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Spines for amoebas of rational curves

Grigory Mikhalkin and Johannes Rau

Abstract. To every rational complex curve $C \subset (\mathbf{C}^\times)^n$ we associate a rational tropical curve $\Gamma \subset \mathbf{R}^n$ so that the amoeba $\mathcal{A}(C) \subset \mathbf{R}^n$ of C is within a bounded distance from Γ. In accordance with the terminology introduced in [PR], we call Γ the *spine* of $\mathcal{A}(C)$. We use spines to describe tropical limits of sequences of rational complex curves.

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1. Introduction

As suggested by Gelfand, Kapranov and Zelevinsky [GKZ], an algebraic variety V in the complex torus $(\mathbf{C}^\times)^n = (\mathbf{C} \setminus \{0\})^n$ can be visualized through its *amoeba*. Namely, consider the map $\mathrm{Log} : (\mathbf{C}^\times)^n \to \mathbf{R}^n$ defined by $\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|)$. The image $\mathrm{Log}(V)$ is called the amoeba of V. It possesses many geometric properties reflecting those of V. Furthermore, amoebas can be used as intermediate geometric objects between complex and tropical varieties, cf. [Mik1]. Passare and Rullgård [PR] have identified a tropical variety (called *the Passare–Rullgård spine*) inside $\mathrm{Log}(V)$ in the case when $V \subset (\mathbf{C}^\times)^n$ is a hypersurface, i.e., $\dim V = n - 1$.

In the paper we focus on the case when V is a rational curve. In this case we associate to V a tropical rational curve in $\mathbf{R}^n$, called *spine*, whose distance to $\mathrm{Log}(V)$ (in Hausdorff metric on sets in $\mathbf{R}^n$) is universally bounded in terms of the degree of V. Our spine is not necessarily contained in $\mathrm{Log}(V)$.

In the case $n = 2$ a rational curve $V \subset (\mathbf{C}^\times)^2$ is a hypersurface, so the Passare–Rullgård spine of V is also defined as a tropical curve in $\mathbf{R}^2$. Nevertheless this tropical curve does not have to be a rational curve (see Remark 2.3).

We are freely using some basic notions from tropical geometry here. For details, we refer the reader to [MR, MS].

2. The main statements

A *complex rational curve* in $(\mathbf{C}^\times)^n$ is a holomorphic map $f : S \to (\mathbf{C}^\times)^n$ from a Riemann sphere with k labelled punctures $S = \mathbf{CP}^1 \setminus \{\alpha_1, \dots, \alpha_k\}$ to $(\mathbf{C}^\times)^n$. To each puncture α_i we associate the integer vector $\delta(\alpha_i) \in \mathbf{Z}^n$ whose j-th coordinate is given by the order of vanishing $-\operatorname{ord}_{\alpha_i}(z_j \circ f)$. The sequence of vectors $\Delta(f) = (\delta(\alpha_1), \dots, \delta(\alpha_k)) \in \mathbf{Z}^{k \times n}$ is called the *toric degree* of $f : S \to (\mathbf{C}^\times)^n$. Note that $\sum_{i=1}^k \delta(\alpha_i) = 0$.

A *tropical rational curve* in $\mathbf{R}^n$ is a tropical morphism $h : \Gamma \to \mathbf{R}^n$, where $\overline{\Gamma}$ is a compact smooth rational tropical curve with k labelled ends $a_1, \dots, a_k$ and $\Gamma = \overline{\Gamma} \setminus \{a_1, \dots, a_k\}$. This amounts to the following list of properties:

- the graph $\overline{\Gamma}$ is a tree with k labelled ends $a_1, \dots, a_k$;
- the open subset Γ carries a complete inner metric such that each leaf and bounded edge is isometric to $[0, \infty)$ and $[0, l(e)]$, respectively. In the second case, $l(e) \in \mathbf{R}_>$ is called the *length* of e;
- the map h is affine on each edge;
- for each oriented edge e, the vector of derivatives $\partial h(e)$ with respect to travelling along e with unit speed is integer, $\partial h(e) \in \mathbf{Z}^n$;
- at each vertex $v \in \Gamma^\circ$, if $e_1, \dots, e_k$ denote the adjacent outgoing edges, the *balancing condition*
$$\sum_{i=1}^m \partial h(e_i) = 0$$
is satisfied.

Let $l_1, \dots, l_k$ denote the leaves adjacent to the ends $a_1, \dots, a_k$, oriented towards the ends, and set $\delta(a_i) := \partial h(l_i)$, $i = 1, \dots, k$. The sequence of vectors $\Delta(h) = (\delta(a_1), \dots, \delta(a_k)) \in \mathbf{Z}^{k \times n}$ is called the *toric degree* of $h : \Gamma \to \mathbf{R}^n$. The balancing condition implies $\sum_{i=1}^k \delta(a_i) = 0$.

We consider the coordinate-wise logarithm map

$$\begin{aligned} \operatorname{Log} \colon (\mathbf{C}^\times)^n &\to \mathbf{R}^n, \\ (z_1, \dots, z_n) &\mapsto (\log|z_1|, \dots, \log|z_n|). \end{aligned}$$

The image of a complex curve X under this map is called the *amoeba* of X.

Tropical spines. Our first main theorem states that the amoeba of a complex rational curve of given toric degree can be approximated by a tropical rational curve of the same degree up to a constant which only depends Δ, but *not* on the specific curve.

Let us fix a collection of integer vectors $\Delta = (\delta_1, \dots, \delta_k)$, $\delta_i \in \mathbf{Z}^n$ such that $\sum_{i=1}^{k} \delta_i = 0$, called a *toric degree* in the following.

Theorem 2.1. *For any toric degree Δ, there exists a positive constant $\epsilon = \epsilon(\Delta) \geq 0$ having the following property. For any complex rational curve $f : S \to (\mathbf{C}^\times)^n$ of toric degree Δ, there exists a tropical rational curve $h : \Gamma \to \mathbf{R}^n$ of toric degree Δ such that*

$$\text{(1)} \qquad \operatorname{Log}\big(f(S)\big) \subset U_\epsilon\big(h(\Gamma)\big) \quad \text{and} \quad h(\Gamma) \subset U_\epsilon\Big(\operatorname{Log}\big(f(S)\big)\Big).$$

Here, $U_\epsilon(X)$ denotes the ϵ-neighbourhood of a set X in $\mathbf{R}^n$.

Remark 2.2. Since all norms on $\mathbf{R}^n$ are equivalent, the statement of the theorem does not depend on the choice of norm. In practice, we will work with the maximum norm $\|\,.\,\|_\infty$.

