Quintuples of positive integers whose sums in pairs or in triples are squares

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QUINTUPLES OF POSITIVE INTEGERS WHOSE SUMS IN PAIRS OR IN TRIPLES ARE SQUARES

by Ajai CHOUDHRY

ABSTRACT. This paper is concerned with the two diophantine problems of finding five positive integers such that their sums either in pairs or in triples are perfect squares. The known parametric solutions of these problems are very cumbersome and not given explicitly in the published literature. Moreover, these solutions lead to numerical examples involving quite large integers. In this paper we first prove that there is a one-to-one correspondence between quintuples whose pairwise sums are squares and quintuples whose sums in triples are squares. Next we obtain all quintuples, with three or four of the five integers being distinct, such that their sums in pairs or in triples are perfect squares. We also obtain parametrized families of quintuples of distinct polynomials of degrees 6, 10, and 12 such that their sums in pairs or in triples are squares. These solutions yield quintuples of positive integers that are much smaller than the solutions generated by the known parametric solutions.

1. INTRODUCTION

The problem of finding positive integers whose sums in pairs or in triples are perfect squares dates back to the time of Diophantus who gave a method of dividing a given number into four parts such that the sum of any three of them is a perfect square and considered other related problems (see Heath [4, pp. 158, 210–212]). Several other diophantine problems requiring that two or more linear functions be made perfect squares (for instance, finding three integers such that their pairwise sums and differences are all perfect squares) have also attracted considerable attention (see Dickson [2, Chapter XV, pp. 443–458]).

This paper is concerned with the following two similar diophantine problems:

(i) to find five positive integers such that the sum of any two of them is a perfect square;

(ii) to find five positive integers such that the sum of any three of them is a perfect square.

A solution to the first problem was found by Baker in 1839 (as quoted by Dickson [2, p. 455]). Thatcher [8] describes several analytical solutions which yield numerical examples involving fairly large integers — the smallest such solution involves integers with 9 and 10 digits. He also mentions a solution, found by computer trials, containing integers with at most 6 digits [8, p. 27]. Lagrange [1]) gave a fairly general multi-parameter solution as well as a simplification of it but these solutions are very cumbersome and not given explicitly by the author [5]. In fact, even Lagrange's simplification leads to a parametric solution in terms of univariate polynomials of degree 30.

A solution to the second problem was first attempted in 1848 by Gill (see [2, p. 456] and [9]) who also gave extremely cumbersome formulae in terms of trigonometric functions to find the required integers. Gill could not find any solution in positive integers but suggested that his formulae could lead to such a solution. Wagon used Gill's formulae and modern computation techniques to find a solution consisting of five positive integers involving 48 and 49 digits [3, p. 268], and subsequently he found a solution consisting of integers of 20 and 21 digits [9].

We will show in Section 2 that these two diophantine problems are related to each other. In fact there is a one-to-one correspondence between quintuples whose pairwise sums are squares and quintuples whose sums in triples are squares. In Section 3 we first find all quintuples of rational numbers, three or four of which are distinct, such that their pairwise sums are squares, and we then find parametrized families of quintuples of distinct polynomials of degrees 6, 10 and 12 with their pairwise sums being squares. In Section 4 we apply the aforementioned one-to-one correspondence to the quintuples found in Section 3 to obtain quintuples whose sums in triples are squares. The solutions obtained in this paper are much simpler than the solutions of these problems published till now. We thus obtain numerical examples of quintuples of positive integers that are much smaller than the numerical examples generated by the parametric solutions found earlier, and which have the property that their sums in pairs or in triples are perfect squares. In Section 5 we state some open problems and briefly mention the known results related to them.

2. Preliminary remarks and lemmas

We first note that any solution to either of the two problems in rational numbers leads, on multiplying through by a suitable perfect square, to a

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[1]) a 20^{th} century namesake of the great 18^{th} century mathematician

solution in integers. It therefore suffices to obtain quintuples of rational numbers with the desired property.

We obtain in Lemma 1 the complete solution in rational numbers of a diophantine chain and use this result in Lemma 2 to obtain a very simple solution to the problem of finding all quadruples of rational numbers such that their pairwise sums are perfect squares. In Lemma 3 we show that the problem of finding $r+s$ rational numbers such that the sum of any r of them is a perfect k^{th} power is equivalent to the the problem of finding $r+s$ rational numbers such that the sum of any s of them is a perfect k^{th} power. It follows that the problem of finding five rational numbers the sum of any three of which is a square is equivalent to the problem of finding five rational numbers whose sums in pairs are squares. Wagon [9] has referred to this equivalence without giving any proof.

