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PROJECTIONS AND RELATIVE HYPERBOLICITY

by Alessandro SISTO

ABSTRACT. We give an alternative definition of relative hyperbolicity based on properties of closest-point projections on peripheral subgroups. We also derive a distance formula for relatively hyperbolic groups, similar to the one for mapping class groups.

INTRODUCTION

The notion of (strong) relative hyperbolicity first appeared in [Gro87] and has been further studied in [Bow99, Far98, Osi06], where equivalent characterizations have been given. The main aim of this paper is to introduce a new characterization of relatively hyperbolic groups in terms of projections on left cosets of peripheral subgroups. The properties we will consider are similar to those in [Beh06, AK11] and are used in [Sis11] in a more general setting. The characterization we will give is similar to the characterization of tree-graded spaces given in [Sis], the link being provided by asymptotic cones in view of results in [DS05]. Our characterization only involves the geometry of the Cayley graph, alongside the ones given in [DS05] and [Dru09]. Also, the statement deals with the more general setting of metric relative hyperbolicity (i.e. asymptotic tree-gradedness with the established terminology).

We defer the exact statement to Section 2, see Definitions 2.1, 2.11 and Theorem 2.14.

We will use projections also to provide an analogue for relatively hyperbolic groups of the distance formula for mapping class groups [MM00].

Let G be a relatively hyperbolic group and let $\mathcal{P}$ be the collection of all left cosets of peripheral subgroups. For $P \in \mathcal{P}$, let π_P be a closest point projection map onto P. Denote by $\widehat{G}$ the coned-off graph of G, that is to say the metric graph obtained from a Cayley graph of G by adding an edge connecting each pair of (distinct) vertices contained in the same left coset of peripheral subgroup. Let $\{\{x\}\}_L$ denote x if $x > L$, and 0 otherwise. We write $A \approx_{\lambda,\mu} B$ if $A/\lambda - \mu \leq B \leq \lambda A + \mu$.

THEOREM 0.1 (Distance formula for relatively hyperbolic groups). *There exists L_0 so that for each $L \geq L_0$ there exist λ, μ so that the following holds. If $x, y \in G$ then*

$$d(x,y) \approx_{\lambda,\mu} \sum_{P \in \mathcal{P}} \{\{d(\pi_P(x), \pi_P(y))\}\}_L + d_{\widehat{G}}(x,y). \tag{0.1}$$

This formula is used in [MS12] to study quasi-isometric embeddings of relatively hyperbolic groups in products of trees. It is useful for applications to know that projections admit alternative descriptions, see Lemma 1.13. In subsection 3.1 we will give a sample application of the distance formula and show that a quasi-isometric embedding between relatively hyperbolic groups coarsely preserving left cosets of peripheral subgroups gives a quasi-isometric embedding of the corresponding coned-off graphs (the reader may wish to compare this result with [Hru10, Theorem 10.1]).

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1. BACKGROUND ON RELATIVELY HYPERBOLIC GROUPS

DEFINITION 1.1. A geodesic complete metric space X is *tree-graded* with respect to a collection $\mathcal{P}$ of closed geodesic subsets of X (called *pieces*) if the following properties are satisfied:

(T_1) two different pieces intersect in at most one point,

(T_2) each geodesic simple triangle is contained in one piece.

Tree-graded spaces can be characterized in terms of closest-point projections on the pieces. Let us denote by X a complete geodesic metric space and by $\mathcal{P}$ a collection of subsets of X. Consider the following properties.

DEFINITION 1.2. A family of maps $\Pi = \{\pi_P \colon X \to P\}_{P\in\mathcal{P}}$ will be called *projection system for* $\mathcal{P}$ if, for each $P \in \mathcal{P}$,

$(P1)$ for each $r \in P$, $z \in X$, $d(r,z) = d(r, \pi_P(z)) + d(\pi_P(z), z)$,

$(P2)$ π_P is locally constant outside P,

$(P3)$ for each $Q \in \mathcal{P}$ with $P \neq Q$, we have that $\pi_P(Q)$ is a point.

DEFINITION 1.3. A geodesic is $\mathcal{P}$*-transverse* if it intersects each $P \in \mathcal{P}$ in at most one point. A geodesic triangle in X is $\mathcal{P}$*-transverse* if each side is $\mathcal{P}$-transverse.

$\mathcal{P}$ is *transverse-free* if each $\mathcal{P}$-transverse geodesic triangle is a tripod.

THEOREM 1.4 ([Sis]). *Let X be a complete geodesic metric space and let $\mathcal{P}$ be a collection of subsets of X. Then X is tree-graded with respect to $\mathcal{P}$ if and only if $\mathcal{P}$ is transverse-free and there exists a projection system for $\mathcal{P}$.*

The following properties have also been considered in [Sis]. Properties $(P1)$ and $(P2)$ are equivalent to $(P'1)$ and $(P'2)$.

LEMMA 1.5. *Properties $(P1)$ and $(P2)$ can be substituted by:*

$(P'1)$ *for each $P \in \mathcal{P}$ and $x \in P$, $\pi_P(x) = x$,*

$(P'2)$ *for each $P \in \mathcal{P}$ and for each $z_1, z_2 \in X$ such that $\pi_P(z_1) \neq \pi_P(z_2)$,*

$$d(z_1,z_2) = d(z_1, \pi_P(z_1)) + d(\pi_P(z_1), \pi_P(z_2)) + d(\pi_P(z_2), z_2).$$

The reader unfamiliar with asymptotic cones is referred to [Dru02].

CONVENTION 1. *Throughout the paper we fix a non-principal ultrafilter μ on $\mathbf{N}$. We will denote ultralimits by $\mu - \lim$ and by $C(X,(p_n),(r_n))$ the asymptotic cone of X with respect to the ultrafilter μ, the sequence of basepoints (p_n) and the sequence of scaling factors (r_n).*

DEFINITION 1.6 ([DS05]). The geodesic metric space X is *asymptotically tree-graded* with respect to the collection of subsets $\mathcal{P}$ if all its asymptotic cones, with respect to the fixed ultrafilter, are tree-graded with respect to the collection of the ultralimits of elements of $\mathcal{P}$.

