How to turn a tetrahedron into a cube and similar transformations

L'Enseignement Mathématique (2) 59 (2013), 115–131

HOW TO TURN A TETRAHEDRON INTO A CUBE AND SIMILAR TRANSFORMATIONS

by G. C. SHEPHARD

ABSTRACT. Suppose that the surface of a polyhedron P_1 is cut in such a way that it can be opened out flat to form a connected region R in the plane and that R, by introducing suitable folds, can be made into the net of a polyhedron P_2. Then we write $P_1 \Rightarrow P_2$ and say that P_1 is *transformed* into P_2. In this paper we give many examples of the transformation of polyhedra and investigate the properties of the relation $\Rightarrow$.

1. INTRODUCTION

Consider the following example. In Figure 1(a) we show a strip of four congruent acute-angled triangles (heavy solid and dashed lines). This is a *net* of a tetrahedron T because, if the strip is cut out of paper or similar material, folded along the (heavy) dashed lines, and edges with the same labels (x, y or z) are joined together we obtain a model of (the surface of) the tetrahedron T. Now cut the model along the (lighter) solid lines indicated in Figure 1(a). It will be found that the surface can be opened out flat in the plane, yielding the shape shown in Figure 1(b). This can be made into a *net* of a cube C, because if it is folded along the dashed lines, and edges with the same labels $(a, b, c, \dots, g)$ are joined together, we obtain the surface of the cube C. Thus we have turned the tetrahedron T into the cube C !

We may say that T is *transformed* into the cube C and write $T \Rightarrow C$. In general, if the surface of a polyhedron P is cut in such a way that it can be opened out flat in the plane (without overlaps) to form a connected region R, then R is called a *development* of P. If this development only involves cuts along the edges of P then, following Akiyama [1], we refer to R as an *edge-development*, or *net* of P. With this terminology $P_1 \Rightarrow P_2$ means that

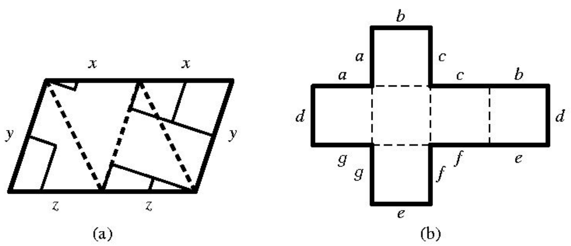


FIGURE 1

some development of P_1 is an edge-development of P_2. The relation $P_1 \Rightarrow P_2$ implies that P_1 and P_2 have the same surface area — it does not imply that they are isometric. Further, though trivially $P \Rightarrow P$, except in special cases, the relation $\Rightarrow$ is neither symmetric nor transitive (see Section 4).

The tetrahedron T in the above example will be called an *almost regular* tetrahedron since it has, as its faces, four congruent acute-angled triangles. (We note, in passing, that there exists no tetrahedron whose faces are four congruent obtuse-angled triangles or four congruent right-angled triangles.) A tetrahedron is *almost regular* if it has three pairs of opposite edges of equal length. If these lengths are a, b and c then we denote the tetrahedron by $T(a, b, c)$. Thus the tetrahedron in the above example is $T(2, \sqrt{5}/2, 3\sqrt{2}/2)$ where the cube C has edge-length 1.

It was shown in [8] that some nets may represent more than one polyhedron if the edges to be joined are not labelled (see Section 4). We have labelled the edges of the polygons in Figure 1 but, in general, we omit the labels unless there is the possibility of ambiguity.

2. REGULAR-FACED TESSELLATION POLYHEDRA

THEOREM 1. *If $T \Rightarrow P$ where T is an almost regular tetrahedron, then P must be a tessellation polyhedron, that is, a polyhedron of which some net R of P tiles the plane.*

The idea of a *tessellation polyhedron* was first introduced in [9]. Considering the net R of P as a closed set, "tiles the plane" means that the plane may be covered by copies of R without gaps or overlaps. That is, the union of copies of R is the whole plane, and the intersection of any two copies of R is either empty or of zero area. If such an intersection is non-empty it will be a union of edges, or parts of edges, of copies of R.

Proof. This result follows immediately from the following remarkable theorem (see [1]):

AKIYAMA'S THEOREM. *Every development of an almost regular tetrahedron T tiles the plane.*

In fact Akiyama stated and proved this theorem only for regular tetrahedra, but there is no difficulty in extending the result to almost regular tetrahedra.