LEMMA 1. *The complete solution in rational numbers of the diophantine chain*

$$a_1^2+b_1^2=a_2^2+b_2^2=a_3^2+b_3^2\,, \tag{2.1}$$

is given by

$$\begin{aligned}
a_1 &= p_1q_1r_1+p_1q_2r_2-p_2q_1r_2+p_2q_2r_1\,,\\
a_2 &= p_1q_1r_1+p_1q_2r_2+p_2q_1r_2-p_2q_2r_1\,,\\
a_3 &= p_1q_1r_1-p_1q_2r_2+p_2q_1r_2+p_2q_2r_1\,,\\
b_1 &= -p_1q_1r_2+p_1q_2r_1-p_2q_1r_1-p_2q_2r_2\,,\\
b_2 &= -p_1q_1r_2+p_1q_2r_1+p_2q_1r_1+p_2q_2r_2\,,\\
b_3 &= p_1q_1r_2+p_1q_2r_1-p_2q_1r_1+p_2q_2r_2\,,
\end{aligned} \tag{2.2}$$

where p_i, q_i, r_i, $i=1,2$, are arbitrary rational parameters.

Proof. Using a complete solution of the equation $x^2+y^2=z^2+w^2$ similar to that given by Mordell [7, p. 15], the complete solutions of equations $a_1^2+b_1^2=a_2^2+b_2^2$ and $a_1^2+b_1^2=a_3^2+b_3^2$ may be written as

$$\begin{aligned}
a_1 &= p_1m_1+p_2m_2\,, & a_2 &= p_1m_1-p_2m_2\,,\\
b_1 &= p_1m_2-p_2m_1\,, & b_2 &= p_1m_2+p_2m_1\,,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
a_1 &= r_1s_1+r_2s_2\,, & a_3 &= r_1s_1-r_2s_2\,,\\
b_1 &= r_1s_2-r_2s_1\,, & b_3 &= r_1s_2+r_2s_1\,,
\end{aligned} \tag{2.4}$$

respectively where m_i, p_i, r_i, s_i, $i=1,2$, are arbitrary rational parameters. For these two solutions to be consistent, we equate the two values of a_1

given by (2.3) and (2.4) as well as the two values of b_1, and on solving the resulting equations for m_1, m_2, we get

$$(2.5)\qquad \begin{aligned} m_1 &= (p_1r_1s_1+p_1r_2s_2-p_2r_1s_2+p_2r_2s_1)/(p_1^2+p_2^2),\\ m_2 &= (p_1r_1s_2-p_1r_2s_1+p_2r_1s_1+p_2r_2s_2)/(p_1^2+p_2^2). \end{aligned}$$

Thus the complete solution of the diophantine chain (2.1) is given by (2.3) and (2.4) where m_1, m_2 are given by (2.5) and $p_i, r_i, s_i, i = 1, 2,$ are arbitrary parameters. We now simplify this solution by expressing the two independent rational parameters s_1, s_2 in terms of two new independent rational parameters q_1, q_2 defined by the following invertible linear transformation:

$$(2.6)\qquad s_1 = p_1q_1+p_2q_2, \qquad s_2 = p_1q_2-p_2q_1.$$

This gives (2.2) as the complete solution of (2.1) in terms of arbitrary rational parameters $p_i, q_i, r_i, i = 1, 2$.

LEMMA 2. *All quadruples of rational numbers whose pairwise sums are squares are given by*

$$(2.7)\qquad n_1 = s-a_1^2, \quad n_2 = s-a_2^2, \quad n_3 = s-a_3^2, \quad n_4 = a_1^2+b_1^2-s,$$

where $s = (a_1^2+a_2^2+a_3^2)/2$ *and* a_1, a_2, a_3, b_1 *are rational numbers defined by* (2.2).

Proof. If (n_1, n_2, n_3, n_4) is any such quadruple, the sum $\sum_{i=1}^4 n_i$ can be expressed in three ways as a sum of two squares, and thus there exist rational numbers $a_i, b_i, i = 1, 2, 3,$ satisfying the diophantine chain (2.1). The result now follows readily by solving the linear conditions satisfied by n_1, n_2, n_3, n_4 and using the result of Lemma 1.