Remark 2.3. In [PR], the authors associate to any complex hypersurface $V_f \subset (\mathbf{C}^\times)^n$ a tropical hypersurface $S_f \subset \operatorname{Log}(V_f) \subset \mathbf{R}^n$, called the *spine* of V_f, and show that S_f is a deformation retract of $\operatorname{Log}(V_f)$. The construction overlaps with ours in the case $n = 2$, i.e., when $C = V_f$ is a planar curve. However, note that in general S_f can be of too large genus. In particular, assuming that C is rational (as considered in this paper), the spine S_f is not necessarily rational (i.e. parametrised by a tropical rational curve $h : \Gamma \to \mathbf{R}^2$). A counterexample can be constructed from a counterexample to the similar statement that reducibility of C does not imply reducibility of S_f. To find such an example, we may arrange a generic line $L_1 \subset (\mathbf{C}^*)^2$ and the Cremona transform of a second line $\operatorname{Cr}(L_2)$ such that the union of their amoebas forms a contractible domain in $\mathbf{R}^2$ while the two spines S_1 and S_2 intersect transversally (in two points). In this case, among the tropical curves contained in $\operatorname{Log}(L_1 \cup \operatorname{Cr}(L_2))$ and of correct degree, there is a unique reducible curve (namely $S_1 \cup S_2$) as well as a unique curve being a deformation retract of $\operatorname{Log}(L_1 \cup \operatorname{Cr}(L_2))$ (namely the spine of $L_1 \cup \operatorname{Cr}(L_2)$). Since these two curves are not equal the claim follows. As mentioned above, the example can be modified to the case of an irreducible rational curve by completing the picture as indicated by the dashed lines. The related question to which extent S_f displays the singularities of V_f has been studied in [Lan] in the case of generalized simple Harnack curves.

Despite this behaviour of Passare–Rullgård spines, one may proceed in spirit of Theorem 2.1 and try to find universal bounds $\epsilon = \epsilon(\operatorname{NP}(f))$, only depending on the Newton polytope of f, such that

$$\text{(2)} \qquad \operatorname{Log}(V_f) \subset U_\epsilon(S_f) \quad \text{and} \quad S_f \subset U_\epsilon\big(\operatorname{Log}(V_f)\big).$$

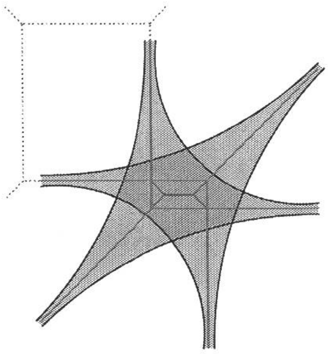

FIGURE 1
The union of two amoebas coming from a line L_1 and the Cremona transform $\mathrm{Cr}(L_2)$ of a second line L_2. In red, we depict the union of the Passare–Rullgård spines of the individual curves. The Passare–Rullgård spine of the the union $L_1 \cup \mathrm{Cr}(L_2)$ differs from this reducible tropical curve by the green edges. It gives an irreducible tropical curve which is a deformation retract of $\mathrm{Log}(L_1 \cup \mathrm{Cr}(L_2))$.

To our knowledge such bounds are currently not known. If instead of the Passare–Rullgård spine the naive tropicalization of f (replacing all coefficients a_i by $\log|a_i|$) is used, such bounds have been established (at least for the first inclusion of Equation 2) in [Mik2, EPR, For].

Tropical limits. Using Theorem 2.1 we can describe all possible *tropical limits* of families of rational complex curves of toric degree Δ. Such a description is important in the context of correspondence theorems between complex and tropical curves.

Let $\Delta = (\delta_1, \dots, \delta_k)$ be a toric degree. Let D be a tree with k labelled leaves. We can uniquely decorate the oriented edges of D with integer vectors $\delta(e)$ such that

- the leaf labelled by i (oriented outwards) is decorated by δ_i,
- an oppositely oriented edge $-e$ carries the vector $\delta(-e) = -\delta(e)$,
- around each vertex v, the vectors $\delta(e)$ of adjacent edges, oriented outwards, sum up to zero and hence form a toric degree denoted Δ_v.

A subset of vertices S is called *allowable* if there exists an assignment of non-negative non-all-zero numbers $(a(e) : e \text{ non-leaf})$ such that for any $v, w \in S$ we have

$$\sum_{e \subset [v,w]} a(e)\delta(e) = 0.$$

Here, $[v, w]$ denotes the oriented simple path from v to w. A collection of toric degrees obtained as $(\Delta_v)_{v\in S}$ for an allowable vertex set S is called a *degeneration* of Δ. An example is given in Figure 2.

Let $(t_m)_{k\in\mathbf{N}}$ be a sequence of positive real numbers converging to $+\infty$. Let $f_m : S_m \to (\mathbf{C}^\times)^n$ be a sequence of complex rational curves of fixed toric degree Δ. We set

$$\mathcal{A}_m := \operatorname{Log}_{t_m}\big(f_m(S_m)\big) = \frac{1}{\log(t_m)} \operatorname{Log}\big(f_m(S_m)\big).$$

Our result describes the possible limits of such sets in the Hausdorff sense. For precise definitions, we refer to Section 5.

Theorem 2.4.

(a) *Any sequence of complex rational curves $f_m : S_m \to (\mathbf{C}^\times)^n$ of toric degree Δ contains a subsequence such that the sets $\mathcal{A}_m$ converge to a Hausdorff limit $A \subset \mathbf{R}^n$ (including $A = \varnothing$).*

(b) *In this case, the Hausdorff limit A is of the form*

$$A = h_1(\Gamma_1) \cup \cdots \cup h_s(\Gamma_s)$$

for tropical rational curves $h_i : \Gamma_i \to \mathbf{R}^n$ of toric degree Δ_i such that $(\Delta_1, \ldots, \Delta_s)$ is a degeneration of Δ.

(c) *If $s > 1$, the tropical curves $(h_1, \ldots, h_s)$ can be chosen from a sublocus of dimension strictly less than $n + k - 3$ in the parameter spaces of all tuples of curves of degree $(\Delta_1, \ldots, \Delta_s)$.*

Note that $n + k - 3$ is the dimension of the parameter space of rational complex/tropical curves of toric degree Δ.

The toric degree Δ also defines a homology class $\Delta_X \in H_2(X)$ in any compact toric variety X. This class can be obtained as the homology class of the closure of a complex curve of toric degree Δ in $(\mathbf{C}^\times)^n \subset X$. The group $H_2(X)$ also contains elements representable by curves contained in the toric boundary $\partial X = X \setminus (\mathbf{C}^\times)^n$. The element $\Delta_X \in H_2(X)$ can be represented by reducible (stable) rational curves with some components in ∂X. Theorem 2.4 can be used to produce examples of tropical curves that cannot appear as tropical limits of complex curves of degree Δ_X without components in ∂X.

Example 2.5. Consider the second Hirzebruch surface Σ_2 and the class $2E + 4F \in H_2(\Sigma_2)$, where E and F denote the class of the -2-curve and a fibre, respectively. The discriminant $\mathcal{D} \subset |2E + 4F|$ consists of two components: The closure of the locus of irreducible rational curves, and the locus of reducible

