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## Note on the distortion of $(2, q)$ -torus knots

Luca STUDER

**Abstract.** We show that the distortion of the  $(2, q)$ -torus knot is not bounded linearly from below.

**Mathematics Subject Classification (2010).** Primary: 57M25.

**Keywords.** Distortion, torus knots.

### 1. Introduction

The notion of distortion was introduced by Gromov [GPL]. If  $\gamma$  is a rectifiable simple closed curve in  $\mathbf{R}^3$ , then its distortion  $\delta$  is defined as

$$\delta(\gamma) = \sup_{v, w \in \gamma} \frac{d_\gamma(v, w)}{|v - w|},$$

where  $d_\gamma(v, w)$  denotes the length of the shortest arc connecting  $v$  and  $w$  in  $\gamma$  and  $|\cdot|$  denotes the euclidean norm on  $\mathbf{R}^3$ . For a knot  $K$ , its distortion  $\delta(K)$  is defined as the infimum of  $\delta(\gamma)$  over all rectifiable curves  $\gamma$  in the isotopy class  $K$ . Gromov [Gro] asked in 1983 if every knot  $K$  has distortion  $\delta(K) \leq 100$ . The question was open for almost three decades until Pardon gave a negative answer. His work [Par] presents a lower bound for the distortion of simple closed curves on closed PL embedded surfaces with positive genus. Pardon showed that the minimal intersection number of such a curve with essential discs of the corresponding surface bounds the distortion of the curve from below. In particular for the  $(p, q)$ -torus knot he showed that  $\delta(T_{p,q}) \geq \min(p, q)/160$ . By considering a standard embedding of  $T_{p,p+1}$  into a torus of revolution one obtains  $\delta(T_{p,p+1}) \leq \text{const} \cdot p$ , hence for  $q = p + 1$  Pardon's result is sharp up to a constant.

An alternative proof for the existence of families with unbounded distortion was given by Gromov and Guth [GG]. In both works the answer to Gromov's question was obtained by estimating the conformal length, which is up to a

constant a lower bound for the distortion of rectifiable closed curves. However, the conformal length is in general not a good estimate for the distortion. For example, one finds easily an embedding of the  $(2, q)$ -torus knot with conformal length  $\leq 100$  and distortion  $\geq q$  by looking at standard embeddings into a torus of revolution with suitable dimensions. In particular, neither Pardon's nor Gromov and Guth's arguments yield lower bounds for  $\delta(T_{2,q})$ . While Pardon conjectures that  $\lim_{q \rightarrow \infty} \delta(T_{2,q}) = \infty$  and that there are to his knowledge no known embeddings of  $T_{2,q}$  with sublinear distortion [Par, p. 638], Gromov and Guth [GG, p. 2588] write that the distortion of  $T_{2,q}$  appears to be approximately  $q$ . In this article we show that the growth rate of  $\delta(T_{2,q})$  is in fact sublinear in  $q$ .

**Theorem.** *Let  $q \geq 50$ . Then  $\delta(T_{2,q}) \leq 7q/\log q$ . In particular the distortion of the  $(2, q)$ -torus knot is not bounded linearly from below.*

With the same technique as used in this article and somewhat more effort one can give an embedding  $\gamma_q$  of  $T_{2,q}$  with  $\delta(\gamma_q) \sim \frac{\pi}{2} \frac{q}{\log q}$ . Moreover, a more technical proof yields that this asymptotical upper bound for  $\delta(T_{2,q})$  is sharp for those embeddings of  $T_{2,q}$  that project orthogonally onto a standard knot diagram. This leads to the following question.