LEMMA 3. *If* $n_1, n_2, \ldots, n_{r+s}$ *are* $r+s$ *rational numbers such that the sum of any* r *of them is a perfect* k^{th} *power, then the rational numbers* $N_1, N_2, \ldots, N_{r+s}$, *defined by* $N_j = (\sum_{i=1}^{r+s} n_i)/s - n_j$, $j = 1, 2, \ldots, (r+s)$, *are such that the sum of any* s *of them is a perfect* k^{th} *power. Conversely if the rational numbers* $N_1, N_2, \ldots, N_{r+s}$ *are such that the sum of any* s *of them is a perfect* k^{th} *power, then the rational numbers* $n_i = (\sum_{j=1}^{r+s} N_j)/r - N_i$, $i = 1, 2, \ldots, (r+s)$, *are such that the sum of any* r *of them is a perfect* k^{th} *power.*

Proof. If the rational numbers $n_1, n_2, \ldots, n_{r+s}$ are such that the sum of any r of them is a perfect k^{th} power, and the numbers $N_j, j = 1, 2, \ldots, (r+s)$, are defined as stated in the lemma, then taking $N_{j_1}, N_{j_2}, \ldots, N_{j_s}$ as any s of the $r+s$ numbers N_j, we have

$$\begin{aligned} N_{j_1}+N_{j_2}+\cdots+N_{j_s} &= \textstyle\sum_{i=1}^{r+s} n_i-(n_{j_1}+n_{j_2}+\cdots+n_{j_s}) \\ &= \text{sum of } r \text{ of the numbers } n_1, n_2, \ldots, n_{r+s} \\ &= \text{a } k^{\text{th}} \text{ power}. \end{aligned}$$

The converse is similarly seen to be true.

COROLLARY 1. *All n-tuples of rational numbers, whose sums taken $n-1$ at a time are k^{th} powers, are given by $N_j = (\sum_{i=1}^{n} m_i^k)/(n-1) - m_j^k$, $j=1, 2, \ldots, n$.*

Proof. This follows immediately from the lemma by taking $r=1$, $s=n-1$ and the rational numbers n_i as m_i^k, $i=1, 2, \ldots, n$.

Given a quintuple of rational numbers such that their pairwise sums are perfect squares, we may apply Lemma 3 to obtain quintuples whose sums in triples are perfect squares. It is to be noted, however, that positive quintuples whose pairwise sums are squares do not generally yield positive quintuples whose sums in triples are squares and vice versa. In fact, finding quintuples of positive integers whose sums in triples are squares is much more difficult as compared to finding positive quintuples whose pairwise sums are squares. This is also illustrated by the sizes of the known solutions to these problems as already mentioned in the Introduction.

3. QUINTUPLES WHOSE PAIRWISE SUMS ARE SQUARES

It is a trivial exercise to find all quintuples of rational numbers only two of which are distinct and whose pairwise sums are squares. We give below in Section 3.1 all quintuples of rational numbers, exactly three of which are distinct, whose pairwise sums are squares. In Section 3.2 we give all such quintuples in which exactly four of the numbers are distinct and finally in Section 3.3 we obtain, in parametric terms, several quintuples of distinct rational numbers all of whose pairwise sums are squares.

3.1 Quintuples in which three numbers are distinct

THEOREM 1. *All quintuples of rational numbers, three of which are distinct and whose pairwise sums are squares, are given by*

$$\begin{aligned} n_1 &= n_2 = n_3 = (qu+pv)^2/2\,, \\ n_4 &= \{(2p^2-q^2)u^2+2pquv+(2q^2-p^2)v^2\}/2\,, \\ n_5 &= (q^2u^2-6pquv+p^2v^2)/2\,, \end{aligned} \tag{3.1}$$

where p, q, u, v, are arbitrary rational parameters, and by

$$\begin{aligned} n_1 &= n_2 = (p^2r^2-6pqrs+q^2s^2)^2p^2/2\,, \\ n_3 &= n_4 = (p^2r^2+2pqrs-7q^2s^2)^2p^2/2\,, \\ n_5 &= -\{(p^2-2q^2)p^2r^2-(6p^2-4q^2)pqrs+(9p^2-2q^2)q^2s^2\} \\ &\quad \times\{p^2r^2+2pqsr-(8p^2-q^2)s^2\}/2\,, \end{aligned} \tag{3.2}$$

where p, q, r and s are arbitrary rational parameters.

