2}e^{i\pi 0^2} + e^{i\pi(\frac{1}{2})^2} + \tfrac{1}{2}e^{i\pi 1^2} - \tfrac{1}{2}e^{i\pi(0^2-2\cdot 0)} - e^{i\pi((\frac{1}{2})^2-2\cdot\frac{1}{2})} - \tfrac{1}{2}e^{i\pi(1^2-2\cdot 1)}$$
$$= \tfrac{1}{2}(e^0 - e^0) + (e^{\pi i/4} - e^{3\pi i/4}) + \tfrac{1}{2}(e^{\pi i} - e^{-\pi i}) = 2e^{i\pi/4}\,,$$

giving $e^{i\pi/4}$ for the integral. It seems just short of miraculous.

It would have been slightly simpler to have followed Patterson and calculated

$$\int_{-\infty}^{\infty} e^{2\pi i x^2}\, dx\,,$$

which would not have required a separation into even and odd, but I wanted to make this calculation look like later ones.

## 9. Dirichlet's formula

We want to define $e^{i\pi c x^2}$ as a periodic distribution, or in other words evaluate

$$\int_{-\infty}^{\infty} e^{i\pi c x^2} f(x)\, dx\,,$$

where $f(x)$ is a smooth periodic function. We shall do this in two different ways for certain values of $c$.

I begin with the first step in Dirichlet's argument, by far the easier half. Suppose $F(x)$ to be any smooth function on $\mathbf{R}$. Formally, by expressing $f$ in terms of its Fourier series we get

$$\int_{-\infty}^{\infty} F(x) f(x)\, dx = \int_{-\infty}^{\infty} F(x)\Big(\sum_m f_m\, e^{2\pi i m x}\Big)\, dx$$
$$= \sum f_m \Big(\int_{-\infty}^{\infty} F(x)\, e^{2\pi i m x}\, dx\Big) = \sum_m f_m\, \widehat{F}(-m)\,.$$

This is not difficult to justify in our case, and leads to the formula

$$\int_{-\infty}^{\infty} e^{\pi i c x^2} f(x)\, dx = I_c \cdot \Big(\sum_m f_m e^{-i(\pi/c)m^2}\Big)\,.$$

Thus integration against $e^{i\pi c x^2}$ can always be defined as a distribution on $\mathbf{R}/\mathbf{Z}$, no matter what real value $c$ takes, in terms of spectral data. In doing this, Dirichlet became the first of many to use a trick that is still extraordinarily fruitful.

Now for Dirichlet's second step, which is by far the more intricate. The point now is to come up with a second way to evaluate the integral in the last formula. I do not know if there are explicit formulas for arbitrary values of $c$. I doubt it — these distributions smell chaotic to me if $c$ is, say, highly irrational. But if we set $c = -n/2$ we get, referring to equation (2),

$$\int_{-\infty}^{\infty} e^{-i\pi(n/2)x^2} f(x)\,dx = I_c\Big(\sum_m f_m e^{2\pi i m^2/n}\Big), \qquad \text{with } I_c = \sqrt{\frac{2}{n}}\,\frac{1-i}{\sqrt{2}}\,,$$

which is a kind of infinite Gauss sum. That's promising. It is especially promising because the terms are periodic in $m$ modulo $n$:

$$e^{2\pi i(m+kn)^2/n} = e^{2\pi i(m^2+2kmn+k^2n^2)/n} = e^{2\pi i m^2/n}.$$

What Dirichlet in effect proves, although of course stated very differently, is this very general result:

THEOREM. *If $F_{m+kn} = F_m$ for all $k$, the distribution*

$$f \longmapsto \sum_m f_m F_m$$

*on $C_c^\infty(\mathbf{R}/\mathbf{Z})$ is equal to*

$$\sum_0^{n-1} a_k\,\delta_{k/n} \qquad \textit{with } a_k = \sum_{m=0}^{n-1} F_m e^{2\pi i k m/n}\,.$$

In particular

$$a_0 = \frac{1}{n}\sum_{m=0}^{n-1} F_m\,.$$

In our case, we have $F_m = I_{-n/2}e^{2\pi i m^2/n}$, and $a_0$ will be, up to a simple factor, the Gauss sum.