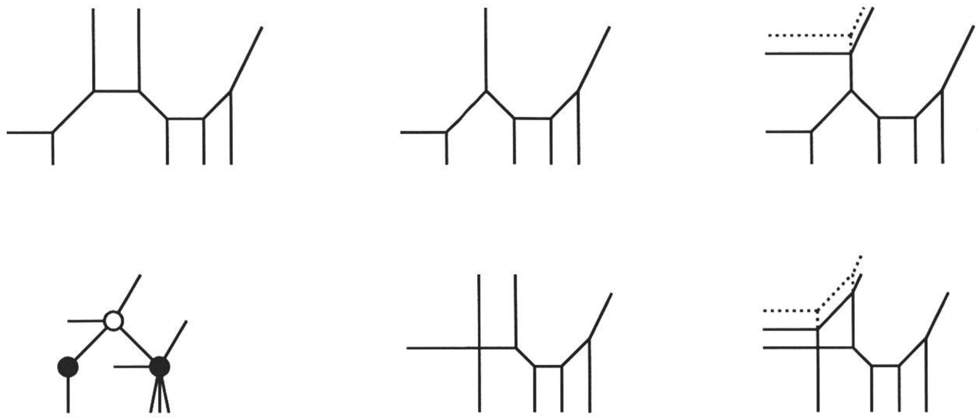

FIGURE 2

Three tropical curves of type $|E + 4F|$ in Σ_2. Since $((0,-1)^4,(1,2),(0,1)^2,(-1,0))$ is not a degeneration of $\Delta = ((0,-1)^4,(1,2)^2,(-1,0)^2)$, C_1 cannot occur as limit of (irreducible) complex rational curves in $|2E + 4F|$. Conversely, C_2 and C_3 occur as limits of tropical curves of toric degree Δ as shown on their right hand side. The degree of C_2 is $((0,-1)^4,(1,2),(0,2),(-1,0))$ and may be written as a degeneration of Δ using a single allowable vertex S. The pair of degrees of the reducible curve C_3 is $((0,1),(0,-1))$ and $((0,-1)^3,(1,2),(0,1),(-1,0))$. It is a degeneration of Δ induced by the tree D and allowable vertices S (in black) displayed on the bottom left hand side. The path connecting the two vertices in S consists of two edges of direction $(0,\pm 1)$, and we can choose $a(e) = 1$ for these two edges and $a(e) = 0$ for all others.

curves $E + |E + 4F|$. Both components have dimension 7. Tropical curves of degree $E + |E + 4F|$ (actually, since we restrict to $\mathbf{R}^2$, in $|E + 4F|$) also form a 7-dimensional family. By Theorem 2.4, the ones that appear as limits of irreducible complex curves form a subfamily of dimension at most 6. Figure 2 shows examples of curves which appear or do not appear as such limits.

3. Spines of lines

For every $n \in \mathbf{N}$, we set

$$\Delta_n := (-e_0, -e_1, \dots, -e_n),$$

where $e_1, \dots, e_n$ denotes the standard basis of $\mathbf{R}^n$ and $-e_0 = e_1 + \dots e_n = (1, \dots, 1)$. Complex and tropical curves of degree Δ_n are called *(non-degenerate) lines*. For lines, we number the punctures, ends, and leaves, respectively, from 0 to n.

Complex lines. By choosing a coordinate z for $\mathbf{CP}^1$ such that $\alpha_0 = \infty$, we can parametrize any complex line $L \subset (\mathbf{C}^\times)^n$ by a map

(3) $f : \mathbf{CP}^1 \setminus \{\infty, \alpha_1, \dots, \alpha_n\} \to (\mathbf{C}^\times)^n, \qquad z \mapsto \big(\kappa_1(z-\alpha_1), \dots, \kappa_n(z-\alpha_n)\big).$

We call L *calibrated* if $\kappa_1 = \dots = \kappa_n$.

Tropical lines. Let us recall the basic properties of tropical lines:

- If $h : \Gamma \to \mathbf{R}^n$ is a tropical line, then h is injective. Indeed, the balancing condition implies that, as we follow the path from a_i to a_0 with unit speed, the function $x_i \circ h$ has constant derivative 1. Since any point $p \in \Gamma$ lies on at least one such path, the injectivity follows.
- Throughout the following, we will identify Γ with its image and use the notation $\Gamma \subset \mathbf{R}^n$ (suppressing h).
- Given $p = (p_1, \dots, p_n) \in \Gamma$, let e denote the oriented edge pointing from p towards a_0. Then the direction vector $\partial h(e) \in \mathbf{Z}^n$ has only 0 and 1 as entries. A coordinate x_i corresponding to an entry 1 is called a *local coordinate* for Γ at p. Given $k \in \mathbf{R}$, the linear tropical polynomials $\mu(x) = k + \max\{x_i - p_i, 0\}$, for any local coordinate x_i, restrict to the same function on Γ. The *linear modification* of Γ at p (of height k) is the unique line $\widetilde{\Gamma} \subset \mathbf{R}^{n+1}$ which contains the graph of μ. More concretely, $\widetilde{\Gamma}$ is the union of the graph of μ with the ray in direction $-e_{n+1}$ emanating from (p, k).
- The inverse operation to modification is called contraction. Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection forgetting x_n. Then the image $\Gamma' = \pi(\Gamma)$ of any line $\Gamma \subset \mathbf{R}^n$ is a line in $\mathbf{R}^{n-1}$, called the *contraction* of Γ (along x_n). Let $p \in \Gamma'$ be the image of the contracted leaf l_n. Then Γ is the linear modification of Γ' at p (for a suitable height k). In particular, the contraction map $\pi : \Gamma \to \Gamma'$ is a bijection when restricted to $\Gamma \setminus l_n^\circ$.
- A tropical line Γ is *calibrated* if the leaf l_0 is contained in the (usual) line $\mathbf{R}e_0$ (emanating from the origin). Given $p \in \Gamma$, note that Γ is calibrated if and only if $p_i = p_j$ for any two local coordinates x_i and x_j at p. Moreover, the modification of a calibrated line is calibrated if and only, in the notation from above, $k = p_i$ and hence $\mu(x) = \max\{x_i, p_i\}$.

Spines for amoebas of lines. We consider the (shifted) geometric series

$$\epsilon_n = 2\log(2) \sum_{i=0}^{n-2} 3^i = \log(2)(3^{n-1} - 1)$$

with initial value $\epsilon_1 = 0$. Note that $\epsilon_n = 3\epsilon_{n-1} + 2\log(2)$ for all $n \in \mathbf{N}$.

Theorem 3.1. *Let $L \subset (\mathbf{C}^\times)^n$ be a complex line. Then there exists a tropical line $\Gamma \subset \mathbf{R}^n$ and a map $\phi : L \to \Gamma$ such that*

$$\| \operatorname{Log}(q) - \phi(q) \|_\infty \leq \epsilon_n$$

for all $q \in L$. Moreover, if L is calibrated, there exists a calibrated Γ such that the statement holds.

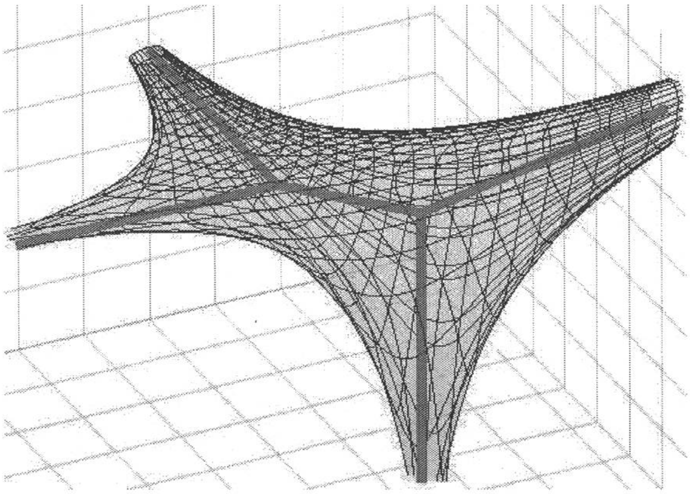

FIGURE 3
The amoeba of a complex line $L \subset (\mathbf{C}^\times)^3$ together with an approximating tropical line $\Gamma \in \mathbf{R}^3$. The line L is parametrised by $z \mapsto (z, z+1, z-2i)$. The vertices of Γ are $(0,0,0)$ and $(\log(2), \log(2), 0)$.