DEFINITION 1.7. The finitely generated group G is *hyperbolic relative* to its subgroups $H_1, \dots, H_n$, called *peripheral subgroups*, if its Cayley graphs are asymptotically tree-graded with respect to the collection of all left cosets of the H_i's.

Let X be asymptotically tree-graded with respect to $\mathcal{P}$. We recall below some useful lemmas from [DS05] that will be used later.

When A is a subset of the metric space X, the notation $N_d(A)$ will denote the closed neighborhood of radius d around A, i.e.

$$N_d(A) = \{x \in X \mid d(x, A) \leq d\}.$$

LEMMA 1.8 ([DS05, Theorem 4.1 $-(\alpha_2)$]). *There exists $M \geq 0$ with the following property. If γ is a geodesic connecting x to y, and $d(x,P), d(y,P) \leq d(x,y)/3$ for some $P \in \mathcal{P}$, then $\gamma \cap N_M(P) \neq \varnothing$.*

LEMMA 1.9 ([DS05, Lemma 4.7]). *For each $H \geq 0$ there exists B such that* $\operatorname{diam}(N_H(P) \cap N_H(Q)) \leq B$ *for each $P, Q \in \mathcal{P}$ with $P \neq Q$.*

We will also need that each $P \in \mathcal{P}$ is *quasi-convex*, in the following sense.

LEMMA 1.10 ([DS05, Lemma 4.3]). *There exists t such that for each $L \geq 1$ each geodesic connecting $x, y \in N_L(P)$ is contained in $N_{tL}(P)$.*

If G is hyperbolic relative to $H_1, \dots, H_n$, its *coned-off* graph, denoted $\widehat{G}$, is obtained from a Cayley graph of G by adding edges connecting vertices lying in the same left coset of peripheral subgroup.

By [Far98], $\widehat{G}$ is hyperbolic and the following property holds.

PROPOSITION 1.11 (BCP property). *Let α, β be geodesics in $\widehat{G}$, for G relatively hyperbolic, and let $\mathcal{P}$ be the collection of all left cosets of peripheral subgroups of G. There exists c with the following properties.*

(1) *If α contains an edge connecting vertices of some $P \in \mathcal{P}$ but β does not, then such vertices are at distance at most c in G.*
(2) *If α and β contain edges $[p_\alpha, q_\alpha], [p_\beta, q_\beta]$ (respectively) connecting vertices of some $P \in \mathcal{P}$, then $d_G(p_\alpha, p_\beta), d_G(q_\alpha, q_\beta) \leq c$.*

1.1 Geodesics and projections

Convention 2. *In this subsection X is an asymptotically tree-graded space with respect to a collection of subsets $\mathcal{P}$. Sometimes we will restrict to X a Cayley graph of a relatively hyperbolic group, and in that case $\mathcal{P}$ will always be the collection of left cosets of peripheral subgroups.*

The following definition is taken from [DS05, Definition 4.9].

Definition 1.12. If $x \in X$ and $P \in \mathcal{P}$, define the almost projection $\pi_P(x)$ to be the subset of P of points whose distance from x is less than $d(x,P)+1$.

The following lemma gives two alternative characterizations of the maps π_P.

Lemma 1.13.
(1) *If α is a continuous (K,C)-quasi-geodesic connecting x to $P \in \mathcal{P}$ then for each $D \geq D_0 = D_0(K,C)$ there exists M so that the first point in $\alpha \cap N_D(P)$ is at distance at most M from $\pi_P(x)$.*
(2) (Bounded Geodesic Image) *If X is the Cayley graph of G, there exists M so that if $\widehat{\gamma}$ is a geodesic in $\widehat{G}$ connecting $x \in G$ to $P \in \mathcal{P}$ then the first point in $\widehat{\gamma} \cap P$ is at distance at most M from $\pi_P(x)$.*

Proof. (1) The *saturation* of a geodesic is the union of the geodesic and all $P \in \mathcal{P}$ whose μ-neighborhood intersects the geodesic (for some appropriately chosen μ). By [DS05, Lemma 4.25] there exists $R = R(K,C)$ so that if γ is a geodesic and the (K,C)-quasi-geodesic α connects points in the saturation $Sat(\gamma)$ of γ, then α is contained in the R-neighborhood of $Sat(\gamma)$.

We can apply this when α is as in our statement and γ is a geodesic from x to $\pi_P(x)$. Let $D \geq \mu, R$ and let p be the first point in $\alpha \cap N_D(P)$. There are two cases to consider. If $p \in N_R(\gamma)$ then we are done as $\mathrm{diam}(\gamma \cap N_{D+R}(P)) \leq D+R$ and $p, \pi_P(x) \in N_R(\gamma) \cap N_D(P)$. Otherwise there exists $P' \neq P$ so that $P' \subseteq Sat(\gamma)$ and $p \in N_R(P')$. By [DS05, Lemma 4.28] there exists $B = B(D)$ so that $N_D(P) \cap N_R(P') \subseteq N_B(\gamma)$. As noticed earlier $\mathrm{diam}(\gamma \cap N_{B+D}(P)) \leq B+D$ and $\pi_P(x), p \in N_B(\gamma) \cap N_D(P)$, so we are done.

(2) Let $\widehat{\gamma}_0$ be a geodesic in $\widehat{G}$ connecting x to $\pi_P(x)$ and denote by p the first point in $\widehat{\gamma} \cap P$, and let $\widehat{\gamma}_1$ be any geodesic from x to P intersecting P only in its endpoint q. By adding an edge to $\widehat{\gamma}_1$ connecting q to $\pi_P(x)$ we are

in a situation where we can apply the BCP property to get a uniform bound on $d(p,q)$. So, it is enough to prove the statement for $\widehat{\gamma}=\widehat{\gamma}_0$. By [Hru10, Lemma 8.8], we can bound by some constant, say B, the distance from p to a geodesic γ in G from x to $\pi_P(x)$. As in the first part, we have $p,\pi_P(x)\in N_B(\gamma)\cap N_D(P)$, a set whose diameter can be bounded by $B+D$.