*Proof.* Dirichlet works pretty hard to get this result, but modern terminology makes it easier. The Fourier transform is an isomorphism of $C^\infty(\mathbf{R}/\mathbf{Z})$ with the space $\mathcal{S}(\mathbf{Z})$ of rapidly decreasing functions on $\mathbf{Z}$. A *distribution* on $\mathbf{R}/\mathbf{Z}$ is a linear map from the space $C^\infty(\mathbf{R}/\mathbf{Z})$ to $\mathbf{C}$ satisfying some mild continuity condition. The *Fourier transform* is defined on distributions by the equation

$$\langle \widehat{\Phi}, f\rangle = \langle \Phi, \widehat{f}\rangle$$

for $f$ in $\mathcal{S}(\mathbf{Z})$. The space of distributions is the dual of $C^\infty(\mathbf{R}/\mathbf{Z})$, hence isomorphic to the space of functions of moderate growth on $\mathbf{Z}$.

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The *Dirac distribution* $\delta_x$ takes $f$ to $f(x)$. Its Fourier transform is the sequence $(e^{-2\pi imx})$. If $x = k/n$ then this is the sequence $(e^{-2\pi ikm/n})$, which is periodic on $\mathbf{Z}$ with period $n$. Dimensions match. So those distributions whose Fourier coefficients are of period $n$ are those of support on $\frac{1}{n}\mathbf{Z}$.

The $m$-th Fourier coefficient of $\sum_{k=0}^{n-1} a_k \delta_{k/n}$ is

$$F_m = \sum_k a_k e^{-2\pi ikm} .$$

In other words, the Fourier transform from the space spanned by the $\delta_{k/n}$ to that spanned by the functions on $\mathbf{Z}$ of period $n$ may be identified with that on $\mathbf{Z}/n$, except for the factor $1/\sqrt{n}$.

The inverse map takes the periodic function $(F_m)$ to

$$\sum_{k=0}^{n-1} a_k \delta_{k/n} , \qquad \text{where } a_k = \frac{1}{n} \sum_{m=0}^{n-1} F_m e^{2\pi ikm/n} .$$

This concludes the proof of the theorem. □

So the effect of integration against $e^{i\pi(n/2)x^2}$ is to take $f$ to some sum

$$\sum_{k \bmod n} a_k f\left(\frac{k}{n}\right).$$

The coefficient $a_0$ will be up to some simple factor the Gauss sum. This observation gives us, unfortunately, no practical idea of how to find the coefficients $a_k$ explicitly. Dirichlet did find explicit expressions, and we shall follow him in a moment.

To me the really remarkable thing about Dirichlet's paper is that he saw no difficulty in integrating one smooth function against another and getting a finite sum like this.

Here is Dirichlet's main result:

THEOREM. *For $n$ in $\mathbf{N}$, $x$ in $\mathbf{R}$ we have*

$$\sum_{k\in\mathbf{Z}} e^{-i\pi(n/2)(x+k)^2} = \frac{1}{n}\left[(1+i^{-n})\delta_0 + \sum_{m=1}^{n-1}\big(F_0(m/n) + F_1(m/n)\big)\delta_{m/n}\right]$$

*with*

$$F_0(x) = e^{-i\pi(n/2)x^2} \quad \text{and} \quad F_1(x) = e^{-i\pi(n/2)(x+1)^2} .$$

I postpone the proof for a moment. If we combine this theorem with the previous result, looking only at the coefficient of $\delta_0$, we get

$$\frac{1}{n}(1+i^{-n}) = I_{-n/2}\sqrt{n}\,\gamma_n(1) = \frac{1}{n}\sqrt{\frac{2}{n}}\,\frac{1-i}{\sqrt{2}}\sqrt{n}\,\gamma_n(1),$$

from which we deduce at last Gauss' formula

$$\gamma_n(1) = \frac{1+i^{-n}}{1-i},$$

concluding the proof of quadratic reciprocity.