Proof. We prove the statement for calibrated lines by induction on n. The general statement obviously follows from the calibrated case after applying translations in $(\mathbf{C}^\times)^n$ and $\mathbf{R}^n$.

For $n = 1$, we have $L = \mathbf{C}^\times$ and hence $\Gamma = \mathbf{R}$ and $\phi = \operatorname{Log} : \mathbf{C}^\times \to \mathbf{R}$ satisfy the requirements.

For the induction step $n-1 \to n$, let us start with a given calibrated complex line $L \subset (\mathbf{C}^\times)^n$. We denote by $L' \subset (\mathbf{C}^\times)^{n-1}$ the calibrated complex line obtained as the closure of the image of L under the projection forgetting the last coordinate z_n. The closure contains the point $w = (w_1, \dots, w_{n-1})$ corresponding to the puncture α_n. Since L is calibrated, the coordinates on L are related by $z_n = z_i - w_i$ for $i = 1, \dots, n-1$.

By the induction assumption, there exists a calibrated tropical line $\Gamma' \subset \mathbf{R}^{n-1}$ and a map $\phi' : L' \to \Gamma'$ such that $\| \operatorname{Log}'(q) - \phi'(q) \|_\infty \leq \epsilon_{n-1}$ for all $q \in L'$. Here, we use Log and Log$'$ to denote the log map on n and $n-1$ variables, respectively. We set $p = (p_1, \dots, p_{n-1}) := \phi'(w)$. We define the tropical line $\Gamma \subset \mathbf{R}^n$ as the modification of Γ' at p corresponding to the function $\mu(x) = \max\{x_i, p_i\}$ for a local coordinate x_i at p. By the remarks on page 383, Γ is calibrated and does not depend on the choice of local coordinate x_i. For now, let us fix such x_i.

In the next step, we define the map $\phi : L \to \Gamma$. We distinguish two cases depending on whether $\phi'(q)$ is close to p or not. For $q = (q_1, \dots, q_n) \in L$ we set

$$\phi(q) := \begin{cases} \big(p, \min\{\log|q_n|, p_i\}\big) & \text{if } \|\phi'(q) - p\|_\infty \le 2\epsilon_{n-1} + \log(2), \\ \Big(\phi'(q), \mu\big(\phi'(q)\big)\Big) & \text{otherwise.} \end{cases}$$

Note that $\phi(q) \in \Gamma$ by construction. It remains to prove that $\|\operatorname{Log}(q) - \phi(q)\|_\infty \le \epsilon_n$ for all $q \in L$. Before continuing, let us collect two consequences of the induction assumption for reference:

$$\big|\log|q_i| - \phi'(q)_i\big| \le \epsilon_{n-1}, \tag{4}$$

$$\big|\log|w_i| - p_i\big| \le \epsilon_{n-1}. \tag{5}$$

We proceed in several cases.

Case 1. Assume that $\|\phi'(q) - p\|_\infty \le 2\epsilon_{n-1} + \log(2)$. Since this implies

$$\|\operatorname{Log}'(q) - p\|_\infty \le \|\operatorname{Log}'(q) - \phi'(q)\|_\infty + \|\phi'(q) - p\|_\infty \le 3\epsilon_{n-1} + \log(2) < \epsilon_n,$$

by the definition of $\phi(q)$ it suffices to show $\log|q_n| \le p_i + \epsilon_n$. To do so, we apply the case assumption again to the i-th coordinates, providing

$$|\phi'(q)_i - p_i| \le 2\epsilon_{n-1} + \log(2). \tag{6}$$

Combining Equation 4 and Equation 6, we get

$$\big|\log|q_i| - p_i\big| \le 3\epsilon_{n-1} + \log(2) \tag{7}$$

and hence

$$|q_n| = |q_i - w_i| \le |q_i| + |w_i| \le (2e^{3\epsilon_{n-1}} + e^{\epsilon_{n-1}})e^{p_i} \le 3e^{3\epsilon_{n-1}}e^{p_i} < e^{\epsilon_n}e^{p_i}. \tag{8}$$

Here, the second inequality uses Equation 5 and Equation 7 and the third inequality follows from $e^{3\epsilon_{n-1}} \ge e^{\epsilon_{n-1}}$ since $\epsilon_{n-1} \ge 0$. Finally, this implies

$$\log|q_n| = \log|q_i - w_i| \le p_i + \epsilon_n, \tag{9}$$

as required.

Case 2. Let us now assume $\|\phi'(q) - p\|_\infty > 2\epsilon_{n-1} + \log(2)$. By definition of $\phi(q)$, we need to show that $|\log|q_n| - \mu(\phi'(q))| \le \epsilon_n$. We subdivide this case further as follows (see Figure 4).

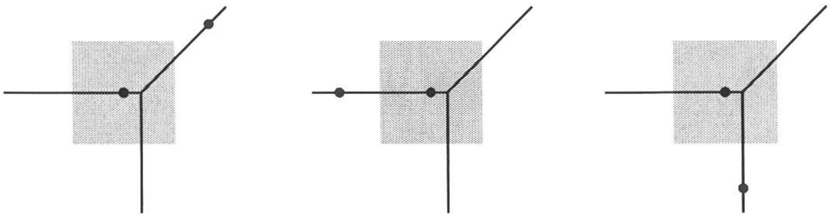

FIGURE 4
The three subcases 2.1–3 in the proof of Theorem 3.1 with $\delta = 2\epsilon_{n-1} + \log(2)$

Subcase 2.1. There exists $i \in \{1, \dots, n-1\}$ such that $\phi'(q)_i - p_i > 2\epsilon_{n-1} + \log(2)$. Note that when following the path in L' from p to $\phi'(q)$, any coordinate increases at most as much as the local coordinates at p. Thus we may assume without loss of generality that x_i is a local coordinate at p and hence $\mu(\phi'(q)) = \phi'(q)_i$. Using Equations 4 and 5, we obtain

$$\log|q_i| - \log|w_i| > \log(2)$$

or, equivalently, $|q_i| > 2|w_i|$. The triangle inequalities for $q_n = q_i - w_i$ give

$$|q_n| \leq |q_i| + |w_i| < |q_i| + \frac{1}{2}|q_i| < 2|q_i|,$$
$$|q_n| \geq |q_i| - |w_i| > |q_i| - \frac{1}{2}|q_i| > \frac{1}{2}|q_i|,$$

and hence

$$\big|\log|q_n| - \log|q_i|\big| < \log(2).$$

Together with Equation 4, we get $|\log|q_n| - \phi'(q)_i| < \epsilon_{n-1} + \log(2) < \epsilon_n$.

Subcase 2.2. There exists a local coordinate x_i at p such that $p_i - \phi'(q)_i > 2\epsilon_{n-1} + \log(2)$. The reciprocal previous argument implies

$$\big|\log|q_n| - \log|w_i|\big| < \log(2)$$

and hence $|\log|q_n| - p_i| < \epsilon_{n-1} + \log(2) < \epsilon_n$, and we are done.