Proof. The repeated number(s) in the quintuple must necessarily be of the type $k^2/2$ and so we may take the quintuple either as $k^2/2$, $k^2/2$, $k^2/2$, $x_1^2-k^2/2$, $x_2^2-k^2/2$ or as $k^2/2$, $k^2/2$, $m^2/2$, $m^2/2$, $x^2-k^2/2$.

In the first case, we must choose k, x_1 and x_2 such that $x_1^2+x_2^2-k^2=t^2$. The complete solution of this equation is well-known and is given in terms of arbitrary parameters p, q, u, v, by

$$x_1=pu+qv\,,\quad x_2=-qu+pv\,,\quad k=qu+pv\,,\quad t=pu-qv\,, \tag{3.3}$$

and this yields the quintuples given by (3.1). As a numerical example, taking $p=2$, $q=1$, $u=-4$, $v=1$, we get the quintuple 2, 2, 2, 47, 34.

In the second case, we must choose k, m and x such that the following conditions are satisfied:

$$\begin{aligned} k^2/2+m^2/2 &= y^2\,, \\ x^2-k^2/2+m^2/2 &= z^2\,. \end{aligned} \tag{3.4}$$

Adding these equations, we get $m^2+x^2=y^2+z^2$ for which the complete solution is given, as above, by

$$m=pu+qv\,,\quad x=-qu+pv\,,\quad y=pu-qv\,,\quad z=qu+pv\,, \tag{3.5}$$

and on substituting these values in the first of the two equations (3.4), we get the condition

$$k^2-p^2u^2+6puqv-q^2v^2=0\,. \tag{3.6}$$

This may be considered as a quadratic equation in three variables k, u, v, with a known solution $k = p$, $u = 1$, $v = 0$. Thus the complete solution of equation (3.6) is readily found and is given by

$$\begin{aligned} k &= p^2r^2 - 6pqrs + q^2s^2\,, \\ u &= -p^2r^2 + q^2s^2\,, \\ v &= -2ps(pr - 3qs)\,. \end{aligned} \tag{3.7}$$

Substituting these values of u, v in (3.5), we get the values of m and x and we thus obtain the quintuples given by (3.2). Taking $(p, q, r, s) = (2, 1, 1, 1)$, we get the quintuple of positive integers 98, 98, 2, 2, 23.

As in both cases we have found the complete solution of all diophantine equations concerned, (3.1) and (3.2) give all the quintuples with the stated properties.

3.2 Quintuples in which four numbers are distinct

THEOREM 2. *All quintuples of rational numbers, four of which are distinct and whose pairwise sums are squares, are given by*

$$\begin{gathered} n_1 = n_2 = \frac{k^2}{2}, \quad n_3 = k^2\left\{\frac{\phi_1^2(m_1, m_2, m_3)}{\phi_2^2(m_1, m_2, m_3)} - \frac{1}{2}\right\}, \\ n_4 = k^2\left\{\frac{\phi_1^2(m_2, m_3, m_1)}{\phi_2^2(m_1, m_2, m_3)} - \frac{1}{2}\right\}, \quad n_5 = k^2\left\{\frac{\phi_1^2(m_3, m_1, m_2)}{\phi_2^2(m_1, m_2, m_3)} - \frac{1}{2}\right\}, \end{gathered} \tag{3.8}$$

where m_1, m_2 *and* m_3 *are arbitrary rational parameters, and*

$$\begin{aligned} \phi_1(t_1, t_2, t_3) &= t_1^2t_2^2t_3^2 + 2t_1t_2^2t_3^2 - t_1^2t_2^2 - t_1^2t_3^2 + 4t_1t_2t_3^2 + t_2^2t_3^2 \\ &\quad -2t_1t_2^2 + 2t_1t_3^2 + t_1^2 + 4t_1t_2 - t_2^2 - t_3^2 - 2t_1 + 1\,, \\ \phi_2(m_1, m_2, m_3) &= m_1^2m_2^2m_3^2 - m_1^2m_2^2 - m_1^2m_3^2 - m_2^2m_3^2 \\ &\quad -8m_1m_2m_3 + m_1^2 + m_2^2 + m_3^2 - 1\,. \end{aligned} \tag{3.9}$$