## 10. Dirichlet's proof

To prove the theorem, Dirichlet uses his 'basic tool' to calculate the integral by the same technique we have used to evaluate the Fresnel integral.

With $c = -n/2$ we get as the integral over $\mathbf{R}$ the limit of sums

$$\int_0^1 \sum_{k=-N}^{N} e^{-i\pi(n/2)(x+k)^2} f(x)\,dx$$

which we could write as a distribution pairing

$$\Big\langle \sum_k e^{-i\pi(n/2)(x+k)^2}, f \Big\rangle_{[0,1]}.$$

But now we can write (without shame)

$$\begin{aligned}\sum_k e^{-i\pi(n/2)(x+k)^2} &= \sum_k e^{-i\pi(n/2)(x^2+2xk+k^2)} \\ &= e^{-i\pi(n/2)x^2}\Big[\sum_k e^{-i\pi(n/2)(2kx+k^2)}\Big] \\ &= e^{-i\pi(n/2)x^2}\Big[\sum_k e^{-i\pi(n/2)k^2} e^{-\pi inkx}\Big].\end{aligned}$$

Cavalier but justifiable. We are looking at a limit sum of distributions on $[0,1]$ :

$$\sum_k e^{-i\pi(n/2)(x+k)^2} = e^{-i\pi(n/2)x^2}\Big[\sum_k e^{i\pi(n/2)k^2} e^{-i\pi nkx}\Big].$$

As before, the term $e^{-i\pi(n/2)k^2}$ takes on a small number of different values. If $k$ is odd then $k^2$ is 1 modulo 4 and this factor is $e^{-2\pi i(n/4)}$, while if $k$ is even then $k^2$ is 0 modulo 4 and the factor is $1$. Separate the two cases, $k = 2\ell$ and $k = 2\ell+1$.

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The first $(k = 2\ell)$ is

$$e^{-i\pi(n/2)x^2} \cdot 1 \cdot \Big[\sum_\ell e^{-i\pi 2n\ell x}\Big] = \frac{1}{n}\Big(\frac{1}{2}(1 + i^{-n})\,\delta_0 + \sum_{m=1}^{n-1} F_0(m/n)\,\delta_{m/n}\Big)$$

with $F_0(x) = e^{-i\pi(n/2)x^2}$. The coefficient of $\delta_0$ has been calculated as

$$\tfrac{1}{2}\big(e^{-i\pi(n/2)\cdot 0^2} + e^{-i\pi(n/2)\cdot 1^2}\big)\,.$$

The second $(k = 2\ell + 1)$ is

$$\begin{aligned} e^{-i\pi(n/2)x^2} \cdot i^{-n} \cdot \Big[\sum_\ell e^{-i\pi n(2\ell+1)x}\Big] &= e^{-i\pi(n/2)x^2 - i\pi nx} \cdot i^{-n} \cdot \Big[\sum_\ell e^{-i\pi 2n\ell x}\Big] \\ &= \frac{1}{n}\Big(\frac{1}{2}(i^{-n} + 1)\delta_0 + \sum_{m=1}^{n-1} F_1(m/n)\delta_{m/n}\Big) \end{aligned}$$

with

$$F_1(x) = e^{-i\pi(n/2)(x+1)^2}\,.$$

Summing the two contributions, even and odd, we get

$$\sum_k e^{-i\pi(n/2)(x+k)^2} = \frac{1}{n}\Big[(1 + i^{-n})\delta_0 + \sum_{m=1}^{n-1}\big(F_0(m/n) + F_1(m/n)\big)\delta_{m/n}\Big]$$

with

$$F_0(x) = e^{-i\pi(n/2)x^2} \quad\text{and}\quad F_1(x) = e^{-i\pi(n/2)(x+1)^2}\,. \qquad \square$$

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