Subcase 2.3. We have $p_i - \phi'(q)_i > 2\epsilon_{n-1} + \log(2)$ for some i and none of the previous subcases occurs. In this case, the subtree of Γ' spanned by p, $\phi'(q)$ and l_0 (the leaf corresponding to e_0) contains a unique three-valent vertex $r = (r_1, \dots, r_n) \in \Gamma'$. Alternatively, r can be described as the point on the path from p to $\phi'(q)$ at which the coordinate x_i starts to decrease. In particular, $r_i = p_i$. Note that $\|r - p\|_\infty \leq 2\epsilon_{n-1} + \log(2)$, since otherwise this would imply the existence of a coordinate satisfying the conditions of the first subcase. Let x_j be a local coordinate at p. Then $\mu(\phi'(q)) = \phi'(q)_j = r_j$ by construction of r. Moreover, both x_i and x_j are local coordinates at r. Since Γ' is calibrated, this implies $r_i = r_j$. The estimate $|\log|q_n| - p_i| < \epsilon_{n-1} + \log(2)$ from the second subcase is still valid, so we can combine these equations to

$$(10)\quad \big|\log|q_n| - \mu\big(\phi'(q)\big)\big| = \big|\log|q_n| - r_i\big| \\ \leq \big|\log|q_n| - p_i\big| + \big|p_i - r_i\big| < 3\epsilon_{n-1} + 2\log(2) = \epsilon_n.$$

This finishes the third subcase and hence completes the proof. □

Remark 3.2. We made no serious attempt to reach optimality of ϵ_n in any sense. For example, $\epsilon_2 = 2\log(2)$ can obviously be improved to $\log(2)$ (even with respect to the Euclidean metric). Note also that except for the trivial case $n = 1$ the proof in fact yields the strict inequality $\|\operatorname{Log}(q) - \phi(q)\|_\infty < \epsilon_n$.

Theorem 3.1 clearly implies $\operatorname{Log}(L) \subset U_{\epsilon_n}(\Gamma)$. To prove $\Gamma \subset U_{\epsilon_n}(\operatorname{Log}(L))$, we upgrade the statement to show surjectivity of ϕ up to small neighbourhoods around the vertices of Γ.

Theorem 3.3. *The map $\phi : L \to \Gamma$ in Theorem 3.1 can be chosen such that*

$$\Gamma \setminus \bigcup_{v \text{ vertex}} U_{\epsilon_n}(v) \subset \phi(L).$$

Proof. As before, we may restrict to the calibrated case. We use the same induction as in Theorem 3.1. For $n = 1$, the $\phi = \operatorname{Log} : \mathbf{C}^\times \to \mathbf{R}$ is obviously surjective. For the induction step $n - 1 \to n$, we use the same notation as before and set $R = \Gamma \setminus \bigcup_v U_{\epsilon_n}(v)$ and $R' = \Gamma' \setminus \bigcup_{v'} U_{\epsilon_{n-1}}(v')$. The additional induction assumption is $R' \subset (\phi'(L'))$.

Clearly $\pi(R \setminus l_n) \subset R'$. Moreover, for any q with $\phi'(q) \in \pi(S \setminus l_n)$, the "otherwise"-case in the definition of ϕ is used. By the induction assumption, we conclude $R \setminus l_n \subset \phi(L)$.

It remains to show that a point in l_n with last coordinate lower or equal than $p_i - \epsilon_n$ lies in $\phi(L)$. Here x_i is a local coordinate for p. In fact, we will prove the stronger statement that for any $q \in L$ with $\log|q_n| \leq p_i - \epsilon_n$, the "if"-case in the definition of ϕ takes effect. First, note that $p_i \leq p_j$ for all j since Γ' is calibrated. It follows that $\log|q_n| \leq \log|w_j| + \epsilon_{n-1} - \epsilon_n$ for all j. Since $\epsilon_n - \epsilon_{n-1} > \log(2)$, we get $|q_n| < |w_j|/2$. As in previous arguments, this implies $|\log|q_j| - \log|w_j|| < \log(2)$ and hence $|\phi(q)_j - p_j| < 2\epsilon_{n-1} + \log(2)$. This shows $\|\phi'(q) - p\|_\infty \leq 2\epsilon_{n-1} + \log(2)$ and finishes the proof. □

Remark 3.4. With little extra effort, the induction argument can be modified to construct a map $\phi : L \to \Gamma$ with the following properties.

(a) The map ϕ is continuous, proper and surjective.

(b) For all $q \in L$ we have $\| \mathrm{Log}(q) - \phi(q) \|_\infty \leq \epsilon'_n$.

(c) For any $p \in \Gamma$ in the interior of an edge e, the preimage $\phi^{-1}(p)$ is a smoothly embedded circle in L and the homology class $[\phi^{-1}(p)] \in \mathrm{H}_1((\mathbf{C}^\times)^n, \mathbf{Z}) = \mathbf{Z}^n$ is equal to the direction vector of e (for compatible orientations of $\phi^{-1}(p)$ and e).

(d) For any vertex $v \in \Gamma$, the preimage $\phi^{-1}(v) \subset L$ is a compact surface with boundary. The boundary components are in bijection (given by homology classes) with the edges adjacent to v.

Here, the value ϵ'_n can be defined by the recursion $\epsilon'_n = 5\epsilon'_{n-1} + \log(5)$. The induction step can then be modified as follows: Choose $2\epsilon'_{n-1} + \log(2) \leq \delta < 2\epsilon'_{n-1} + \log(5)/2$ such that $\partial U_\delta(p)$ does not contain vertices of Γ' and set

$$\phi(q) := \begin{cases} (p, \log|q_n|) & \text{if } \log|q_n| \leq p_i - \delta, \\ \big(\phi'(q), \mu(\phi'(q))\big) & \text{if } \phi'(q) \notin U_\delta(p). \end{cases}$$

It remains to extend ϕ to

$$B = \big\{ q \in L : \phi'(q) \in \overline{U_\delta(p)} \quad \text{and} \quad \log|q_n| \geq p_i - \delta \big\},$$

which is a connected surface with boundary in L whose boundary components are in bijection with $\Gamma \cap \partial U_\delta((p, p_i))$. It is clear that a map $\phi_B : B \to \Gamma \cap \overline{U_\delta((p, p_i))}$ satisfying properties (a), (c) and (d) exists. Property (b) then follows from previous arguments and

$$\begin{aligned} \|\pi(\phi(q) - \mathrm{Log}(q))\|_\infty &\leq \|\pi(\phi(q)) - p\|_\infty + \|p - \phi'(q)\|_\infty + \|\phi'(q) - \mathrm{Log}'(q)\|_\infty \\ &\leq \delta + \delta + \epsilon'_{n-1} < 5\epsilon'_{n-1} + \log(5) = \epsilon'_n. \end{aligned}$$

Using ϕ_B to extend ϕ to L, we obtain a function which satisfies (a)–(d).

4. Spines of rational curves

Let $\Delta = (\delta_0, \dots, \delta_k)$ be a toric degree in dimension n. We denote by $\psi_\Delta : \mathbf{R}^k \to \mathbf{R}^n$ the linear map which sends the standard basis vector e_i to $-\delta_i$ for all $i = 1, \dots, k$ (this implies $e_0 \to -\delta_0$). In this section, we assume that Δ is *non-degenerate*, that is to say, the map ψ_Δ is surjective.