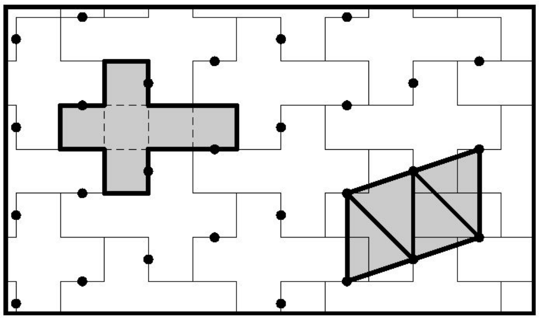


FIGURE 2

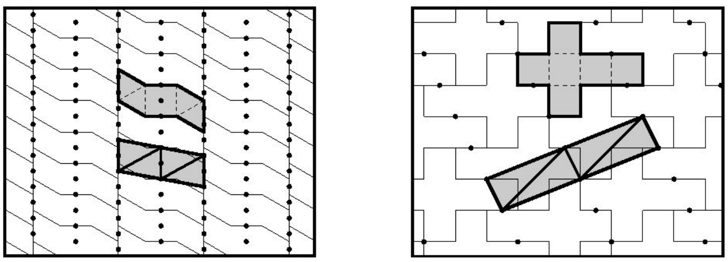
Theorem 1 greatly simplifies the search for polyhedra into which almost regular tetrahedra may be transformed. Initially we shall consider *regular-faced polyhedra*, that is, polyhedra all of whose 2-faces are regular polygons. A complete enumeration of these was given by Norman Johnson in 1966 [7]. They comprise the five regular (Platonic) and thirteen archimedean solids, the n-prisms, and n-antiprisms for ($n \geq 3$), and 92 others. We shall refer to the latter by the numbers J1–J92 assigned to them in Johnson's original paper [7]. This list can also be found in Wikipedia. That Johnson's list is complete was proved by Zalgaller [10]. Of the regular-faced polyhedra it is now known that twenty-two are tessellation polyhedra namely, eight polyhedra all of whose faces are regular (equilateral) triangles (the regular tetrahedron, octahedron, icosahedron, J12, J13, J17, J51 and J84) and twelve with both square and regular triangular faces (J1, J8, J10, J14, J15, J16, J49, J50, J86, J87, J88, J89 and J90). In addition there is the cube with square faces and the hexagonal antiprism with hexagonal and regular triangular faces. The completeness of this enumeration was established in [2].

In Figure 2 we show a tessellation $\mathfrak{T}$ for a net of a cube C and the net of an almost regular tetrahedron is superimposed on it. From this we can deduce the example above: the lines along which the surface of the tetrahedron must be cut to transform it into a cube are the edges of $\mathfrak{T}$ that lie within the net.

For each of these twenty-two tessellation polyhedra P_i there exists an almost regular tetrahedron T_i such that $T_i \Rightarrow P_i$. This statement is established by the diagrams in Figure 3. Each part of this diagram shows a net of P_i, the corresponding tessellation $\mathfrak{T}$ by this net, and, superimposed, a net of T_i. Of course, the tetrahedra T_i are different in each case; their edge-lengths can be read off from the diagrams. All the nets are shaded. Each of the tessellations has symmetry groups of type $p2$ (see [6, § 1.4]) in which the centres of 2-fold symmetry form a (non-rectangular) lattice. Each tetrahedron is a fundamental region (or union of fundamental regions) of the symmetry group of the tessellation $\mathfrak{T}$, all its vertices are points of 2-fold rotational symmetry, and edges of $\mathfrak{T}$ pass through all the vertices of the tetrahedron.

