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A Dynamical Interpretation of the Thomas Precession

By V. L. TELEGDI and R. WINSTON, University of Chicago, Chicago, Illinois, USA

In this note we shall discuss a phenomenon that has on and off occupied the minds of many physicists during the last thirty-five years, i.e. roughly throughout that period during which the man to whom the present volume is dedicated directed and stimulated experimental physics at E.T.H.

The original spin hypothesis of GOUDSMIT and UHLENBECK was to attribute to the electron a mechanical spin $s = {}^1/_2\,\hbar$ and a magnetic moment $\mu = e\hbar/2\,mc$, i.e. a gyromagnetic ratio $g = 2$. This hypothesis was immediately successful in explaining the anomalous Zeeman effect, where a bound electron is acted upon by an external magnetic field B. However, when the same hypothesis was used to explain atomic fine structure – where the effective magnetic field B' is a relativistic consequence of the nuclear electric field E acting on a moving electron – it first appeared that one had to attribute $g = 1$ to the electron in order to obtain agreement with the observed spectra! THOMAS [1][1]), very shortly after the spin hypothesis had been published, succeeded in resolving this apparent discrepancy by noting that relativistically the space axes of the successive instantaneous rest frames of the accelerated electron – wherein B' is supposed to act – precess with respect to an inertial frame where the electron has a velocity $\boldsymbol{v}$ and an acceleration $\dot{\boldsymbol{v}}$, at a rate

$$\boldsymbol{\omega}_T = -(\gamma - 1)\,\boldsymbol{v} \times \dot{\boldsymbol{v}}/v^2 \cong -(\boldsymbol{v} \times \dot{\boldsymbol{v}})/2\,c^2. \tag{1}$$

With the advent of Dirac's covariant wave equation, which supplies all consequences of special relativity so to speak automatically, this curious precession discovered by THOMAS lost some of its importance in atomic physics, although this did not stop the appearance of numerous papers devoted to deriving Eq. (1). Very recently however, in connection with experiments designed for measuring the anomalous g-factor of unbound electrons or mu mesons [2], the THOMAS precession has gained renewed interest. To obtain a correct description of spin motion, its proper application to those cases where the spinning particles move on macroscopic orbits is in fact just as important as it was in the microscopic case [3].

1) The numbers in brackets refer to the References, page 252.

Within the framework of special relativity, the THOMAS precession stems from the fact that two successive pure Lorentz transformations are in general not equivalent to a single pure Lorentz transformation, but to a Lorentz transformation plus a rotation. Thus the apparent precession of the space axes of the instantaneous rest frames of a particle is attributed to a *kinematic* effect. On the other hand, one may describe the effect of the THOMAS precession by saying that the spin of a particle *not* subjected to any physical torque precesses with respect to an intertial frame. Taking this point of view, one finds the kinematical interpretation unsatisfying and is led to seek a *dynamical* one. The THOMAS precession is exhibited by accelerated frames only, and it is natural to explore whether the desired dynamical explanation can be uncovered in the formalism of general relativity.

By considering the elementary action of a magnetic field on the magnetic moment of a particle we can anticipate that the additional precession sought will arise from an analogous action of the axial vector gravitational potential on the spin of the particle. In some sense we are thus led to look for an analogue of Larmor's theorem, for the case where the external field is a gravitational one. Instead of a single point particle with spin, we shall for convenience consider a classical system of particles and allow the dimensions of this system to approach zero. We expect that the conclusions based on such a model will be applicable to "intrinsic" spin as far as this concept can be formulated classically at all.

Consider a system of particles moving in a central potential of unspecified character. As in the derivation of Larmor's theorem, we are interested in the perturbation of such a system by a weak external field, in this case of gravitational character.

The action function for a single particle of mass m in a gravitational field is

$$S = -mc \int ds, \tag{2}$$

where $ds^2 = -g_{\mu\nu}\, dx_\mu\, dx_\nu$ (Greek indices run from 1 through 4, Latin indices from 1 through 3). By a weak external field we mean that $g_{\mu\nu}$ may be expanded as $\delta_{\mu\nu} + \gamma_{\mu\nu}$, and that only linear terms in $\gamma_{\mu\nu}$ need be retained. With this approximation, and also neglecting higher powers of v/c (as is done in the usual derivation of Larmor's theorem), the addition to the Lagrangian becomes, using (2)

$$L_G = m\boldsymbol{g}\cdot\boldsymbol{v}/c - m\Phi, \text{ where } g_i = ic^2\gamma_{4i},\ \Phi = c^2\gamma_{44}/2. \tag{3}$$

We remark here that for a system of particles whose motion is entirely determined by their own gravitational field one must add a term $(^1/_2)\, m\,\Phi\, (v/c)^2$ to L_G, which is comparable in magnitude to $m\boldsymbol{g}\cdot\boldsymbol{v}/c$ (i.e. increase the mass by $(1+\Phi/c^2)$. Using L_G, one proceeds in complete analogy with the standard derivation of Larmor's theorem and finds that the effect of a perturbing homogeneous axial gravitational field $\boldsymbol{G}$ related to $\boldsymbol{g}$ by $\boldsymbol{g} = (^1/_2)\,(\boldsymbol{G}\times\boldsymbol{r})$ is to superimpose a precession with an angular frequency $\Omega = -\boldsymbol{G}/2c$ upon the unperturbed motion of the system. Note that provided that the system is sufficiently small there are no restrictions on the magnitude of $\boldsymbol{G}$ itself. In this we depart from the usual discussion of the Larmor theorem which is restricted to terms linear in the magnetic field. However, it is true in the electromagnetic case as well that the contribution of the quadratic terms vanishes in the limit that the system of particles becomes a point magnetic dipole.