Proof. As noted earlier, the repeated number in the quintuple must be of the type $k^2/2$ and so we may take the five numbers as $k^2/2, k^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2, x_3^2 - k^2/2$. For all pairwise sums to be squares, we must solve the following diophantine equations:

$$x_1^2 + x_2^2 - k^2 = y_3^2\,, \quad x_2^2 + x_3^2 - k^2 = y_1^2\,, \quad x_3^2 + x_1^2 - k^2 = y_2^2\,. \tag{3.10}$$

Each of the three equations in (3.10) may be replaced by two linear equations, and thus equations (3.10) are equivalent to the following six linear equations in variables x_i, y_i, $i = 1, 2, 3$:

$$
\begin{array}{lllllll}
 & k - x_1 & = & m_1(x_2 - y_3), & m_1(k + x_1) & = & x_2 + y_3, \\
(3.11) & k - x_2 & = & m_2(x_3 - y_1), & m_2(k + x_2) & = & x_3 + y_1, \\
 & k - x_3 & = & m_3(x_1 - y_2), & m_3(k + x_3) & = & x_1 + y_2,
\end{array}
$$

where m_1, m_2 and m_3 are arbitrary rational parameters. Solving these linear equations for x_i, y_i, $i = 1, 2, 3$, we get the values of x_i which immediately give us the quintuples mentioned in the theorem. As we have obtained the complete solution of equations (3.10), the theorem gives all quintuples with the stated properties.

As a numerical example, taking $m_1 = 2$, $m_2 = 2$, $m_3 = 3$, $k = 6$, we get the quintuple of positive integers 18, 18, 12082, 8082, 5607, all of whose pairwise sums are squares.

3.3 Quintuples in which all five numbers are distinct

To find quintuples of distinct numbers n_i, $i = 1, 2, \dots, 5$, whose pairwise sums are squares, we take n_i, $i = 1, 2, 3, 4$, as given by Lemma 2, and solve the diophantine equations,

$$
(3.12) \qquad n_1 + n_5 = x_1^2, \quad n_2 + n_5 = x_2^2, \quad n_3 + n_5 = x_3^2, \quad n_4 + n_5 = x_4^2.
$$

From the first two of the equations (3.12), we get $n_1 - n_2 = (x_1 - x_2)(x_1 + x_2)$ so that we readily obtain the solution

$$
(3.13) \qquad \begin{aligned}
x_1 &= p_2(r_1q_1 + r_2q_2)h_1 - p_1(r_1q_2 - r_2q_1)/h_1, \\
x_2 &= p_2(r_1q_1 + r_2q_2)h_1 + p_1(r_1q_2 - r_2q_1)/h_1,
\end{aligned}
$$

where h_1 is an arbitrary parameter. Similarly, from the last two of the equations (3.12), we get $n_3 - n_4 = (x_3 - x_4)(x_3 + x_4)$ so that we readily obtain the solution

$$
(3.14) \qquad \begin{aligned}
x_3 &= (p_1 + p_2)\{(r_1 - r_2)q_1 + (r_1 + r_2)q_2\}h_2/2 \\
&\quad + (p_1 - p_2)\{(r_1 + r_2)q_1 - (r_1 - r_2)q_2\}/(2h_2), \\
x_4 &= (p_1 + p_2)\{(r_1 - r_2)q_1 + (r_1 + r_2)q_2\}h_2/2 \\
&\quad - (p_1 - p_2)\{(r_1 + r_2)q_1 - (r_1 - r_2)q_2\}/(2h_2),
\end{aligned}
$$

where h_2 is an arbitrary parameter. Thus, we get $n_5 = x_1^2 - n_1$ from the first two equations, and $n_5 = x_3^2 - n_3$ from the last two equations of (3.12) with x_1, x_3 being given by (3.13) and (3.14). Equating the two values of n_5, we get $x_1^2 - x_3^2 = n_1 - n_3$ which gives the following condition:

$$\begin{aligned}
&[[\{(h_2^2+1)h_1p_1 + h_1(h_2^2+2h_1h_2-1)p_2\}q_1\\
&\quad+\{(h_1h_2^2-h_1-2h_2)p_1+(h_2^2+1)h_1p_2\}q_2]r_1\\
&\quad+[\{(-h_1h_2^2+h_1+2h_2)p_1-(h_2^2+1)h_1p_2\}q_1\\
&\quad+\{(h_2^2+1)h_1p_1+h_1(h_2^2+2h_1h_2-1)p_2\}q_2]r_2]\\
&\times[[\{(h_2^2+1)h_1p_1+h_1(h_2^2-2h_1h_2-1)p_2\}q_1\\
&\quad+\{(h_1h_2^2-h_1+2h_2)p_1+(h_2^2+1)h_1p_2\}q_2]r_1\\
&\quad+[\{(-h_1h_2^2+h_1-2h_2)p_1-(h_2^2+1)h_1p_2\}q_1\\
&\quad+\{(h_2^2+1)h_1p_1+h_1(h_2^2-2h_1h_2-1)p_2\}q_2]r_2]\\
&=16h_1^2h_2^2(p_1q_2-p_2q_1)(p_1q_1+p_2q_2)r_1r_2\,.
\end{aligned}\tag{3.15}$$

Equation (3.15) is of degree 2 in p_1, p_2, of degree 2 in q_1, q_2, and also of degree 2 in r_1, r_2, and may be solved in several ways. For instance, we can easily choose parameters q_1, q_2, such that the coefficient of r_2 in either of the two factors on the left-hand side of (3.15) vanishes and then the factor r_1 cancels out on both sides of (3.15) giving a linear equation in r_1, r_2 which is readily solved. For fixed numerical values of h_1, h_2, this leads to values of n_i, $i = 1, 2, \dots, 5$, in terms of polynomials of degree 12 in the parameters p_1, p_2. We obtain simpler solutions of (3.15) by taking $h_2 = (1-h_1)/(1+h_1)$. With this value of h_2, on transposing all terms to one side and substituting $q_1 = h_1q_2$, the factor $4q_2^2(p_1-h_1p_2)$ cancels out and the resulting linear equation in p_1, p_2 is readily solved so that we obtain the following solution of (3.15):

$$\begin{aligned}
h_2 &= (1-h_1)/(1+h_1)\,, \quad q_1 = q_2h_1\,,\\
p_1 &= h_1(h_1^2+1)(h_1^2-1)^3r_1^2+2h_1^2(h_1^2-1)(h_1^4-4h_1^2-1)r_1r_2\\
&\quad -8h_1^3(h_1^2+1)r_2^2\,,\\
p_2 &= -(h_1^2+1)(h_1^2-1)^3r_1^2-2h_1(h_1^2-1)(h_1^4+4h_1^2-1)r_1r_2\\
&\quad -8h_1^4(h_1^2+1)r_2^2\,.
\end{aligned}\tag{3.16}$$

Two more solutions of (3.15), similarly obtained, are given by

$$\begin{aligned}
&h_2 = (1-h_1)/(1+h_1)\,, \quad q_2 = q_1h_1\,,\\
&r_1 = (h_1^2+1)^2(p_1-h_1p_2)\{(h_1^4+4h_1^2-1)p_1+h_1(h_1^4-4h_1^2-1)p_2\}\,,\\
&r_2 = 2h_1(h_1^2-1)\{(3h_1^4+1)p_1^2-8h_1^3p_1p_2+h_1^2(h_1^4+3)p_2^2\}\,,
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
h_2 &= (1-h_1)/(1+h_1)\,,\\
p_1 &= -h_1(h_1+1)(h_1^2-2h_1-1)q_1-h_1(h_1-1)(h_1^2+1)q_2\,,\\
p_2 &= (h_1+1)(h_1^2+1)q_1-(h_1-1)(h_1^2-2h_1-1)q_2\,,\\
r_1 &= (q_1+h_1q_2)\{(2h_1^3-h_1^2-2h_1-1)q_1^2\\
&\qquad +h_1(h_1^3-2h_1^2+h_1+2)q_2^2\}\,,\\
r_2 &= (h_1q_1-q_2)\{-(h_1^3+h_1)q_1^2+(h_1^2-2h_1-1)^2q_1q_2\\
&\qquad -(h_1^3+h_1)q_2^2\}\,.
\end{aligned}\tag{3.18}$$