The *lift* of a geodesic in $\widehat{G}$ is a path in G obtained by substituting edges labeled by an element of some H_i and possibly the endpoints with a geodesic in the corresponding left coset. The following is a consequence of [Hru10, Lemma 8.8] (or of the distance formula and the second part of Lemma 1.13, but [Hru10, Lemma 8.8] is used in the proof).

PROPOSITION 1.14 (Hierarchy paths for relatively hyperbolic groups). *There exist λ,μ so that if α is a geodesic in $\widehat{G}$ then its lifts are (λ,μ)-quasi-geodesics.*

LEMMA 1.15. *There exists L so that if $d(\pi_P(x),\pi_P(y))\geq L$ for some $P\in\mathcal{P}$ then*

(1) *all (K,C)-quasi-geodesics connecting x to y intersect $B_R(\pi_P(x))$ and $B_R(\pi_P(y))$, where $R=R(K,C)$,*
(2) *all geodesics in $\widehat{G}$ connecting x to y contain an edge in P, when X is a Cayley graph of G.*

Proof. (1) In view of Lemma 1.13(1), in order to show (1) we just have to show that any quasi-geodesic α as in the statement intersects a neighborhood of P of uniformly bounded radius. We can suppose that α is continuous. Let p be a point on α minimizing the distance from P, and let γ be a geodesic from p to P of length $d(p,P)$. The point p splits α in two halves α_1,α_2, and it is easy to show that the concatenation β_i of α_i and γ is a quasi-geodesic with uniformly bounded constants:

LEMMA 1.16. *Let δ_0 be a geodesic connecting q to p and let δ_1 be a (K,C)-quasi-geodesic starting at p. Suppose that $d(q,p)=d(q,\delta_1)$. Then the concatenation δ of δ_0 and δ_1 is a (K',C')-quasi-geodesic, for K',C' depending on K,C only.*

Proof. It is clear that the said concatenation is coarsely lipschitz. Let $I=I_0\cup I_1$ be the domain of δ, where I_0,I_1 are (translations of) the domains of δ_0,δ_1. We will denote by t the intersection of I_0 and I_1 so that $\delta(t) = \delta_0(t) = \delta_1(t) = p$. Let $t_0, t_1 \in I$ and set $x_i = \delta(t_i)$. We can assume $t_i \in I_i$, the other cases being either symmetric or trivial. Suppose first $d(x_0,p) = |t - t_0| \leq |t - t_1|/(2K) - C/2$. In this case $d(x_0,p) \leq d(x_1,p)/2$ so that $d(p,x_1) \leq d(p,x_0) + d(x_0,x_1) \leq d(p,x_1)/2 + d(x_0,x_1)$ and hence $d(p,x_1) \leq 2d(x_0,x_1)$. Then

$$|t_0 - t_1| = |t_0 - t| + |t - t_1| \leq 3d(p,x_1)/2 \leq 3d(x_0,x_1).$$

On the other hand, if $|t - t_0| \geq |t - t_1|/2K - C/2$ then

$$|t_0 - t_1| \leq (2K+1)d(x_0,p) + KC \leq (2K+1)d(x_0,x_1) + KC,$$

as $d(x_0,p) \leq d(x_0,x_1)$.

Let $D_0 = D_0(K,C)$ be as in Lemma 1.13(1). If $d(p,P) > D_0$ then, as $\beta_i \cap N_{D_0}(P) = \gamma \cap N_{D_0}(P)$, by Lemma 1.13(1) we can give a bound on the distance between the projections on P of x and y. If L is large enough it must then be the case that $d(p,P) \leq D_0$, what we needed to conclude the proof of part (1).

(2) Let $\widehat{\gamma}$ be a geodesic in $\widehat{G}$. Part (1) applies in particular to lifts $\widehat{\gamma}$, so that the conclusion follows applying the BCP property to a sub-geodesic of $\widehat{\gamma}$ connecting points close to $\pi_P(x)$ to $\pi_P(y)$ and the geodesic in $\widehat{G}$ consisting of a single edge connecting $\pi_P(x)$ to $\pi_P(y)$.

2. Alternative definition of relative hyperbolicity

In this section we state the analogue of the alternative definition of tree-graded spaces that can be found in [Sis]. Throughout the section let X be a geodesic metric space and let $\mathcal{P}$ be a collection of subsets of X.

We will need the coarse versions of the definitions of projection system and being transverse-free, as defined in [Sis].

DEFINITION 2.1. A family of maps $\Pi = \{\pi_P \colon X \to P\}_{P\in\mathcal{P}}$ will be called *almost-projection system for* $\mathcal{P}$ if there exist $C \geq 0$ such that, for each $P \in \mathcal{P}$,

$(AP1)$ for each $x \in X$, $p \in P$, $d(x,p) \geq d(x,\pi_P(x)) + d(\pi_P(x),p) - C$,

$(AP2)$ for each $x \in X$ with $d(x,P) = d$, $\operatorname{diam}(\pi_P(B_d(x))) \leq C$,

$(AP3)$ for each $P \neq Q \in \mathcal{P}$, $\operatorname{diam}(\pi_P(Q)) \leq C$.

REMARK 2.2. For each $x \in X$ and $P \in \mathcal{P}$, by $(AP1)$ with $p = \pi_P(x)$ we have $d(x,\pi_P(x)) \leq d(x,P) + C$.

2.1 TECHNICAL LEMMAS

First of all, let us prove some basic lemmas. One of the aims will be to prove that properties $(AP1)$ and $(AP2)$ are equivalent to coarse versions of properties $(P'1)$ and $(P'2)$ that will be formulated later.