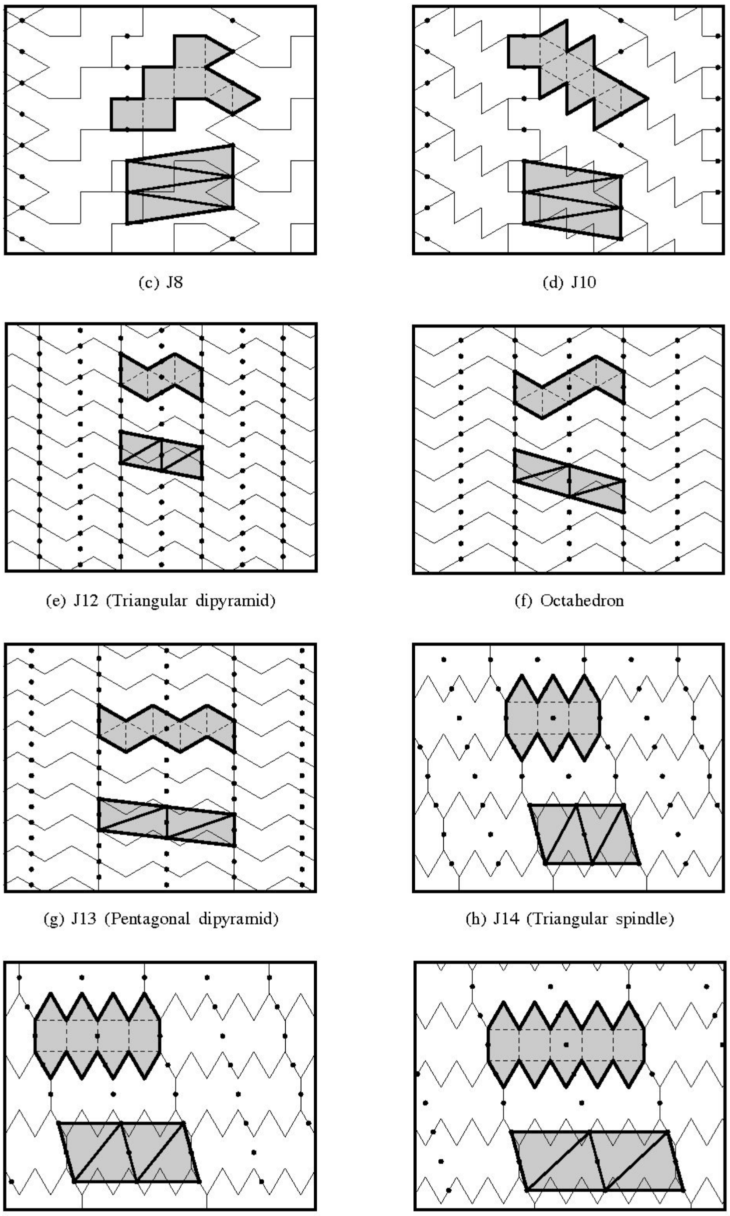
CONJECTURE. *If P is a convex tessellation polyhedron, then $T \Rightarrow P$ for some almost regular tetrahedron T.*

As we have just shown, the conjecture is true for regular-faced polyhedra. The question as to whether the conjecture is true in general is open. In the next section we give further examples of tessellation polyhedra into which a suitable almost regular tetrahedron can be developed, thus giving further evidence of the truth of the conjecture. No counter-examples are known.



(a) J1 (Square pyramid) (b) Cube

FIGURE 3



(c) J8

(d) J10

(e) J12 (Triangular dipyramid)

(f) Octahedron

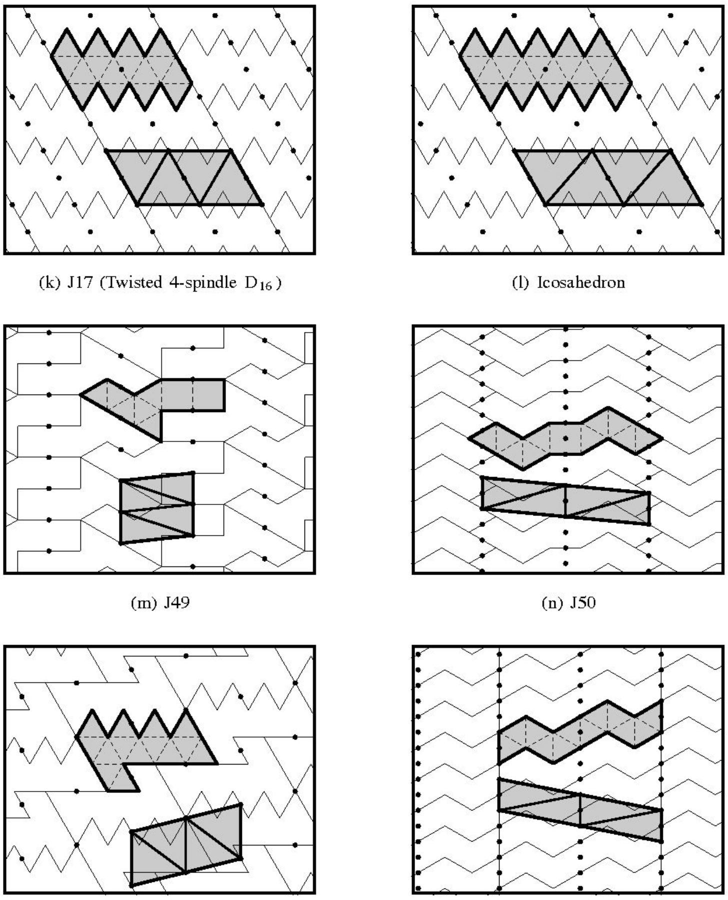
(g) J13 (Pentagonal dipyramid)