**Question.** *Is  $\delta(T_{2,q})$  up to a constant asymptotically equal to  $q/\log q$ ? And if yes, is the constant equal to  $\pi/2$ ?*

## 2. Proof of the Theorem

In order to prove the Theorem we need to give for every odd integer  $q \geq 50$  an embedding  $\gamma$  of the  $(2, q)$ -torus knot with distortion smaller or equal to  $7q/\log q$ . The idea is to use a logarithmic spiral. Let  $S$  be a logarithmic spiral of unit length starting at its center  $0 \in \mathbf{R}^3$  and ending at some  $u \in \mathbf{R}^3$ . An elementary calculation shows that its distortion is equal to  $1/|u|$ . For another path  $\alpha \subset \mathbf{R}^3$  of unit length and diameter  $\leq 2|u|$  with endpoints  $\{v, w\} = \partial\alpha$  we get

$$\delta(\alpha) \geq \frac{d_\alpha(v, w)}{|v - w|} = \frac{1}{|v - w|} \geq \frac{1}{2|u|} = \frac{\delta(S)}{2}.$$

Hence up to at most a factor 2 the logarithmic spiral has the smallest distortion among all paths for a prescribed pathlength-pathdiameter-ratio. It seems therefore natural to pack the  $q$  windings of the  $(2, q)$ -torus knot into a logarithmic spiral in order to minimize distortion.

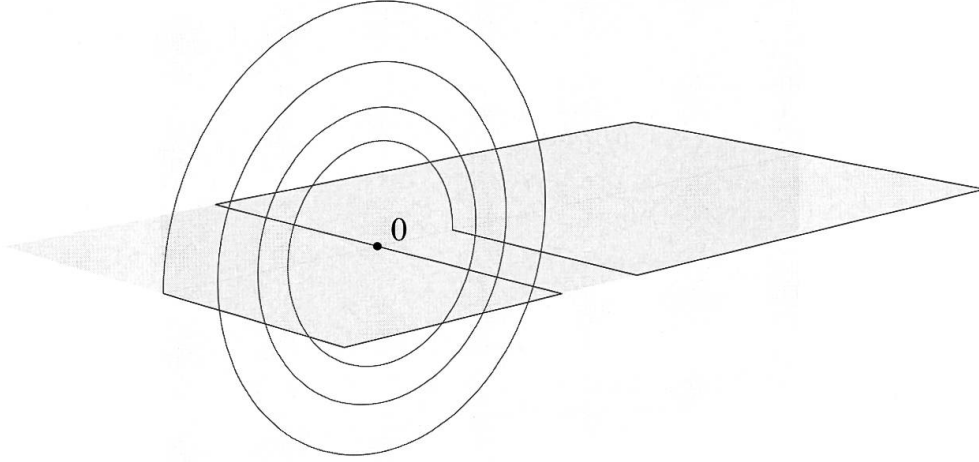


FIGURE 1  
The embedding  $\gamma$  for  $q = 7$

We define the embedding  $\gamma$  as the union of a segment of the logarithmic spiral  $S$  with slope  $k = \log(q)/2\pi q$  and a piecewise linear part  $L$ , see Figure 1. The segment of the logarithmic spiral  $S$  is contained in the vertical  $(x, z)$  plane and parametrized by

$$\varphi : [0, \pi q] \rightarrow \mathbf{R}^2, \quad \varphi(s) = e^{ks} \cdot \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix},$$

see Figures 1 and 2. The segment of the piecewise linear part  $L$  is in the horizontal  $(x, y)$  plane, see Figures 1 and 3. Note that

$$|\varphi(\pi q)| = e^{k\pi q} = \sqrt{q} \quad \text{and} \quad |\varphi(0)| = 1,$$

hence the lengths defining  $L$  in Figure 3 are chosen such that the union  $\gamma$  of  $S$  and  $L$  is the simple closed curve illustrated in Figure 1. The linear segments  $L_1$  and  $L_2$  indicated in Figure 3 are named because of their special role in the following computations.

To see that the obtained curve is an embedded  $(2, q)$ -torus knot, we perturb  $\gamma$ , see Figure 4. This simple closed curve is ambient isotopic in  $\mathbf{R}^3$  to  $\gamma$  and if we project it onto the  $(x, y)$  plane, we see a well known diagram of the  $(2, q)$ -torus knot, see Figure 5.