Now on using the solution (3.16), we obtain quintuples, whose sums in pairs are perfect squares, in terms of polynomials of degree 6 in r_1, r_2 with the coefficients themselves being polynomials in h_1 of degree at most 20. As this solution is cumbersome to write, we give below these quintuples explicitly in the special case when $h_1 = 2$, $r_2 = 1/2$:

$$
(3.19) \quad \begin{aligned}
n_1 &= 911250r_1^6 - 437400r_1^5 - 595512r_1^4 - 59616r_1^3 \\
&\quad +606528r_1^2 + 460800r_1 + 80000\,, \\
n_2 &= -255150r_1^6 - 1117800r_1^5 - 102888r_1^4 + 1590816r_1^3 \\
&\quad +854272r_1^2 + 19200r_1 - 22400\,, \\
n_3 &= 911250r_1^6 + 1312200r_1^5 - 336312r_1^4 - 1269216r_1^3 \\
&\quad -718272r_1^2 - 153600r_1 + 80000\,, \\
n_4 &= 255150r_1^6 + 1117800r_1^5 + 2185137r_1^4 + 1185516r_1^3 \\
&\quad +71172r_1^2 - 19200r_1 + 22400\,, \\
n_5 &= 911250r_1^6 + 4422600r_1^5 + 7050888r_1^4 + 5815584r_1^3 \\
&\quad +3083328r_1^2 + 806400r_1 + 80000\,.
\end{aligned}
$$

Similarly the solutions (3.17) and (3.18) of equation (3.15) lead to quintuples given by polynomials of degrees 6 and 10 respectively in r_1, r_2 where in both cases the coefficients are again polynomials in h_1. By assigning specific numerical values to h_1, we can obtain reasonably simple solutions of our problem.

It is readily seen that the solution (3.19) leads to quintuples of positive integers when r_1 takes any rational value in the intervals $[-3.92, -2.72]$, $[-1.09, -1.03]$, $[-0.26, -0.24]$, $[0.14, 0.21]$ and $[1.03, 1.22]$. We give below two such quintuples obtained by taking $r_1 = -3$ and $r_1 = 1/5$:

(i) 728118962, 41943538, 346540562, 60086663, 29121362.
(ii) 121818578, 17374226, 6459698, 21725783, 264932978.

4. Quintuples whose sums in triples are squares

It follows from Lemma 3 that for any quintuple n_i, $i = 1, 2, \ldots, 5$, whose sums in pairs are squares, there is a corresponding quintuple N_j, $j = 1, 2, \ldots, 5$, given by $N_j = (\sum_{i=1}^{5} n_i)/3 - n_j$, $j = 1, 2, \ldots, 5$, whose sums in triples are squares. Thus, Theorems 1 and 2 of Section 3 together with Lemma 3 immediately give all quintuples of rational numbers, three or four of which are distinct, whose sums in triples are perfect squares. As numerical examples, on applying Lemma 3 to the quintuple obtained by taking $(p, q, u, v) = (2, 1, 3, 12)$ in (3.1), we get the quintuple of positive integers 12, 12, 12, 417, 300 whose sums in triples are squares and on taking $m_1 = -5/2$, $m_2 = -7/2$, $m_3 = -7/6$, $k = 25425$ in Theorem 2, we get the following quintuple of positive integers with this property:

$$10944723\,, \quad 10944723\,, \quad 183660123\,, \quad 461011179\,, \quad 1759323\,.$$

Similarly, from the quintuples obtained in Section 3.3, we immediately obtain parametrized families of quintuples, whose sums in triples are perfect squares, in terms of polynomials of degrees 6, 10 and 12. As a specific example, we note that the quintuples given by (3.19) immediately lead to quintuples, given by univariate sixth degree polynomials, whose sums in triples are squares. However this solution does not yield any numerical examples of quintuples of positive integers. To obtain quintuples of positive integers, we have to take a different value of h_1 in the solution (3.16). In fact, taking $h_1 = 5/6$ and working out quintuples with pairwise sums being squares as described in Section 3.3, and then applying Lemma 3, we get, after a slight change in variables, the quintuples N_j, $j = 1, \ldots, 5$, whose sums in triples are perfect squares and which are given by