(h) J14 (Triangular spindle)

(i) J15 (4-spindle)

(j) J16 (5-spindle)

FIGURE 3



(k) J17 (Twisted 4-spindle D_{16})

(l) Icosahedron

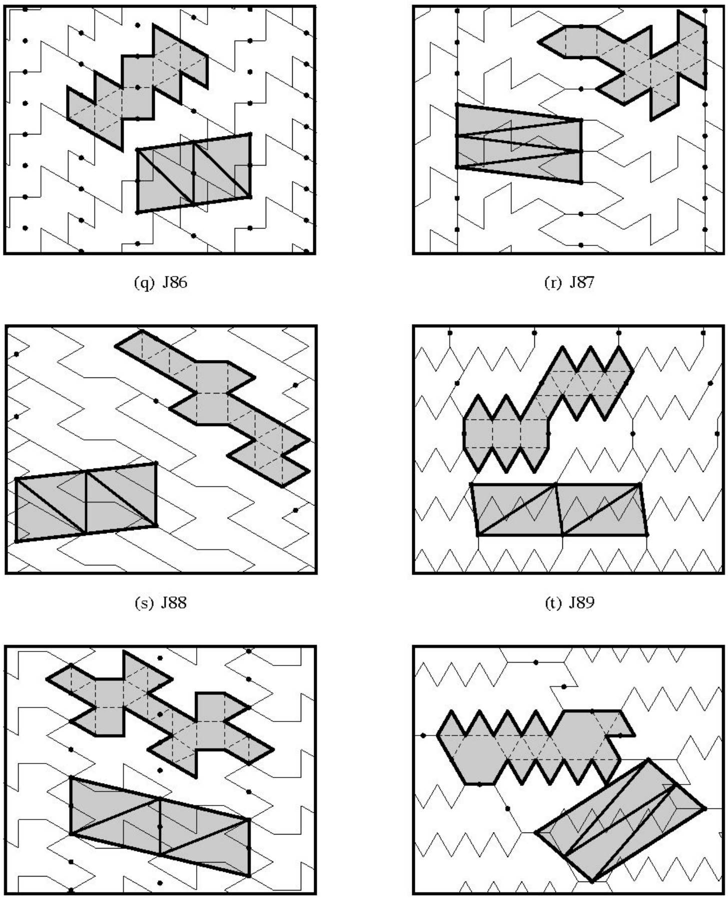
(m) J49

(n) J50

(o) J51 (Deltahedron D_{14})

(p) J84 (Deltahedron D_{12})

FIGURE 3



(q) J86 (r) J87

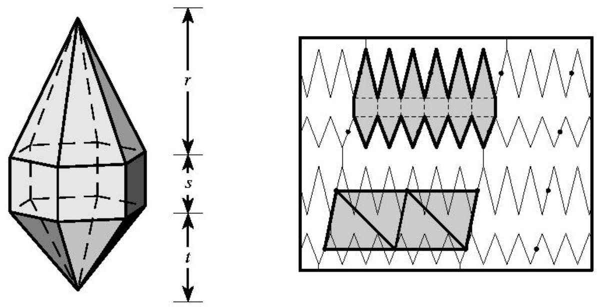
(s) J88 (t) J89

(u) J90 (v) Hexagonal antiprism

FIGURE 3

3. Other tessellation polyhedra

An *n-spindle* ($n \geq 3$) is a polyhedron consisting of two n-pyramids adjoined to opposite faces of an n-prism based on a regular n-gon (see Figure 4(a) for an example with $n = 6$). We may denote this spindle by $P_n(r, s, t)$ where the heights r, s and t of the components, as indicated in the diagram, are any (strictly) positive quantities. It has $3n$ faces: n rectangular faces and $2n$ isosceles triangular faces of two kinds. For $3 \geq n \geq 5$, an n-spindle may be regular-faced (see Figure 3 part (h) for $n = 3$, part (i) for $n = 4$ and part (j) for $n = 5$). In all cases, independent of the choice of n, r, s and t, the spindle is a tessellation polyhedron. See Figure 4(b) which shows the tessellation corresponding to the 6-spindle in Figure 4(a). Moreover, as indicated on the tessellation, there is an almost-regular tetrahedron T with $T \Rightarrow P_n(r, s, t)$.