Consider an almost-projection system for $\mathcal{P}$ and let C be large enough so that $(AP1)$ and $(AP2)$ hold. Let us start by proving that projections are coarsely contractive, in 2 different senses.

LEMMA 2.3.

(1) *Consider some $k \geq 1$ and a path γ connecting x to y such that $d(x,P) \geq kC$ for each $x \in \gamma$. Then $d(\pi_P(x), \pi_P(y)) \leq l(\gamma)/k + C$.*

(2) $d(\pi_P(x), \pi_P(y)) \leq d(x,y) + 6C$.

Proof. (1) Consider a partition of γ in subpaths $\gamma_i = [x_i, y_i]$ of length kC and one subpath $\gamma' = [x', y']$ of length at most kC. By property $(AP2)$ we have $d(\pi_P(x_i), \pi_P(y_i)) \leq C = d(x_i, y_i)/k$ and $d(\pi_P(x'), \pi_P(y')) \leq C$, so

$$\begin{aligned} d(\pi_P(x), \pi_P(y)) &\leq \sum d(\pi_P(x_i), \pi_P(y_i)) + d(\pi_P(x'), \pi_P(y')) \\ &\leq \sum d(x_i, y_i)/k + d(x', y')/k + C \leq l(\gamma)/k + C. \end{aligned}$$

(2) Consider a geodesic γ connecting x to y. If $\gamma \cap N_C(P) = \varnothing$ we can apply the first point. Otherwise, let $\gamma' = [x, x']$ (resp. $\gamma'' = [y', y]$) be a (possibly trivial) subgeodesic such that $\gamma' \cap N_C(P) = x'$ (resp. $\gamma'' \cap N_C(P) = y'$). Applying the previous point to γ' and γ'' and Remark 2.2 we get

$$\begin{aligned} &d(\pi_P(x), \pi_P(y)) \leq \\ &\quad d(\pi_P(x), \pi_P(x')) + d(\pi_P(x'), x') + d(x', y') + d(y', \pi_P(y')) + d(\pi_P(y'), \pi_P(y)) \\ &\qquad \leq (d(x, x') + C) + 2C + d(x', y') + 2C + (d(y', y) + C) = d(x, y) + 6C, \end{aligned}$$

as required.

LEMMA 2.4. *For each r and $c \geq 0$ we have that each $(1, c)$-quasi-geodesic γ from $x \in X$ to $y \in N_r(P)$, for some $P \in \mathcal{P}$, intersects $B_\rho(\pi_P(x))$, where $\rho = 2r + 6C + 5c$. Moreover, any point y' on γ such that $d(x,P) - 2c \leq d(x, y') \leq d(x,P)$ belongs to $B_\rho(\pi_P(x))$.*

Proof. Note that y' as in the statement exists if and only if $d(x,y) \geq d(x,P) - 2c$. Suppose $d(x,y) < d(x,P) - 2c$. In this case $d(\pi_P(x), \pi_P(y)) \leq C$ by $(AP2)$, so $d(y, \pi_P(x)) \leq r + 2C$ (we used Remark 2.2).

Let us now consider the other case. Let $y' \in \gamma$ be such that $d(x,P)-2c \leq d(x,y') \leq d(x,P)$ and let γ' be the sub-quasi-geodesic of γ from x to y'. As $d(y,\pi_P(y)) \leq r+C$ and $d(\pi_P(y'),\pi_P(x)) \leq C$, we have, using $(AP1)$ in the second inequality,

$$\begin{aligned} d(y',y) &\geq d(y',\pi_P(y)) - r - C \geq d(y',\pi_P(y')) + d(\pi_P(y'),\pi_P(y)) - r - 2C \\ &\geq d(y',\pi_P(x)) + d(\pi_P(x),\pi_P(y)) - r - 4C\,. \end{aligned}$$

Also,

$$d(x,y) \leq d(x,\pi_P(x)) + d(\pi_P(x),\pi_P(y)) + r + C\,.$$

As $d(x,y) \geq d(x,y')+d(y',y)-3c$ (since these points lie on a $(1,c)$-quasi-geodesic) and $d(x,y') \geq d(x,P)-2c$, we obtain

$$\begin{aligned} &[d(y',\pi_P(x)) + d(\pi_P(x),\pi_P(y)) - r - 4C] + d(x,P) \\ &\quad \leq d(y',y) + d(y',x) + 2c \leq d(x,y) + 5c \\ &\quad \leq d(x,\pi_P(x)) + d(\pi_P(x),\pi_P(y)) + r + C + 5c \\ &\quad \leq d(x,P) + d(\pi_P(x),\pi_P(y)) + r + 2C + 5c\,. \end{aligned}$$

Therefore,

$$d(y',\pi_P(x)) \leq 2r + 6C + 5c\,.$$

The following can be thought as another coarse version of property $(P1)$.

LEMMA 2.5. *Consider a geodesic γ starting from x and some $P \in \mathcal{P}$ such that $\gamma \cap N_r(P) \neq \varnothing$, for some $r \geq 2C$. Let y be the first point on γ in $N_r(P)$. Then $d(y,\pi_P(x)) \leq 8r + 22C$.*

Proof. If $d(x,y) \leq d(x,P)$, we have $d(\pi_P(x),\pi_P(y)) \leq C$ by $(AP1)$, so $d(y,\pi_P(x)) \leq r+2C$ (we used Remark 2.2). Suppose that this is not the case and let y' be as in the previous lemma. Consider a geodesic $\gamma' = [y,y']$.

By $d(y,\pi_P(y)) \leq r+C$, $d(y',\pi_P(y')) \leq 2r+7C$ (because of Remark 2.2), Lemma 2.3(1) with $k=2$ (recall that $r \geq 2C$ and notice that $\gamma' \cap N_r(P) = \{y\}$), we have

$$\begin{aligned} d(y,y') &\leq d(y,\pi_P(y)) + d(\pi_P(y),\pi_P(y')) + d(\pi_P(y'),y') \\ &\leq 3r + 8C + d(y,y')/2\,. \end{aligned}$$

So, $d(y,y') \leq 6r + 16C$ and $d(y,\pi_P(x)) \leq d(y,y') + d(y',\pi_P(x)) \leq 8r + 22C$.