To obtain the THOMAS precession, we now consider an accelerated frame F' in the absence of permanent gravitational fields, letting the origin of F' follow the origin of our system of particles. In order to conform with the weak field requirement and to display the THOMAS precession in its space axes are related to those of an inertial frame F (in which the motion of the system is specified) by a pure Lorentz transformation. That is, F' is related to F by

$$x_\mu = \xi_\mu(t') + \Lambda_{\mu\kappa}(t')\,x'_\kappa, \tag{4}$$

where t' is the proper time of the motion of the origin, and $\Lambda_{\mu\nu}$ is a pure Lorentz transformation. Denoting $d\Lambda_{\mu\nu}/dt'$ by $\dot{\Lambda}_{\mu\nu}$ and $d\xi_\mu/dt'$ by u_μ, we have $dx_\mu = \Lambda_{\mu\kappa}\,dx'_\kappa + (u_{\mu\kappa} + \dot{\Lambda}_{\mu\kappa}x'_\kappa)\,dt'$, so that we may write $ds^2 = -dx^2_\mu$ as

$$\begin{aligned} ds^2 = {} & -\delta_{ik}\,dx'_i\,dx'_k + (\Lambda_{\mu i}\dot{\Lambda}_{\mu k} - \Lambda_{\mu i}\dot{\Lambda}_{\mu k})\,x'_i dx'_k dt' \\ & + (c^2 - \dot{\Lambda}_{\mu i}\dot{\Lambda}_{\mu k}\,x'_i x'_k + 2\,\dot{u}_\mu\,\Lambda_{\mu k}\,x'_k)\,dt'^2. \end{aligned} \tag{5}$$

We see that for x'_i sufficiently small, the weak field requirement is rigorously satisfied and that $\boldsymbol{g}$ indeed has the form appropriate to a homogeneous axial vector field $(^1/_2)\,(\boldsymbol{G}\times\boldsymbol{r})$, where $G_i/2 = (\dot{\Lambda}_{\mu j}\Lambda_{\mu k} - \Lambda_{\mu j}\dot{\Lambda}_{\mu k})$ (i, j, k permuted cyclically). Also as x'_i tends to zero, $\partial\Phi/\partial x'_i$ takes on the obvious form $\dot{u}_\mu\Lambda_{\mu i}$, the acceleration of the system with respect to an instantaneous inertial rest frame. Therefore in the total Lagrangian the $m\,\Phi$ term of (3) is just cancelled by whatever external forces accelerate the system.

From the above considerations we conclude that the intrinsic angular momentum and magnetic moment of our model of a particle with spin will precess relative to F with the frequency $\Omega = -\boldsymbol{G}/2c$. Expressing G in terms of $\boldsymbol{w}$, the velocity of the system relative to F, by means of the relations

$$\begin{gathered} \Lambda_{ik} = \delta_{ik} + [c^2(\gamma+1)]^{-1}u_i u_k,\ \Lambda_{4i} = -\Lambda_{i4} = iu_i/c, \\ \text{where } \gamma = [1-(w/c)^2]^{-1/2}, \end{gathered} \tag{6}$$

we readily find that

$$G_i/2c = [c^2(\gamma+1)]^{-1}(u_j\dot{u}_k - \dot{u}_j u_k) \text{ or, } = -[(\gamma-1)/w^2]\, \boldsymbol{w}\times\dot{\boldsymbol{w}}, \qquad (7)$$

which, except for notation, is identical with (1).

REFERENCES

[1] L. H. THOMAS, Nature *117*, 514 (1926); Phil. Mag. *3*, 1 (1927).

[2] CRANE, PIDD, and LOUISELL, Bull. Am. Phys. Soc. Ser. II, *3*, 369 (1958); P. S. FARAGO, Proc. Phys. Soc. (London) *72*, 891 (1958); there are as yet no published accounts of the analogous experiments with mu mesons, but such experiments are being pursued actively in several laboratories.

[3] In a formalism (whether classical or quantummechanical) in which the spin is treated covariantly, the Thomas precession never appears explicitely; it is so to speak a fine one has to pay for treating the spin as a 3-vector. This was recognized very early by J. FRENKEL [Z. Physik *37*, 243 (1926)]. In this context see also BARGMANN, MICHEL, and TELEGDI, Phys. Rev. Letters, *2*, 435 (1958).