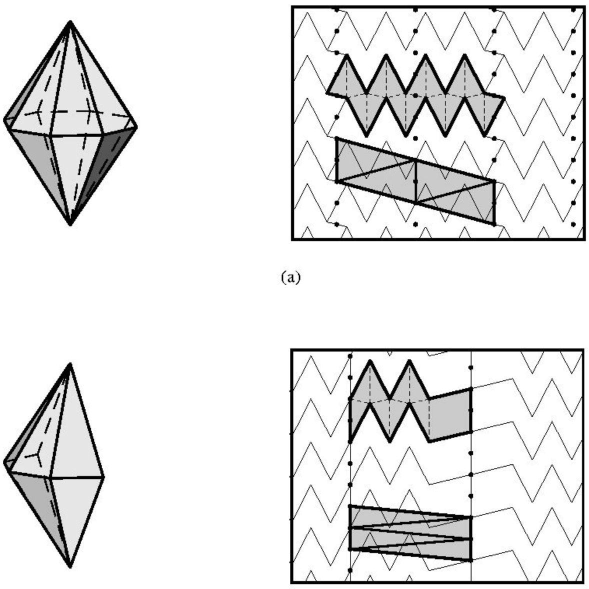


(a) (b)

FIGURE 4

If we put $s = 0$, then we obtain the n-dipyramid (see Figure 5(a) for an example with $n = 8$). This is also a tessellation polyhedron, but only if the heights r and t are equal. If $n = 3$, 4 or 5 then the n-dipyramid may be regular-faced (see Figure 3 part (e) for $n = 3$, part (f) for $n = 4$, the octahedron, and part (g) for $n = 5$). In Figure 5(a) we also show a tessellation for a net of a dipyramid D_6, and indicate the net of an almost regular tetrahedron T such that $T \Rightarrow D_6$. An analogous result holds for all $n \geq 3$.

Further possibilities arise, such as that shown in Figure 5(b). If n is even we can cut an n-dipyramid (here $n = 8$) into two equal parts by a plane through the apexes of the constituent pyramids and four of its edges. The corresponding tessellation and net of an almost regular tetrahedron are also shown in Figure 5(b).



(a)

(b)

FIGURE 5

Further examples of tessellation polyhedra are the twisted spindles. Each of these consists of two n-pyramids adjoined to the n-gonal faces of an n-antiprism, see Figure 6(a) for the case $n = 6$. Here r, s and t must satisfy the inequalities $0 < r(\Lambda - 1) < s$ and $0 < t(\Lambda - 1) < s$ where $\Lambda = (\cos(\pi/n))^{-1}$ if the resulting polyhedron is to be strictly convex. There are $4n$ triangular faces, of three kinds. If $n = 4$ or 5, all the faces can be regular (equilateral) triangles (See Figure 3 part (k) the deltahedron D_{16} for $n = 4$, and part (l), the icosahedron, for $n = 5$). In Figure 6(b) we show a tessellation for a net of this polyhedron (with $n = 6$), and also a net of the almost regular tetrahedron which can be transformed into it.

Certain special cases are of interest. If $r = t = s(\Lambda - 1)^{-1}$, where Λ is defined as above, then the faces of the antiprism and pyramids merge to yield a polyhedron with $2n$ kite-shaped faces which we call a *kite-polyhedron*. See Figure 7(a) for an example for which $n = 6$. In Figure 7(b) we show a tessellation for a net of this kite polyhedron and also indicate a net of the almost regular tetrahedron which can be transformed into it.

In Figure 7(c) we show the case of a twisted spindle with $n = 3$, and $r = s = t$. The six faces are rhombs and the polyhedron is a rhomboid. The corresponding tessellation is shown in Figure 7(d). Notice that the net is an affine image of a net of a cube, and that the rhomboid itself is the affine image of a cube.

In general, a non-singular affine image of a net is *not* a net of any polyhedron. Those nets shown in Figures 4, 5, 6 and 7, have the unexpected property that there exist affine images of each of the nets which are also nets of polyhedra of the same type.

Further examples are provided by multiple polyhedra. We write $T \Rightarrow nP$ if a development of T is the union of n nets of the polyhedron P. An example is shown in Figure 8. Here P is a cube and nets of the various almost regular tetrahedra T for which $T \Rightarrow nP$ are indicated for $n = 1, 2, 3, 4, 5$. Similar constructions clearly apply for larger values of n.

All these examples serve to strengthen confidence in the truth of the conjecture.