We now estimate the distortion of  $\gamma$ . One has to show that

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{7q}{\log q}$$

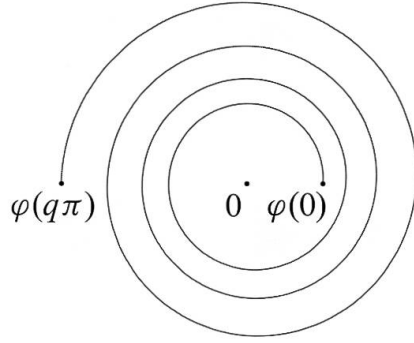


FIGURE 2  
The logarithmic spiral  $S$  in the  $(x, z)$  plane

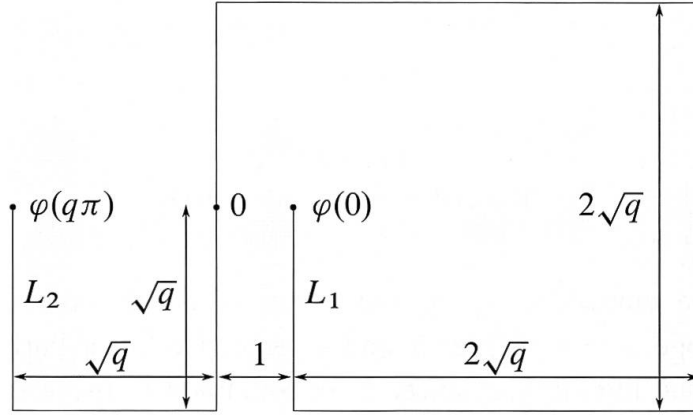


FIGURE 3  
The linear part  $L$  in the  $(x, y)$  plane

for all pairs of points  $v, w \in \gamma$ . An easy computation shows that

$$\frac{1}{k} \cdot \sqrt{2k^2 + 1} = \frac{2\pi q}{\log q} \cdot \sqrt{2(\log q / 2\pi q)^2 + 1} \leq \frac{7q}{\log q}$$

for all integers  $q \geq 2$ . Therefore, it suffices to show that

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{2k^2 + 1}}{k}.$$

In order to do this, we distinguish four cases.

*Case I:*  $v, w \in S$ . Let  $0 \leq s \leq t \leq \pi q$ ,  $v = \varphi(s)$ ,  $w = \varphi(t)$ . From

$$|\varphi'(r)| = \left| \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} k e^{kr} \\ e^{kr} \end{pmatrix} \right| = \left| \begin{pmatrix} k e^{kr} \\ e^{kr} \end{pmatrix} \right| = \sqrt{k^2 + 1} \cdot e^{kr},$$