Let $\Psi_\Delta : (\mathbf{C}^\times)^k \to (\mathbf{C}^\times)^n$ denote the torus homomorphism which is the exponential of ϕ_Δ (hence also surjective). In other words, the diagram

(11)
$$\begin{array}{ccc} (\mathbf{C}^\times)^k & \xrightarrow{\Psi_\Delta} & (\mathbf{C}^\times)^n \\ \text{Log}\downarrow & & \downarrow\text{Log} \\ \mathbf{R}^k & \xrightarrow{\psi_\Delta} & \mathbf{R}^n \end{array}$$

commutes.

The following lemmas state that complex and tropical rational curves of toric degree Δ can be represented as images of lines under Ψ_Δ and ψ_Δ, respectively.

Lemma 4.1. *Given a complex line $L \subset (\mathbf{C}^\times)^k$, the map*

$$f = \Psi_\Delta|_L : L \to (\mathbf{C}^\times)^n$$

is a complex rational curve of toric degree Δ. Any complex rational curve of toric degree Δ can be represented in such a way. Two lines L, L' provide the same rational curve if and only if $L = wL'$ for some $w \in \ker \Psi_\Delta$.

Proof. The uniqueness up to $\ker \Psi_\Delta$ is obvious. Using coordinates $\delta_i = (\delta_i^1, \dots, \delta_i^n)$, the map Ψ_Δ is given by

$$z'_j = z_1^{-\delta_1^j} \cdots z_k^{-\delta_k^j}.$$

This implies $-\operatorname{ord}_{\alpha_i}(z_j \circ f) = \delta_i^j$, as required.

Let $f : S \to (\mathbf{C}^\times)^n$ be a complex rational curve of toric degree Δ. Up to isomorphism, we may assume $S = \mathbf{CP}^1 \setminus \{\infty, \alpha_1, \dots, \alpha_n\}$, with affine coordinate z. By definition of toric degree, we have

$$z_j \circ f = \kappa_j (z - \alpha_1)^{-\delta_1^j} \cdots (z - \alpha_k)^{-\delta_k^j}$$

for some constant $\kappa_j \in \mathbf{C}^\times$. Pick a preimage $(\lambda_1, \dots, \lambda_k)$ of $(\kappa_1, \dots, \kappa_n)$ under Ψ_Δ. Then f factors through Ψ_Δ by the line

(12) $$S \to (\mathbf{C}^\times)^k,$$

(13) $$z \mapsto \big(\lambda_1(z - \alpha_1), \dots, \lambda_k(z - \alpha_k)\big).$$

□

Lemma 4.2. *Given a tropical line $\Gamma \subset \mathbf{R}^k$, the map*

$$h = \psi_\Delta|_\Gamma : \Gamma \to \mathbf{R}^n$$

is a tropical rational curve of toric degree Δ. Up to isomorphism, any tropical rational curve of toric degree Δ can be represented in such a way. Two lines Γ, Γ' provide the same rational curve if and only if $\Gamma = x\Gamma'$ for some $x \in \ker \psi_\Delta$.

Proof. Let $\Gamma \subset \mathbf{R}^k$ be a tropical line. Since ψ_Δ is linear, $\psi_\Delta|_\Gamma : \Gamma \to \mathbf{R}^n$ is clearly a tropical morphism. Moreover, the degree requirements are satisfied since ψ_Δ maps $-e_i \to \delta_i$ for $i = 0, \dots, k$.

Given a tree Γ with complete inner metric and $k+1$ leaves $l_0, \dots, l_k$, an arbitrary base point $p_0 \in \Gamma$ and toric degree Δ, the set of tropical rational curves $h : \Gamma \to \mathbf{R}^n$ of toric degree Δ is in bijection to $\mathbf{R}^n$ via $h \mapsto h(p_0)$. Indeed, since Γ is a tree and since the direction vectors $\partial h(l_i)$ are fixed by Δ, the balancing condition recursively prescribes all direction vectors $\partial h(e)$. To fix $h : \Gamma \to \mathbf{R}^n$, it hence suffices to fix the image of a single point.

Let $h : \Gamma \to \mathbf{R}^n$ be a tropical rational curve of toric degree Δ with base point p_0. Choose a point $x \in \mathbf{R}^k$ such that $\psi_\Delta(x) = h(p_0)$. Applying the previous discussion to Δ_k, there exists a unique tropical line $g : \Gamma \to \mathbf{R}^k$ such that $g(p_0) = x$. Moreover, by construction we have $\psi_\Delta(\partial g(e)) = \partial h(e)$ for any edge e of Γ. Hence, $f = \psi_\Delta \circ g$, as required. The uniqueness property also follows easily from the previous discussion. □

We are now ready to prove the main theorem.

Theorem 2.1. Given a toric degree Δ consisting of $k+1$ vectors, we set $\epsilon' = \epsilon_k \cdot N(\Delta)$, where

$$N(\Delta) = \|\psi_\Delta\|_\infty = \max\left\{ \frac{\|\psi_\Delta(x)\|_\infty}{\|x\|_\infty} : 0 \neq x \in \mathbf{R}^k \right\}.$$

Let $f : S \to (\mathbf{C}^\times)^n$ be a complex rational curve of toric degree Δ. By Lemma 4.1, we may assume that $S \cong L \subset (\mathbf{C}^\times)^k$ is a complex line and $f = \psi_\Delta(x)|_S$. By Theorem 3.1, there exists a tropical line $\Gamma \subset \mathbf{R}^k$ and a map $\phi : L \to \Gamma$ such that $\| \operatorname{Log}(q) - \phi(q) \|_\infty \leq \epsilon_k$ for all $q \in L$. By Lemma 4.2, $h = \psi_\Delta|_\Gamma : \Gamma \to \mathbf{R}^n$ is a tropical rational curve of toric degree Δ. The situation can be summarized in the following diagram (whose left hand side is only commutative up to ϵ_k):

$$(14) \qquad \begin{array}{ccccc} & & \overset{f}{\curvearrowright} & & \\ L & \subset & (\mathbf{C}^\times)^k & \xrightarrow[\Psi_\Delta]{} & (\mathbf{C}^\times)^n \\ {\scriptstyle\phi}\downarrow & {\scriptstyle \leq \epsilon_k} & \downarrow{\scriptstyle \operatorname{Log}} & & \downarrow{\scriptstyle \operatorname{Log}} \\ \Gamma & \subset & \mathbf{R}^k & \xrightarrow{\psi_\Delta} & \mathbf{R}^n \\ & & \underset{h}{\curvearrowright} & & \end{array}$$

Hence, for all $q \in L$,

$$(15) \qquad \big\| \operatorname{Log}\big(f(q)\big) - h\big(\phi(q)\big) \big\|_\infty = \big\| \psi_\Delta\big(\operatorname{Log}(q) - \phi(q)\big) \big\|_\infty \leq N(\Delta)\epsilon_k = \epsilon',$$

which implies $\operatorname{Log}(f(L)) \subset U_{\epsilon'}(h(\Gamma))$.