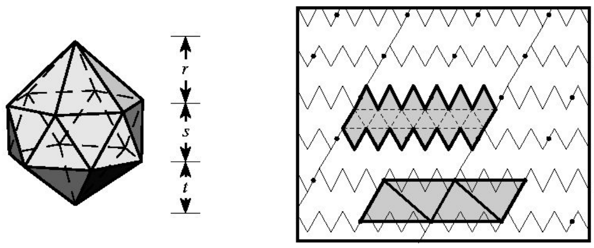
4. THE RELATION $\Rightarrow$

We have remarked that the relation $\Rightarrow$ is reflexive. Now we shall show:

THEOREM 2. *The relation $\Rightarrow$ is not symmetric.*

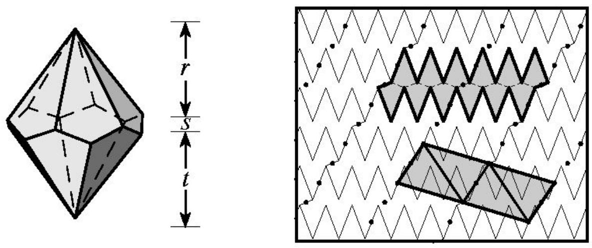
Proof. All we need to prove the theorem is to display one counter-example. In the Introduction we showed that $T \Rightarrow C$ where C is a cube and T is the almost regular tetrahedron $T(2, \sqrt{5}/2, 3\sqrt{2}/2)$. We shall now show that $C \not\Rightarrow T$ where C is a cube and T is *any* almost regular tetrahedron.

To do this we remember that $P_1 \Rightarrow P_2$ means that some development R of P_1 is a net of P_2. Such a development R of P_1 is obtained by cutting the surface of P_1 along the edges of a *hamiltonian tree* $\mathcal{T}$. Because $\mathcal{T}$ is a tree (a graph with no circuits) the resulting region R will be connected and because it is hamiltonian (incident with every vertex of P_1) the region R is planar. Now every tree has at least two edges of which an end-point has valency one. Consider such an end-point. Clearly it must be a vertex of P_1 (if it were an interior point of a face or an edge the cut would not yield the development R). In the case of a cube, since the sum of the angles of the faces at any vertex is $3\pi/2$ the development R must have at least two vertices with this angle. But then R cannot be a net of any almost regular

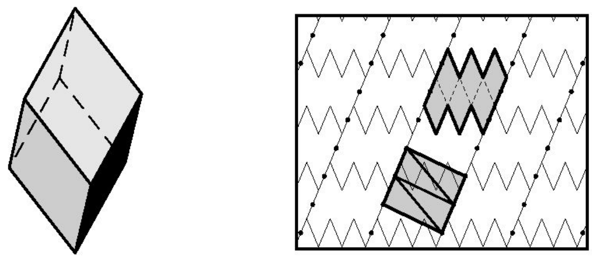


(a) (b)

FIGURE 6



(a) (b)



(c) (d)

FIGURE 7

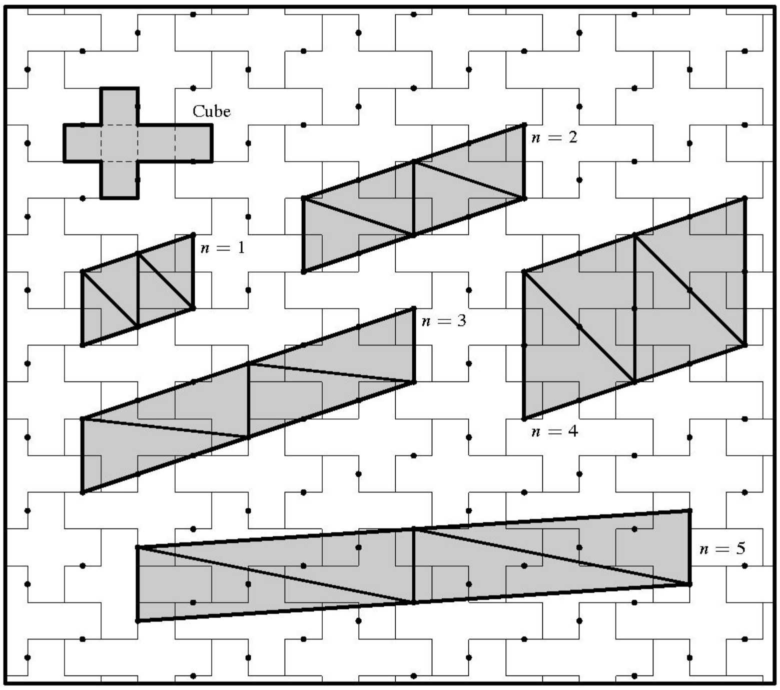


FIGURE 8

tetrahedron T. In fact $C \not\Rightarrow P$ where P is any polyhedron whose faces are all regular (equilateral) triangles (J12, J13, J17, J51, J84, the octahedron and icosahedron) since no net of such a polyhedron can have a vertex angle $3\pi/2$. More generally, if P is a regular-faced polyhedron, it is impossible for $C \Rightarrow P$ unless P has at least three squares or two octagons as faces. Thus $C \Rightarrow P$ is impossible for all the "Modified Platonic Solids" J58–J64 in Johnson's list [7]. It is also impossible in the case of the 8-prism and 8-antiprism, each with two octagons, since the octagons are not adjacent.