$$\begin{aligned}
N_1 &= -3r_1(r_1-2)(124547746638400r_1^3 - 182188063433281r_1^2 \\
&\qquad +364376126866562r_1 + 90223974086400)\,,\\
N_2 &= 131290275600000r_1^6 - 881152253388800r_1^5 - 741639023231757r_1^4 \\
&\qquad +3678538394903028r_1^3 - 3340198492079028r_1^2 \\
&\qquad -1110929686099200r_1 + 840257763840000\,,\\
N_3 &= -3r_1(r_1-2)(112779967608000r_1^3 - 249856043998081r_1^2 \\
&\qquad +499712087996162r_1 - 189861947193600)\,,\\
N_4 &= 4412812041000r_1^6 + 151518703248000r_1^5 + 456681910914714r_1^4 \\
&\qquad -2165067546482856r_1^3 + 2538709945634856r_1^2 \\
&\qquad -327155916384000r_1 + 28241997062400\,,\\
N_5 &= 3r_1(r_1-2)(237327433992000r_1^3 + 107459583602881r_1^2 \\
&\qquad -214919167205762r_1 - 99637973107200)\,,
\end{aligned}$$

where r_1 is an arbitrary parameter. This leads to examples of quintuples in positive integers when r_1 takes any rational value in the intervals $[-0.51, -0.394]$ and $[1.59, 1.67]$. Two numerical examples of such quintuples, obtained by taking $r_1 = -1/2$ and $r_1 = 5/3$, and suitably scaling the results, are as follows:

(i) 333763076484003, 367756719375563, 1208620204287123, 720132418225155, 761277923316603.

(ii) 1345142316326403, 538999066496163, 6263536168803, 776258666609790, 267105220607523.

These solutions, involving integers of at most 16 digits, are much smaller than the aforementioned solutions found by Wagon. We obtain even smaller examples of such quintuples by using other solutions found in Section 3.3. For instance, using the solution (3.18) of equation (3.15), and taking $h_1 = 4$, $q_2 = 1$, we first get quintuples whose pairwise sums are squares, and on applying Lemma 3, we get quintuples, given by univariate tenth degree polynomials in q_1, whose sums in triples are squares. Denoting by $(c_0, c_1, \ldots, c_n)$ the polynomial $c_0 q_1^n + c_1 q_1^{n-1} + \cdots + c_n$, we may write these quintuples of polynomials as follows:

$$
\begin{array}{rrrr}
(-3697822725, & -30327296550, & -56160813021, & -103842240102, \\
-109817488116, & -165730367070, & -99417915405, & -35308230390, \\
55807039179, & 87672646728, & 53387465904), & \\
(8304231675, & -6594487350, & 21824444595, & 31373372970, \\
78621501420, & 134536973490, & 152014224771, & 162074905722, \\
116525894859, & 81312977736, & 47011228848), & \\
(8304231675, & 33036490650, & 55814738835, & 133337577150, \\
102847820292, & 160077561894, & 73296428355, & 124922040426, \\
146173940811, & 76326695496, & 47011228848), & \\
(1245841050, & 11387154900, & 60350887875, & 42790166280, \\
120489158862, & 19198226304, & -43338050685, & -174461502564 \\
-210383902134, & -146449532304, & -88263469152), & \\
(3133191675, & 9542732250, & 24913193475, & 62984304090, \\
134480289132, & 212627208834, & 338342997555, & 263447246826, \\
221460896331, & 67481132616, & 41263209648). &
\end{array}
$$

We get quintuples consisting only of positive numbers if we assign to q_1 any rational value in the interval $[-6.43, -3.23]$. Two numerical examples, obtained by taking $q_1 = -5$ and $q_1 = -7/2$ and suitably scaling the results, are as follows:

$$
\begin{array}{llll}
\text{(i)} & 689438025051\,, & 8653578146587\,, & 2494643376462\,, \\
& 1026253246587\,, & 1576737123387\,. & \\
\text{(ii)} & 102912949803\,, & 508481852619\,, & 31576638603\,, \\
& 117938277678\,, & 70988581803\,. &
\end{array}
$$

5. Some open problems

The two problems of quintuples considered above may be extended in two ways. We could ask for six or more integers whose sums in pairs or in triples are perfect squares. At present only one numerical example of a sextuple of integers is known such that all sums in pairs are perfect squares [6, p. 94]. It would be interesting to determine whether or not there exist infinitely many such examples of sextuples whose pairwise sums are all perfect squares.

We could also ask for sets of integers such that their sums in pairs or in triples are perfect cubes. While we know of examples of quadruples such that all the six pairwise sums are cubes as well as quintuples such that nine of the ten pairwise sums are cubes [1], no quintuple is at present known for which all the ten pairwise sums are cubes.

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