we get

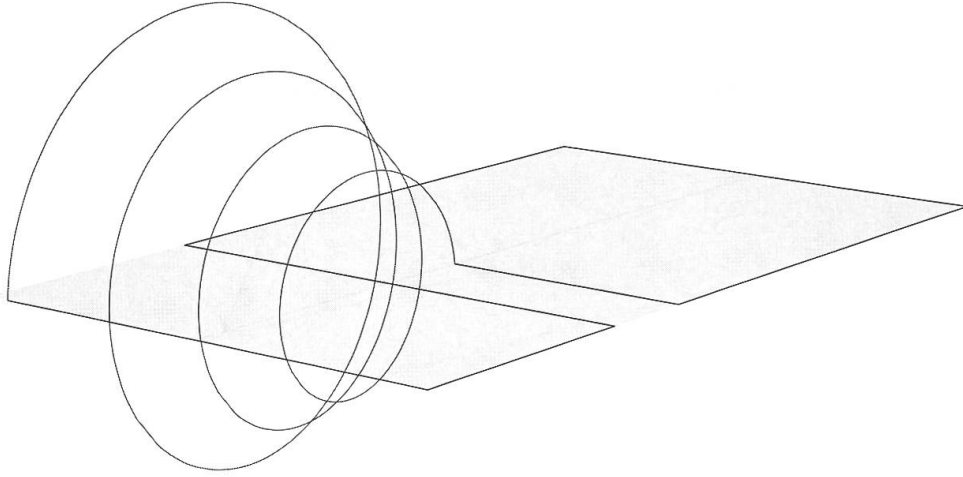


FIGURE 4  
Perturbation of  $\gamma$

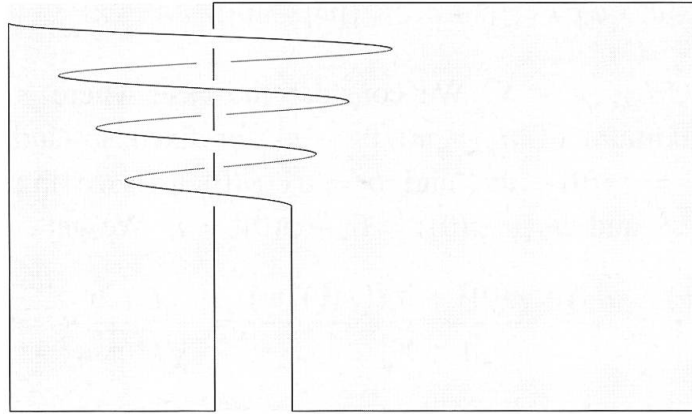


FIGURE 5  
Projection onto the  $(x, y)$  plane

$$\begin{aligned}
 d_\gamma(v, w) &\leq d_S(v, w) \\
 &= \int_s^t |\varphi'(r)| dr \\
 &= \sqrt{k^2 + 1} \int_s^t e^{kr} dr \\
 &= \frac{\sqrt{k^2 + 1}}{k} \cdot (e^{kt} - e^{ks}) \\
 &= \frac{\sqrt{k^2 + 1}}{k} \cdot (|\varphi(t)| - |\varphi(s)|) \\
 &= \frac{\sqrt{k^2 + 1}}{k} \cdot (|w| - |v|).
 \end{aligned}$$

Since  $|w - v| \geq |w| - |v|$ , we conclude that

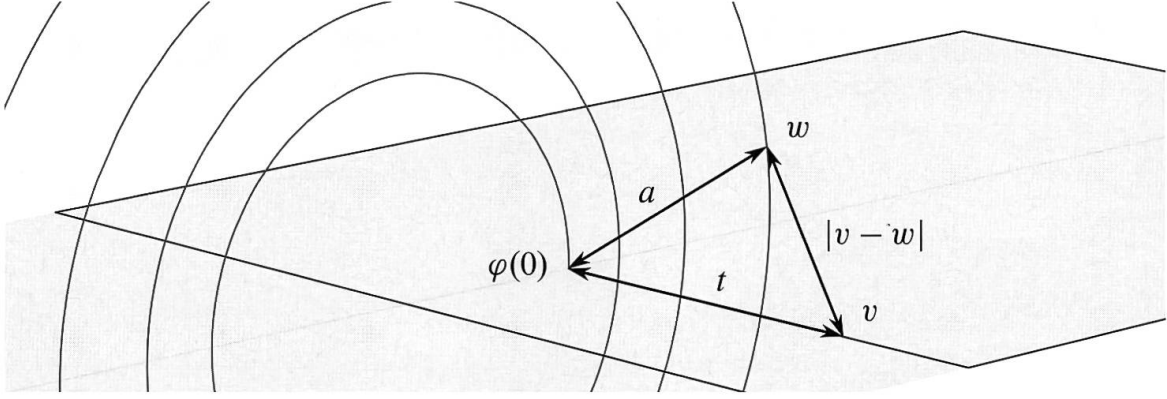


FIGURE 6  
Visualization of Case 2

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{k^2 + 1}}{k} \cdot \frac{(|w| - |v|)}{(|w| - |v|)} = \frac{\sqrt{k^2 + 1}}{k}.$$