A modification of the above argument shows that the region R must have at least $v(P_1)-p(P_1)$ vertices, where $v(P_1)$ is the number of vertices of P_1 and $p(P_1)$ is the number of vertices at which the sum of the angles of the faces is π.

In the case of a cube, $v(C) = 8$, $p(C) = 0$ and so any development R will have at least eight vertices. But R cannot be a net of *any* tetrahedron since all such nets (as is easily verified) have at most six vertices. This proves the assertion.

THEOREM 3. *The relation $\Rightarrow$ is not transitive.*

Proof. As in Theorem 2, we need only display one counterexample. Although a polyhedron P may have several nets, each of which may lead to several tessellations, with a very few exceptions (such as, for example J1, J12, J13, J17, J84, octahedron and icosahedron) there are only a finite number of triples (a, b, c) such that $T(a, b, c) \Rightarrow P$. However, it is not difficult to show that for any given (a, b, c) there are infinitely many triples (r, s, t) such that $T(r, s, t) \Rightarrow T(a, b, c)$. Clearly we may choose (r, s, t) so that $T(r, s, t) \not\Rightarrow P$. Then we have $T(r, s, t) \Rightarrow T(a, b, c)$ and $T(a, b, c) \Rightarrow P$ but these do not imply $T(r, s, t) \Rightarrow P$. Hence $\Rightarrow$ is not transitive.

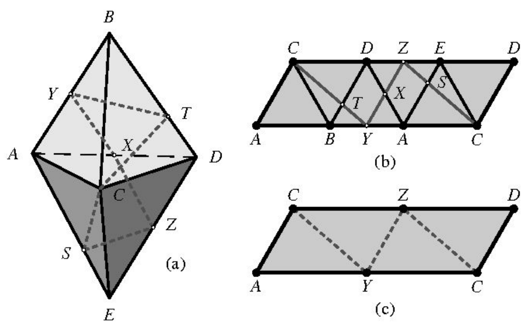


FIGURE 9

From the above we infer that there are very few examples of $P_1 \Rightarrow P_2$ where P_1 is *not* a tetrahedron. One possible reason for this is that there seems to be no theorem for general polyhedra analogous to Theorem 1. We conclude this section with two examples where $P_1 \Leftrightarrow P_2$.

In Figure 9(b) we show a development R of the triangular dipyramid J12 (Figure 9(a)). Here R is a parallelogram. This can be split into four congruent triangles in such a way as to form the net of an almost regular tetrahedron $T = T(1, 3/2, \sqrt{7}/2)$ (Figure 9(c)). Hence J12 $\Rightarrow T$.

Other examples arise when two distinct polyhedra have identical (unlabelled) nets. One such, reproduced from [8], is shown in Figure 10. The net, shown in Figure 10(a) and (b), is a chain of eight triangles with edges of two different lengths. (In the diagram the lengths are 1 and 1.25). In these figures

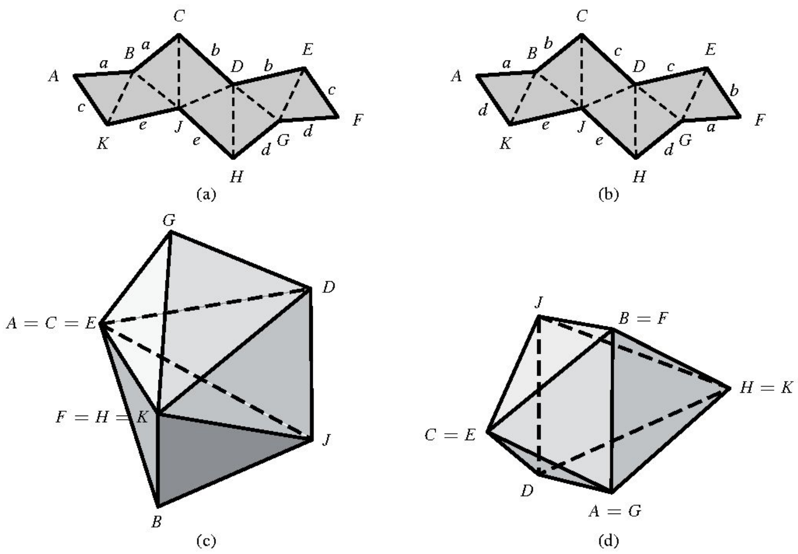


FIGURE 10

we have indicated two distinct labellings of the edges. If the similarly labelled edges in Figure 10(a) are joined, we obtain the net of a stack polyhedron S (Figure 10(c)), whereas the labelling in Figure 10(b) leads to the (non-regular) octahedron O in Figure 10(d). As the nets of the polyhedra are identical, clearly $S \Rightarrow O$. Also $O \Rightarrow S$. A few other examples of distinct polyhedra having identical (unlabelled) nets are known, but there is no general theory on how these can be constructed. In all these cases the relation is symmetric, and we may write $S \Leftrightarrow O$.