Set $R = \Gamma \setminus \bigcup_v U_{\epsilon_n}(v)$. By Theorem 3.3, we have $h(R) \subset h(\phi(L)) \subset U_{\epsilon'}(\mathrm{Log}(f(L)))$. Finally, for $p \in \Gamma \setminus R$, there exists $p' \in R$ such that $\|p - p'\|_\infty < (k-1)\epsilon_k$, since Γ has $k-1$ vertices. Choose $q' \in L$ with $\phi(q') = p'$. Then

$$\big\|h(p) - \mathrm{Log}\big(f(q')\big)\big\|_\infty \leq \|h(p) - h(p')\|_\infty + \big\|h(p') - \mathrm{Log}\big(f(q')\big)\big\|_\infty < k\epsilon'.$$

Hence, for $\epsilon = \epsilon(\Delta) = k \cdot \epsilon'$ we proved $h(\Gamma) \subset U_\epsilon(\mathrm{Log}(f(L)))$, which finishes the proof. □

Remark 4.3. Clearly, Remark 3.4 can be extended to the general case in the sense that for any complex rational curve $f : S \to (\mathbf{C}^\times)^n$ of toric degree Δ, there exists a tropical rational curve $h : \Gamma \to \mathbf{R}^n$ of toric degree Δ and a map $\phi : S \to \Gamma$ which satisfies properties (a)–(d) (after substituting $\|\mathrm{Log}(f(q)) - h(\phi(q))\|_\infty \leq \epsilon'(\Delta)$ and $f_*[\phi^{-1}(p)]$ at the obvious places). Here, $\epsilon'(\Delta) = N(\Delta)\epsilon'_k$.

5. Tropical limits of amoebas

Given two subsets $A, B \subset \mathbf{R}^n$, we set the *Hausdorff distance* of A and B to

$$d(A,B) = \inf\big\{\delta : A \subset U_\delta(B), B \subset U_\delta(A)\big\}.$$

Note that $d(A,B)$ can be infinite in general. If we restrict to non-empty closed subsets of a compact set $K \subset \mathbf{R}^n$, then $d(A,B) \in \mathbf{R}_\geq$ and the Hausdorff distance defines a metric. A sequence of subsets $A_m \subset \mathbf{R}^n$ *converges to the Hausdorff limit* $A \subset \mathbf{R}^n$ if A is closed and for any compact set $K \subset \mathbf{R}^n$ the sequence $d(A_m \cap K, A \cap K)$ converges to 0. In this case $A = \lim A_m$ is unique, since it is unique on each compact K. Note that we include the case $A = \varnothing$, which is to say, for any compact $K \subset \mathbf{R}^n$, there exists $k_0 \in \mathbf{N}$ such that $A_m \cap K = \varnothing$ for all $k \geq k_0$.

Let $f_m : S_m \to (\mathbf{C}^\times)^n$ be a sequence of rational complex curves as in the assumptions of Theorem 2.4. By Theorem 2.1, there exists a sequence of tropical rational curves $h_m : \Gamma_m \to \mathbf{R}^n$ of toric degree Δ such that

$$\mathcal{A}_m \subset U_{\epsilon/\log(t_m)}\big(h_m(\Gamma_m)\big) \qquad \text{and} \qquad h_m(\Gamma_m) \subset U_{\epsilon/\log(t_m)}(\mathcal{A}_m). \tag{16}$$

This implies that the sequence of Hausdorff distances $d(\mathcal{A}_m, h_m(\Gamma_m))$ converges to zero. Obviously, this is still true after restricting to compact subsets K. We get the following corollary.

Corollary 5.1. *The sequence $\mathcal{A}_m$ converges to the Hausdorff limit A if and only if $h_m(\Gamma_m)$ converges to the Hausdorff limit A.*

In other words, Theorem 2.1 reduces the proof of Theorem 2.4 to the study of Hausdorff limits of tropical curves.

Fix a toric degree $\Delta = (\delta_1, \dots, \delta_k)$ in $\mathbf{R}^n$, an (abstract) tree G with m leaves labelled by $\{1, \dots, m\}$ and a marked vertex $v_0 \in G$.

In analogy to our conventions for Γ, the leaves are considered to be half-edges without one-valent end vertices. Let us furthermore assume that G does not contain two-valent vertices except for the case $-\bullet-$. Then the space $\mathcal{M}(\Delta, G)$ of isomorphism classes of rational tropical curves of toric degree Δ and combinatorial type G (allowing edge lengths 0 for convenience) is parametrised by

$$\mathcal{M}(\Delta, G) \cong \mathbf{R}^n \times (\mathbf{R}_{\geq})^{k-3}.$$

Here, the factor $\mathbf{R}^n$ parametrizes the position of the marked vertex $h(v_0)$, and the second factor encodes the lengths of the non-leaf edges of G. Again, there is one exception, namely

$$\mathcal{M}\big((\delta, -\delta), -\bullet-\big) \cong \mathbf{R}^n / \mathbf{R}\delta.$$

Lemma 5.2. *Let $h_m : \Gamma_m \to \mathbf{R}^n$ be a sequence of rational tropical curves of toric degree Δ and combinatorial type G converging in $\mathcal{M}(\Delta, G)$ to a tropical curve $h : \Gamma \to \mathbf{R}^n$. Then the sets $h_m(\Gamma_m)$ converge to the Hausdorff limit $h(\Gamma)$.*

Proof. This follows immediately from the fact that the positions of all vertices and edges of $h_m(\Gamma_m)$ depend linearly (hence continuously) on the parameters in $\mathbf{R}^n \times (\mathbf{R}_{\geq})^{k-3}$. □

Consider the following construction.

(a) Mark some of the edges of G, including all leaves, by the symbol ∞.

(b) Insert a two-valent vertex in some of the ∞-marked edges. If so, mark both new edges by ∞ again.

(c) Decompose G into pieces $G_1, \dots, G_s$ by cutting each interior ∞-marked edge into two halves. The ends of the pieces G_i can by canonically labelled by toric degrees Δ_i.

(d) Mark a vertex $v_i \in G_i$ for $i = 1, \dots, s$.

(e) Pick a set $S \subset \{1, \dots, s\}$ such that there exists an assignment of non-negative non-all-zero numbers $(a(e) : e\ \infty\text{-marked non-leaf})$ such that for $i, j \in S$ we have

$$\sum_{\substack{e \subset [v_i, v_j] \\ \infty\text{-marked}}} a(e)\delta(e) = 0.$$

Here, $[v_i, v_j]$ denotes the oriented simple path from v_i to v_j.

We call such a construction (and the result $((\Delta_i, G_i)_{i\in S})$) a *degeneration* of (Δ, G). Clearly, $(\Delta_i)_{i\in S}$ is a degeneration of Δ in the sense of the definition given before Theorem 2.4 (set D to be the contraction of G along all non-∞-edges). There is an associated linear map of parameter spaces (defined over $\mathbf{Z}$)

$$L : \mathcal{M}(\Delta, G) \to \prod_{i\in S} \mathcal{M}(\Delta_i, G_i), \tag{17}$$

$$(\Gamma, h) \mapsto (\Gamma_i, h_i)_{i\in S}, \tag{18}$$

given by $h_i(v_i) = h(v_i)$ (identifying $v_i \in G_i$ with $v_i \in G$) and keeping the edge lengths for all edges which are still present. Clearly, the definition extends in the obvious way to the case when some G_i are $- \bullet -$. The image $L(\mathcal{M}(\Delta, G))$ is a rational subcone of $\mathbf{R}^N \times (\mathbf{R}_{\geq})^M$ (for suitable N, M) of dimension less than or equal to $\dim(\mathcal{M}(\Delta, G)) \leq n + k - 3$. Moreover, assuming there exist ∞-marked non-leaves, the vector $(a(e) : e$ ∞-marked non-leaf) gives rise to a non-trivial kernel element for L, hence $\dim L(\mathcal{M}(\Delta, G)) < \mathcal{M}(\Delta, G)$. We can summarize the discussion so far by concluding that in order to prove Theorem 2.4, using Corollary 5.1 it suffices to show the following statement.