COROLLARY 2.6. *Consider a geodesic γ from x to y and some $P \in \mathcal{P}$ such that $\gamma \cap N_r(P) = \{y\}$, for some $r \geq 2C$. Then $l(\gamma) \leq d(x,P) + 8r + 23C$ and $\pi_P(\gamma) \subseteq B_{8r+30C}(\pi_P(x))$.*

Proof. Using the previous lemma, $l(\gamma) = d(x,y) \leq d(x,\pi_P(x)) + d(\pi_P(x),y) \leq d(x,P) + C + (8r + 22C)$. The second part is an easy consequence of this fact, using $(AP2)$ and Lemma 2.3(2).

COROLLARY 2.7. *Let γ be a geodesic from x_1 to x_2. Then* $\mathrm{diam}(\gamma \cap N_r(P)) \leq d(\pi_P(x_1), \pi_P(x_2)) + 18r + 62C$ *for each $r \geq 2C$ and $P \in \mathcal{P}$.*

Proof. Let x_1', x_2' be the first and last point in $\gamma \cap N_r(P)$. By Corollary 2.6, we have $d(\pi_P(x_i), \pi_P(x_i')) \leq 8r + 30C$. So,

$$d(\pi_P(x_1), \pi_P(x_2)) \geq d(x_1', x_2') - 2(8r + 30C) - 2(r + C) = d(x_1', x_2') - 18r - 62C.$$

As $d(x_1', x_2') = \mathrm{diam}(\gamma \cap N_r(P))$, this is what we wanted.

We will consider the following coarse analogs of properties $(P'1)$ and $(P'2)$.

$(AP'1)$ There exists $C \geq 0$ such that for each $x \in X$, $d(x, \pi_P(x)) \leq d(x,P) + C$.

$(AP'2)$ There exists $C \geq 0$ with the property that for each $x_1, x_2 \in X$ such that $d(\pi_P(x_1), \pi_P(x_2)) \geq C$, we have

$$d(x_1, x_2) \geq d(x_1, \pi_P(x_1)) + d(\pi_P(x_1), \pi_P(x_2)) + d(\pi_P(x_2), x_2) - C.$$

LEMMA 2.8. $(AP1) + (AP2) \iff (AP'1) + (AP'2)$.

DEFINITION 2.9. We will say that C is a projection constant if the properties $(AP1), (AP2), (AP'1), (AP'2)$ hold with constant C.

Proof. $\Leftarrow$: Fix C large enough so that $(AP'1), (AP'2)$ hold. Property $(AP1)$ is not trivial only if $d(\pi_P(x), x)$ is large, and in this case it follows from $(AP'2)$ setting $x_1 = x$ and $x_2 = \pi_P(x) = p$ and taking into account $d(\pi_P(p), p) \leq C$. Let us show property $(AP2)$. Note that $d(\pi_P(x), \pi_P(x')) > C$ implies $d(x, x') > d(x,P) - 2C$. We want to exploit this fact. Set $d = d(x,P)$. Note that if $x' \in B(x,d)$, then there exists $x'' \in B_{d-2C}$ such that $d(x', x'') \leq 2C$ and one of of the following 2 cases holds:

- $x' \in N_{6C}(P)$, or
- $d(x'', P) \geq 4C$.

In the first case either $d(\pi_P(x'), \pi_P(x'')) < C$ or

$$d(x', \pi_P(x')) + d(\pi_P(x'), \pi_P(x'')) + d(\pi_P(x''), x'') - C \leq d(x', x'') \leq 2C,$$

and so $d(\pi_P(x'), \pi_P(x'')) \leq 3C$. In the second case $d(x', x'') \leq d(x', P) - 2C$, and so $d(\pi_P(x'), \pi_P(x'')) \leq C$.

These considerations yield $\operatorname{diam}(\pi_P(B_d(x))) \leq 4C$.

$\Rightarrow$: We already remarked that $(AP'1)$ holds. Let $C > 0$ be large enough so that $(AP1)$ and $(AP2)$ hold. We will prove the following, which implies $(AP'2)$ setting $c = 0$ and which will be useful later.

LEMMA 2.10. *If $d(\pi_P(x_1), \pi_P(x_2)) \geq 8C + 8c + 1$, for some $c \geq 0$ and $P \in \mathcal{P}$, then any $(1, c)$-quasi-geodesic γ from x_1 to x_2 intersects $N_{2C}(P)$ and $B_{10C+5c}(\pi_P(x_i))$.*

Proof. Once we show that $\gamma \cap N_{2C}(P) \neq \varnothing$, we can apply Lemma 2.4 to obtain $B_{10C+5c}(\pi_P(x_i)) \cap \gamma \neq \varnothing$.

Set $d_i = d(x_i, P)$. We have $B_{d_1}(x_1) \cap B_{d_2}(x_2) = \varnothing$, for otherwise we would have $d(\pi_P(x_1), \pi_P(x_2)) \leq 2C$. Let z_i be a point on γ such that $d_i - 2c \leq d(x_i, z_i) \leq d_i$. Suppose by contradiction that $[z_1, z_2] \cap N_{2C}(P) = \varnothing$. Then $d(\pi_P(z_1), \pi_P(z_2)) \leq d(z_1, z_2)/2 + C$ by Lemma 2.3(1), and in particular $d(z_1, z_2)/2 \geq 5C + 8c + 1$ (notice that $d(\pi_P(z_1), \pi_P(x_i)) \leq C$). So,

$$\begin{aligned}
d(x_1, x_2) &\leq d(x_1, \pi_P(x_1)) + d(\pi_P(x_1), \pi_P(z_1)) \\
&\qquad + d(\pi_P(z_1), \pi_P(z_2)) + d(\pi_P(z_2), \pi_P(x_2)) + d(\pi_P(x_2), x_2) \\
&\leq (d(x_1, P) + C) + C + (d(z_1, z_2)/2 + C) + C + (d(x_2, P) + C) \\
&\leq d(x_1, z_1) + d(z_1, z_2) + d(z_2, x_2) + 5C + 4c - d(z_1, z_2)/2 \\
&\leq (d(x_1, x_2) + 4c) + 5C + 4c - d(z_1, z_2)/2 < d(x_1, x_2),
\end{aligned}$$

a contradiction. Therefore $[z_1, z_2] \cap N_{2C}(P) \neq \varnothing$ and in particular $\gamma \cap N_{2C}(P) \neq \varnothing$, as required.