*Case 2:*  $v \in L_1 \cup L_2$ ,  $w \in S$ . We consider the case where  $v \in L_1$ . The idea is to find the maximum of  $d_\gamma(v, w)/|v - w|$  for fixed  $w$  and varying  $v$ . Let  $t = |v - \varphi(0)|$ ,  $a = |\varphi(0) - w|$ , and  $b = d_S(\varphi(0), w)$ , see Figure 6. Note that  $|v - w| = \sqrt{t^2 + a^2}$  and  $d_\gamma(v, \varphi(0)) = |v - \varphi(0)| = t$ . We get

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{d_\gamma(v, \varphi(0)) + d_S(\varphi(0), w)}{|v - w|} = \frac{t + b}{\sqrt{t^2 + a^2}} =: f(t).$$

Deriving  $f$  with respect to  $t$  yields a unique critical point at  $t = a^2/b$ :

$$0 = f'(t) = \frac{a^2 - bt}{(a^2 + t^2)^{3/2}} \iff t = a^2/b.$$

Since  $a^2/b$  is the only critical point,  $f(\infty) = 1 \leq b/a = f(0)$  and

$$f(0) = \frac{b}{a} \leq \frac{\sqrt{a^2 + b^2}}{a} = \frac{\frac{a^2}{b} + b}{\sqrt{(\frac{a^2}{b})^2 + a^2}} = f(a^2/b),$$

$a^2/b$  must be a global maximum. Consequently we get

$$\begin{aligned} \frac{d_\gamma(v, w)}{|v - w|} &\leq \frac{\sqrt{a^2 + b^2}}{a} \\ &= \sqrt{1 + \left(\frac{b}{a}\right)^2} \\ &= \sqrt{1 + \left(\frac{d_S(\varphi(0), w)}{|\varphi(0) - w|}\right)^2} \end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{Case 1}}{\leq} \sqrt{1 + \left(\frac{\sqrt{k^2+1}}{k}\right)^2} \\
&= \frac{\sqrt{2k^2+1}}{k}.
\end{aligned}$$

In the case where  $v \in L_2$ , we make the estimate with the path that connects  $v$  with  $w$  through  $\varphi(\pi q)$ . It works exactly the same and yields the same estimate.

*Case 3:*  $v, w \in L$ . Consider Figure 3 and note that all pairs of points  $v, w \in L$  of euclidean distance  $< 1$  are either on the same linear segment or on neighboring linear segments of  $L$ . It is easy to see that such pairs of points cannot cause distortion  $> \sqrt{2}$ . For the pairs of points  $v, w \in L$  of euclidean distance  $\geq 1$  we get

$$\frac{d_Y(v, w)}{|v - w|} \leq d_L(\varphi(0), \varphi(\pi q)) = 11\sqrt{q} + 1.$$

A direct calculation shows that

$$11\sqrt{q} + 1 \leq \frac{2\pi q}{\log q} = \frac{1}{k}$$

for  $q \geq 50$ .

*Case 4:*  $v \in L \setminus (L_1 \cup L_2), w \in S$ . Note that for these pairs of points we have  $|v - w| \geq |w|$ . We estimate  $d_Y(v, w)$  using results of Cases 1 and 3:

$$\begin{aligned}
d_Y(v, w) &\leq d_L(v, \varphi(0)) + d_S(\varphi(0), w) \\
&\leq \frac{1}{k} + \frac{\sqrt{k^2+1}}{k} \cdot (|w| - 1) \\
&\leq \frac{\sqrt{k^2+1}}{k} \cdot |w|.
\end{aligned}$$

We conclude that

$$\frac{d_Y(v, w)}{|v - w|} \leq \frac{\frac{\sqrt{k^2+1}}{k} \cdot |w|}{|w|} = \frac{\sqrt{k^2+1}}{k},$$

which finishes the proof.

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