5. FINAL REMARKS

Nets of polyhedra have been known for nearly five hundred years [5] and hundreds of examples of nets of polyhedra can be found on the internet. A complete set of nets for all regular and archimedean polyhedra can be found in [4]. However, there seems to be very little mathematical literature on the subject and many fundamental questions remain unanswered. We now state some of these:

(A) *Does every convex polyhedron have a net?*

It is a common experience of model builders that the construction of a net for a convex polyhedron P fails because, after slitting along the edges of P that form a hamiltonian tree, and opening out the surface in the plane, the resulting region R has overlaps, and so is not a net. One reason that this problem has not been solved is that until recently, it seems that few mathematicians realised that there was any problem at all! And even if they did, they assumed that the answer was positive. One of the first explicit statements of the problem appears in [8]. It is trivial to show there is a negative answer to the question to the corresponding problem for polyhedra with non-convex faces, and several mathematicians have also shown that the answer is negative for non-convex polyhedra with convex polygons as faces, see [3], but the question for convex polyhedra in general remains open.

(B) *Given a set of (closed) convex polygons in the plane which are disjoint (except for their edges) and whose union is a connected set. Under what conditions do these polygons form the net of a convex polyhedron?*

More generally:

(C) *Given a plane polygon R, what are necessary and sufficient conditions for it to be possible to introduce fold lines into R so as to make it into a net of some polyhedron P ?*

One might hope that, if the polygon is convex, the answer would be positive since many convex developments are known (see [8]). But notice that the answer is negative without some clarification of the types of polyhedra that are allowed. For example, it seems that a rectangle cannot be made into the net of any strictly convex polyhedron (but can be made into the net of a (flat) degenerate tetrahedron bounded by four congruent right-triangles).

Another problem related to (B), is to determine the number of distinct nets of a given polyhedron. Again, careful definitions are required if this question is to be meaningful. In [1] Professor Jin Akiyama shows that a regular tetrahedron has two distinct nets, a cube has eleven distinct nets, and a regular octahedron also has eleven distinct nets. But if the faces are labelled, or otherwise distinguished, (for example if the polyhedron is distorted so that the faces are of slightly different shapes) then the number of nets is considerably larger. For example, in the case of a labelled cube, the number of distinct nets is $11 \times 6 \times 4 = 264$. The number of nets of the regular dodecahedron and regular icosahedron (43,380) have been determined by S. Bouzette and F. Vandamme and also by Ch. Hippenmeyer (see [3] for references to the relevant works). In [2] the number of nets of twenty-one of the regular-faced polyhedra is given. In each case, it is not known how many of these are "genuine" nets since they may involve "overlaps" and it remains an unsolved problem to determine the numbers of "genuine" nets. All these numbers are for "labelled polyhedra". They are surprisingly large, for example there are over five billion distinct nets of the polyhedron J44 (the gyroelongated triangular bicupola). The numbers were determined by computer; for details see [2]. There seems to be no general theory, and all the numbers have been found empirically, either by counting hamiltonian edge-trees, or the number of dual trees.

There are further problems relating to tessellation polyhedra. As stated above, a complete list of regular-faced tessellation polyhedra is given in [2]. This list was determined by computer. It would be interesting to decide whether these polyhedra, and those in Section 3, are the *only* tessellation polyhedra, and so whether the conjecture in Section 2 is correct. There are many other problems, such as:

(D) *Does there exist a convex polyhedron P of which no net tiles the plane (so it is* not *a tessellation polyhedron) but two distinct nets of P do so, thus forming a dihedral tiling in the sense of* [6, p. 23] *?*

If such a polyhedron exists we may call it a 2-*tessellation polyhedron* and, in a similar manner an *n-tessellation polyhedron* can be defined. No n-tessellation polyhedra are known for any $n \geq 2$ though their discovery would appear to be a not very difficult problem.

Finally, it would be interesting to discover more examples of unlabelled nets (other than those shown in Figure 10), which, if labelled in a suitable manner can represent two (or more) distinct polyhedra.

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(Reçu le 30 janvier 2012)

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