2.2 MAIN RESULT

DEFINITION 2.11. A $(1, c)$-quasi-geodesic triangle Δ is *$\mathcal{P}$-almost-transverse with constants K, D* if, for each $P \in \mathcal{P}$ and each side γ of Δ, $\operatorname{diam}(N_K(P) \cap \gamma) \leq D$.

$\mathcal{P}$ is *asymptotically transverse-free* if there exist λ, σ such that for each $D \geq 1$, $K \geq \sigma$ the following holds. If Δ is a geodesic triangle which is $\mathcal{P}$-almost-transverse with constants K, D, then Δ is λD-thin.

Recall that a triangle is δ-thin if any point on one of its sides is at distance at most δ from the union of the other two sides.

The definition of being asymptotically transverse-free only involves geodesic triangles. But, as we will see, if there exists an almost-projection system for $\mathcal{P}$, then we can deduce something about $(1,c)$-quasi-geodesic triangles as well.

DEFINITION 2.12. $\mathcal{P}$ is *strongly asymptotically transverse-free* if there exist λ,σ such that for each $c,D\geq 1$, $K\geq\sigma c$ the following holds. If Δ is a $(1,c)$-quasi-geodesic triangle which is $\mathcal{P}$-almost-transverse with constants K,D, then Δ is $\lambda(D+c)$-thin.

LEMMA 2.13. *If $\mathcal{P}$ is asymptotically transverse-free and there exists an almost-projection system for $\mathcal{P}$, then $\mathcal{P}$ is strongly asymptotically transverse-free.*

Proof. Let C be a projection constant for $\mathcal{P}$ and let λ_0,σ_0 be the constants such that $\mathcal{P}$ is asymptotically transverse-free with those constants. We will show that $\mathcal{P}$ is strongly asymptotically transverse-free for $\sigma=10C+5$. Let Δ be a $(1,c)$-quasi-geodesic triangle, for $c\geq 1$, which is $\mathcal{P}$-almost-transverse with constants $K\geq\sigma c, D\geq 1$, and let $\{\gamma_i\}$ be its sides.

Consider $x,y\in\gamma_i$. We want to prove that any geodesic γ from x to y is $\mathcal{P}$-almost-transverse with "well-behaved" constants. Let us start by proving that $d(\pi_P(x),\pi_P(y))\leq D+20C+10c+1$ for each $P\in\mathcal{P}$. In fact, if that were not the case, by Lemma 2.10 we would have that γ_i intersects $B_{10C+5c}(\pi_P(x))$, $B_{10C+5c}(\pi_P(x))$, so $\operatorname{diam}(\gamma_i\cap N_{10C+5c}(P))\geq D+1$ (a contradiction as $\sigma c\geq 10C+5c$). By Corollary 2.7 (we can assume $\sigma_0\geq 2C$), we have $\operatorname{diam}(\gamma\cap N_{\sigma_0}(P))\leq D+18\sigma_0+82C+10c+1$ for each $P\in\mathcal{P}$.

By the fact that $\mathcal{P}$ is asymptotically transverse-free, we obtain that each geodesic triangle whose vertices lie on γ_i is λ'-thin, for $\lambda'=\lambda_0(D+18\sigma_0+82C+10c+1)$. This is all that is needed to apply verbatim the proof of [BH99, Theorem III.H.1.7] (which roughly states that in a hyperbolic space quasi-geodesics are at finite Hausdorff distance from geodesics). The constants appearing in the proof are explicitly determined in terms of the hyperbolicity constant δ (λ' plays the role of δ) and the quasi-geodesics constants λ,ϵ (in our case $\lambda=1$, $\epsilon=c$), and one can easily check that the bound on the Hausdorff distance can be chosen to be linear in $\delta+\epsilon$, when fixing $\lambda=1$ (and, say, for $\delta,\epsilon\geq 1$). One can also obtain this remark by a scaling argument.

Hence, each side of Δ is at Hausdorff distance bounded linearly in $(D+c)$ from the sides of a triangle whose thinness constant is linear in $(D+c)$, so we are done.

THEOREM 2.14. *The geodesic metric space X is asymptotically tree-graded with respect to the collection of subsets $\mathcal{P}$ if and only if $\mathcal{P}$ is asymptotically transverse-free and there exists an almost-projection system for $\mathcal{P}$.*

Proof. $\Leftarrow$: Consider an asymptotic cone $Y = C(X,(p_n),(r_n))$ of X and consider the collection $\mathcal{P}'$ of ultralimits of elements of $\mathcal{P}$ in Y. It is quite clear that elements of $\mathcal{P}'$ are geodesic, by the assumptions on $\mathcal{P}$. Also, it is very easy to see that an almost projection system for $\mathcal{P}$ induces a projection system for $\mathcal{P}'$.

Let us prove that $\mathcal{P}'$ is transverse-free. Consider a geodesic triangle Δ in Y. We would like to say that its sides are ultralimits of geodesics in X. This is not the case, but, as shown in the following lemma, it is not too far from being true.