Theorem 5.3. *Any sequence of tropical rational curves $h_m : \Gamma_m \to \mathbf{R}^n$ of toric degree Δ contains a subsequence converging to a Hausdorff limit A (including $A = \varnothing$). The limit A is of the form*

$$A = h_1(\Gamma_1) \cup \cdots \cup h_s(\Gamma_s)$$

for a combinatorial type G, a degeneration $((\Delta_1, G_1), \ldots, (\Delta_s, G_s))$ of (Δ, G) and a tuple of tropical rational curves $(h_1, \ldots, h_s) \in L(\mathcal{M}(\Delta, G))$.

Proof. Since the number of trees with k labelled leaves is finite, we can assume that (h_m) has constant combinatorial type G. Throughout the following, we will identify the vertices and edges of G with the corresponding vertices and edges of Γ_m. In particular, given an non-leaf edge e or vertex v of G, we write $l_m(e)$ and $h_m(v)$ for the length and position of the corresponding edge and vertex in Γ_m, respectively. We denote by $\mathbf{R}^n \cup \{\infty\}$ the one-point compactification of $\mathbf{R}^n$. By compactness, we may assume that

- for any vertex $v \in G$, $h_m(v)$ converges in $\mathbf{R}^n \cup \{\infty\}$,
- for any non-leaf edge $e \subset G$, $l_m(e)$ converges in $[0, +\infty]$.

We now describe an explicit degeneration of (Δ, G) (see Figure 5). We mark all leaves and all edges with $\lim l_m(e) = +\infty$ by ∞ (step (a)). For any such edge

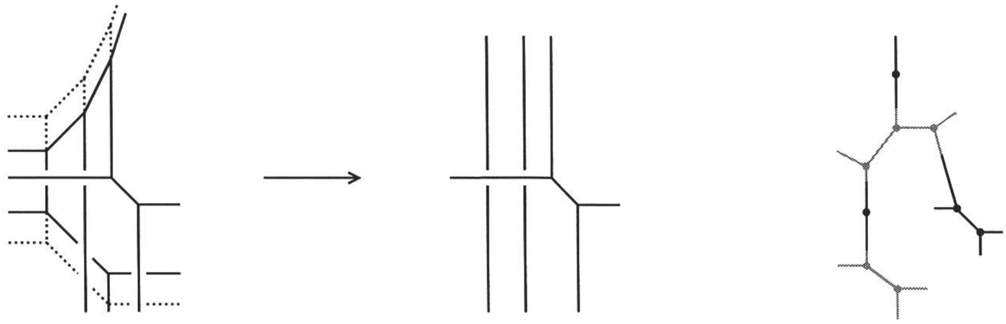

FIGURE 5
The Hausdorff limit of a sequence of tropical rational curves in $\mathbf{R}^2$. On the right hand side, the combinatorial type and its degeneration are depicted. The gray parts are the ones we forget in step (e).

e, we insert a two-valent vertex if and only if all adjacent vertices diverge and there exists a sequence $x_m \in e^\circ \subset\in \Gamma_m$ such that $h_m(x_m)$ is bounded (step (b)). Passing to a subsequence, we may assume that $h_m(x_m)$ converges in $\mathbf{R}^n$. For each two-valent vertex v we fix such a sequence and set $h_m(v) = h_m(x_m)$. Let $G_1, \dots, G_s$ denote the pieces after cutting all interior ∞-edges into halves (step (c)). We mark a vertex $v_i \in G_i$ for all $i = 1, \dots, s$ (step (d)). Finally, we set $S = \{i : \lim h_m(v_i) \in \mathbf{R}^n\}$ (step (e)). In other words, we forget all the pieces G_i for which $\lim h_m(v_i) = \infty$.

Note that since vertices in the same piece G_i are connected via edges with finite limit length, S does not depend on the choice of marked vertices v_i. Setting $((h_{i,m})_{i \in S}) = L(h_m)$, we obtain a sequence of tuples of rational tropical curves of toric degrees $((\Delta_i)_{i \in S})$ contained in $L(\mathcal{M}(\Delta, G))$. By construction, the limit $\lim L(h_m)$ in $\prod_{i \in S} \mathcal{M}(\Delta_i, G_i)$ exists. We denote it by $((h_i)_{i \in S})$. Since $L(\mathcal{M}(\Delta, G))$ is closed, it also lies in $L(\mathcal{M}(\Delta, G))$.

Let us prove that S is allowable. Note that the constructed degeneration is non-trivial if and only if it produces at least one interior ∞-edge. This, in turn, holds true if and only if the sequence $r_m = \max\{l_m(e) : e \text{ non-leaf}\}$ diverges. Then the sequence $(l_m(e)/r_m : e \text{ non-leaf})$ is bounded. Let $(a(e) : e \text{ non-leaf})$ denote an accumulation point. Note that $a(e) = 1 \neq 0$ for at least one ∞-edge e, and that $a(e) = 0$ for any edge not marked by ∞. For any pair $i \neq j \in S$, we have

$$h_m(v_j) - h_m(v_i) = \sum_{e \subset [v_i, v_j]} l_m(e)\delta(e).$$

Dividing by r_m and taking limits, we obtain $\sum_{e \subset [v_i, v_j]} a(e)\delta(e) = 0$, as required.

To finish the proof, it remains to show that the sets $h_m(\Gamma_m)$ converge in the

Hausdorff sense to

$$A = \bigcup_{i \in S} h_i(\Gamma_i).$$

By Lemma 5.2, we have $\lim_{m\to\infty} h_{i,m}(\Gamma_{i,m}) = h_i(\Gamma_i)$ in the Hausdorff sense. It follows that $\lim A_m = A$ with $A_m = \bigcup_{i\in S} h_i(\Gamma_i)$. Let $K \subset \mathbf{R}^n$ be a compact set. For any vertex $v \in G$ which is forgotten during the degeneration construction, we have $h_m(v) = \infty$ and hence $h_m(v) \notin K$ for sufficiently large m. Let $e \subset G$ be an edge which is forgotten during the degeneration. Then e is not subdivided in step (b) and both vertices of e converge to ∞. If $\lim l_m(e) \neq +\infty$, this implies $h_m(e) \cap K = \varnothing$ for large m by the vertex argument. If $\lim l_m(e) = +\infty$, the same is true since by assumption that there does not exist a sequence of points x_m on e with bounded $h_m(x_m)$. For any other edge e, at least one of the adjacent vertices v (possibly after subdividing e into two edges in step (b)) satisfies $\lim h_m(v) \neq \infty$. Then $v \in G_i$ for some $i \in S$ and $h_m(e) \cap K = h_{i,m}(e) \cap K$ for large m. It follows that $h_m(\Gamma) \cap K = A_m \cap K$ for sufficiently large m, and the claim follows. □

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