LEMMA 2.15. *Any geodesic $\gamma\colon [0,l] \to Y$ is the ultralimit of a sequence (γ_n) of $(1,c_n)$-quasi-geodesics, where $\mu - \lim c_n/r_n = 0$.*

Proof. By [FLS11, Lemma 9.4], γ is a ultralimit of lipschitz paths γ_n. Let c_n be the least real number so that γ_n is a $(1,c_n)$-quasi-geodesic. As the ultralimit of (γ_n) is a geodesic, it is readily seen that $\mu - \lim c_n/r_n = 0$.

Using this lemma, we obtain that Δ, the geodesic triangle we are considering, is the ultralimit of some triangles Δ_n of X whose sides are $(1,c_n)$-quasi-geodesics and $\mu - \lim c_n/r_n = 0$ (as Δ is $\mathcal{P}'-$transverse). Suppose that Δ is $\mathcal{P}'-$transverse, and let λ,σ be as in the definition of being strongly asymptotically transverse-free. Let $K_n = \sigma c_n$ and notice that Δ_n must be μ-a.e. $\mathcal{P}$-almost-transverse with constants K_n, D_n, where $\mu - \lim D_n/r_n = 0$. In particular, Δ_n is κ_n-thin, where $\kappa_n = \lambda(D_n + c_n)$ so that $\mu - \lim \kappa_n/r_n = 0$. This implies that Δ is a tripod, and hence we showed that $\mathcal{P}'$ is transverse-free. We proved that both conditions of Theorem 1.4 are satisfied for Y and $\mathcal{P}'$, therefore Y is tree-graded with respect to $\mathcal{P}'$. As Y was any asymptotic cone of X, the proof is complete.

$\Rightarrow$: For each $P \in \mathcal{P}$, define π_P in such a way that for each $x \in X$ we have $d(\pi_P(x),x) \leq d(x,P)+1$. Property $(AP'1)$ is obvious. Property $(AP'2)$ follows directly from Lemma 1.15(1).

Let us prove ($AP3$) (we will use the lemma once again). Let B be a uniform bound on the diameters of $N_H(P) \cap N_H(Q)$ for $P \neq Q \in \mathcal{P}$ (see Lemma 1.9), where $H = \max\{tM, L\}$ for t as in Lemma 1.10. Fix $P, Q \in \mathcal{P}$, $P \neq Q$. Suppose that there exist $x, y \in Q$ such that $d(\pi_P(x), \pi_P(y)) \geq 2L + B + 1$. Consider a geodesic $[x, y]$. It is contained in $N_{tM}(Q)$. Consider points x', y' on $[x, y]$ such that $d(x', \pi_P(x)) \leq L$, $d(y', \pi_P(y)) \leq L$. Then $d(x', y') \geq d(\pi_P(x), \pi_P(y)) - 2L \geq B + 1$. This is in contradiction with $\operatorname{diam}(N_H(P) \cap N_H(Q)) \leq B$.

These considerations readily imply ($AP3$).

We are left to show that $\mathcal{P}$ is asymptotically transverse-free. Suppose that there is no λ such that $\mathcal{P}$ satisfies the definition of being asymptotically transverse-free with $\sigma = tM$ for M as in Lemma 1.8 and t as in Lemma 1.10. Then we have a diverging sequence (r'_n) and geodesic triangles Δ_n which are $\mathcal{P}$-almost-transverse with constants K, D_n and optimal thinness constant $r_n = r'_n D_n$. Let $\alpha_n, \beta_n, \gamma_n$ be the sides of Δ_n. We can assume that there exists $p_n \in \alpha_n$ with $d(p_n, \beta_n \cup \gamma_n) = r_n$. Consider $Y = C(X, (p_n), (r_n))$, and let α, β, γ be the geodesics (or geodesic rays, or geodesic lines) in Y induced by $(\alpha_n), (\beta_n), (\gamma_n)$. Also, let $\mathcal{P}'$ be the collection of pieces for Y as in the definition of asymptotic tree-gradedness. We claim that for each $P \in \mathcal{P}'$, $|\alpha \cap P| \leq 1$ (and same for β, γ). This easily leads to a contradiction. In fact, suppose that α, β, γ all have finite length. Then they form a transverse geodesic triangle that is not a tripod, a contradiction. If at least one of them is infinite, we can reduce to the previous case observing that transverse geodesic rays in Y at finite Hausdorff distance eventually coincide, so that we can cut off parts of α, β, γ to get once again a transverse geodesic triangle that is not a tripod.

So, suppose that the claim does not hold. Then we can find sequences of points $(x_n), (y_n)$ on (α_n) and a sequence (P_n) of elements of $\mathcal{P}$ so that $\mu - \lim d(x_n, P_n)/r_n, \mu - \lim d(y_n, P_n)/r_n = 0$ but $\mu - \lim d(x_n, y_n)/r_n > 0$. By Lemmas 1.8 and 1.10, the portion of α_n between x_n and y_n intersects $N_M(P_n)$, so that it contains a subgeodesic in $N_{tM}(P_n)$. It is easily seen that the length l_n of the maximal such subgeodesic has the property that $\mu - \lim l_n/r_n > 0$, in contradiction with $\operatorname{diam}(N_{tM}(P_n) \cap \alpha) \leq D_n$.

3. Distance formula

Let G be a relatively hyperbolic group and let $\mathcal{P}$ be the collection of all left cosets of peripheral subgroups. For $P \in \mathcal{P}$, let π_P be a closest point projection map onto P. Denote by $\widehat{G}$ the coned-off graph of G. Let $\{\{x\}\}_L$ denote x if $x > L$, and 0 otherwise. We write $A \approx_{\lambda,\mu} B$ if $A/\lambda - \mu \leq B \leq \lambda A + \mu$.

THEOREM 3.1 (Distance formula for relatively hyperbolic groups). *There exists L_0 so that for each $L \geq L_0$ there exist λ, μ so that the following holds. If $x, y \in G$ then*

$$d(x,y) \approx_{\lambda,\mu} \sum_{P \in \mathcal{P}} \{\{d(\pi_P(x), \pi_P(y))\}\}_L + d_{\widehat{G}}(x,y)\,. \tag{3.1}$$

Proof. Let us start with a preliminary fact. There exists σ so that whenever γ_i, for $i = 1,2$, is a geodesic with endpoints in $N_D(P_i)$ for some $P_i \in \mathcal{P}$ with $P_1 \neq P_2$ we have $\operatorname{diam}(\gamma_1 \cap \gamma_2) \leq \sigma = \sigma(D)$. (This is similar to [Hru10, Lemma 8.10], which could also be used for our purposes.) This follows from quasi-convexity of the peripheral subgroups (Lemma 1.10) combined with the existence of a bound depending only on δ on the diameter of $N_\delta(P_1) \cap N_\delta(P_2)$ (Lemma 1.9). So, we have the following estimate for D_0, M as in Lemma 1.13(1) for $K = 1$ and $C = 0$ and $\sigma = \sigma(D_0)$:

$$d(x,y) \geq \sum_{\substack{P \in \mathcal{P} \\ d(\pi_P(x), \pi_P(y)) \geq 2\sigma + 2M}} \big(d(\pi_P(x), \pi_P(y)) - 2\sigma - 2M\big)\,. \tag{3.2}$$

Write $A \lesssim_{\lambda,\mu} B$ or $B \gtrsim_{\lambda,\mu} A$ if $A \leq \lambda B + \mu$. In view of (3.2) and the fact that the inclusion $G \to \widehat{G}$ is lipschitz we have the inequality $\gtrsim_{\lambda,\mu}$ in (3.1). Hence we just need to show that any lift $\tilde{\alpha}$ of a geodesic α in $\widehat{G}$ satisfies $l(\tilde{\alpha}) \lesssim_{\lambda,\mu} R$, where R denotes the right hand side of (3.1), with x, y the endpoints of $\tilde{\alpha}$. Let $\alpha_1, \dots, \alpha_n$ be all maximal subgeodesics of $\tilde{\alpha}$ of length at least some large L' contained in some left cosets $P_1, \dots, P_n$. We have

$$l(\tilde{\alpha}) \approx_{\lambda,\mu} \sum l(\alpha_i) + d_{\widehat{G}}(x,y)\,.$$

The endpoints of α_i have uniformly bounded distance from $\pi_{P_i}(x), \pi_{P_i}(y)$ respectively by Lemma 1.13(2).

3.1 Sample application of the distance formula

We now provide an application of the distance formula. We need a preliminary lemma. We keep the notation set above.

PROPOSITION 3.2. *Let* $\phi\colon G_1 \to G_2$ *be a* (K, C)*-quasi-isometric embedding between relatively hyperbolic groups and let* $\mathcal{H}_i$ *be the collection of the left cosets of peripheral subgroups of* G_i. *Suppose that each* $H \in \mathcal{H}_1$ *is mapped in the* C*-neighborhood of some* $H' \in \mathcal{H}_2$ *and that for each* t *there exists* $L = L(t)$ *so that for each* $Q \in \mathcal{H}_2$ *there exists* $P \in \mathcal{H}_2$ *with* $\phi^{-1}(N_t(Q)) \subseteq N_L(P)$. *Then* ϕ *is a* (K', K')*-quasi-isometric embedding at the level of the coned-off graphs, where* $K' = K'(K, C, L(K))$.

Proof. In view of the characterization of projections given in Lemma 1.13(1) and the fact that left cosets of peripheral subgroups are coarsely preserved, we see that for each $x \in G_1$ and $P \in \mathcal{H}_1$ we have that $\pi_{\phi_\#(P)}(\phi(x))$ is at uniformly bounded distance from $\phi(\pi_P(x))$, where $\phi_\#(P) \in \mathcal{H}_2$ contains $\phi(P)$ in its C-neighborhood. Also, observe that if $d(\pi_Q(\phi(x)), \pi_Q(\phi(y)))$ is large, for some $x, y \in G_1$ and $Q \in \mathcal{H}_2$, then $Q = \phi_\#(P)$ for some $P \in \mathcal{H}_1$ so that $d(\pi_P(x), \pi_P(y))$ is large. This is because $\phi([x, y])$ contains a long sub-quasi-geodesic fellow-travelling Q, and the preimage of such a quasi-geodesic has to be contained in the neighborhood of some $P \in \mathcal{H}_1$.

Fix x, y and let $\widehat{\gamma}$ be a geodesic in $\widehat{G}_1$ connecting them. Let $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_n$ be the maximal sub-geodesics of $\widehat{\gamma}$ that do not contain an edge contained in any left coset of peripheral subgroup P so that $d(\pi_P(x), \pi_P(y))$ is larger than some suitable constant M. The lift of $\widehat{\gamma}_i$ is a quasi-geodesic, and in particular the image γ_i' of the lift via ϕ is also a quasi-geodesic. The observations we made at the beginning of the proof and the distance formula imply that γ_i' is a quasi-geodesic in $\widehat{G}_2$ as well. We see then that the image of $\widehat{\gamma}$ through ϕ is made of a collection of quasi-geodesics of $\widehat{G}$ (with uniformly bounded constants) and if M was chosen large enough those quasi-geodesics connect points on a geodesic $\widehat{\alpha}$ in $\widehat{G}$ from $\phi(x)$ to $\phi(y)$ by Lemma 1.15. It is not hard to check that $\phi(\widehat{\gamma})$ crosses these points in the same order as $\widehat{\alpha}$ does, which implies that $\phi(\widehat{\gamma})$ is a quasi-geodesic (again, with uniformly bounded constants). In fact, it suffices to show that γ_i' does not connect points on opposite sides in $\widehat{\alpha}$ of some $\phi_\#(P)$, where $d(\pi_P(x), \pi_P(y)) > M$. If it did, we would have that the projections of the endpoints of γ_i' on $\phi_\#(P)$ are far apart, which implies that the same holds for the endpoints of $\widehat{\gamma}_i$, but this is not the case in view of Lemma 1.13(2).

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