Can hidden variables be excluded in quantum mechanics?

Can Hidden Variables be Excluded in Quantum Mechanics?

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Abstract. The question of the possible existence of hidden variables is re-examined. It is shown that hidden variables can exist only if every proposition (yes-no experiment) is compatible with every other one. It is further shown that this property is in contradiction with empirical facts. The theorem which leads to this conclusion is a strengthening of the theorem of VON NEUMANN on the same subject. The question is raised whether there exist perhaps quantum mechanical systems which admit approximate dispersion-free states.

I. Introduction

In this paper we shall re-examine an old question in a new context viz. the possibility of introducing *hidden variables* in the description of atomic systems.

This question has played an important part in the history of quantum mechanics. The probabilistic interpretation of the Schroedinger wave function, as it was proposed by BORN and elaborated by the so-called Copenhagen school, is so radically different from the deterministic behaviour of classical systems that the description of the atomic systems by Schroedinger wave functions was soon suspected to be incomplete. If it were incomplete, then this would mean that there exist perhaps additional variables not accessible to measurements as the usual observables (and therefore not subject to the restrictions of the uncertainty relations) and such that the system behaves deterministic with respect to this "complete" set of variables.

The question concerning the existence of such hidden variables received an early and rather decisive answer in the form of VON NEUMANN's proof on the mathematical impossibility of such variables in quantum theory[1]). The method of VON NEUMANN consisted in showing that hidden variables which do what they are supposed to do are inconsistent with quantum mechanics. In other words, if such variables existed quantum mechanics would have to be factually false.

The important consequence of VON NEUMANN's analysis is that the question was brought from the speculative level to the empirical level, and so the answer does not have to wait the development of a non-existing theory. Whether quantum mechanics is false can be verified by observation and so far there is no indication that it is and, consequently, there is today no more chance than there was thirty years ago to introduce hidden variables.

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There are several reasons why we propose to re-examine here VON NEUMANN's proof again. First of all there seems to be a renewed interest in a critique of the foundations of quantum mechanics and some of the recent attempts in this direction have not always done full justice to VON NEUMANN. Thus for instance DE BROGLIE[2]) finds the above-mentioned proof essentially trivial and therefore redundant as soon as one admits an uncertainty relation[3]). BOHM in his book[4]) even goes so far as to accuse VON NEUMANN of circular reasoning. If this were true, this "proof" would mean, of course, exactly nothing and would leave all doors open for speculations on a "sub-quantum mechanical level" and a "deeper reality" so dear to the above-mentioned authors[5])[6]).

A second reason why we find a remodelling of VON NEUMANN's proof appropriate at this time is that some of his assumptions concerning the structure of quantum mechanics are too strong and cannot be justified sufficiently well from the empirical evidence alone. We mean here especially those which refer to the properties of observables and states of a physical system. For VON NEUMANN observables are the self-adjoint operators in a Hilbert space with complex coefficients and states are additive functionals on the observables (the expectation values of the observables). In the course of the proof it is used that essentially every projection operator is an observable. This assumption is very difficult to justify and in systems with super-selection rules[7]) it is actually false. It is, however, possible to weaken this assumption to the bare necessity of the empirically required and yet to carry through a reasoning which leads essentially to the same conclusions as before.

Concerning states, it is difficult to justify the additivity of the expectation values on non-compatible observables (especially in the case of continuous spectra). To be sure all quantum mechanical states are actually of this form; there are no other states known. In ordinary quantum mechanics this property of states is a consequence of a deep theorem of GLEASON[8]) which has been proved only a few years ago. In order to avoid all possible objections of circular reasoning we shall, however, not assume the validity of quantum mechanics and therefore we shall not use this additivity property either.

There is a third reason for our work: VON NEUMANN has shown that quantum mechanics would have to be objectively false if another description than the statistical one is possible, but from his result it is not immediately obvious *how* false it would have to be. The deviations of quantum mechanics from the "true theory" could be so minute that ordinary quantum mechanics could still be regarded as a valid approximation to the "true" laws of microphysics. We shall however show that the hidden variable interpretation is only possible if the theory is observably wrong. With this result the hidden variables are empirically refuted.

We emphasize here especially that this result has a range of validity which transcends ordinary quantum mechanics. It is equally applicable, for instance, in quaternion quantum mechanics or any other, as yet unknown form of quantum mechanics.

The important notion is that of *compatibility* or *incompatibility* of certain observations. In ordinary quantum mechanics two observables are incompatible if they do not commute. We shall show that the notion of incompatibility does not depend on this particular representation of observables. Moreover, it is an empirically verifiable property between certain pairs of observables. Loosely stated the main result is simply this: if there exist incompatible observables then hidden variables are not possible.

This result will perhaps suffice to devaluate the reproach that von Neumann's proof contains circular reasoning. On the contrary we hope that it will serve to underline the far-reaching implications of von Neumann's analysis at an early stage of the evolution of quantum mechanics.

We feel this work is today even more relevant than at the time when it was written since the mounting pressure from high energy physics for a modification of our conceptual frame should not lead us to lose sight of the foundations which are secure and on which a future expansion will have to be constructed.

II. The "logic" of quantum mechanics

In ordinary quantum mechanics, observables are represented by self-adjoint linear operators in an infinite-dimensional Hilbert space. The real spectrum of such an operator are the possible values of the observable represented by the operator. The simplest operators of this kind are the projections. Their spectrum consists only of the two points 0 and 1, and they represent so-called yes-no experiments. These are observables which can assume only one of two alternatives which we may designate by 1 or 0, yes or no, true or false. It is easy to exhibit a large number of examples of such observables and it is equally easy to show that the measurement of any measurable physical quantity can be reduced to the determination of a series of yes-no experiments. We shall in the following refer to such yes-no experiments as *propositions* of a physical system.

The propositions of any physical system have a structure which is quite independent of the particular fact that in quantum mechanics they are represented by projection operators. It is to some extent common with classical mechanics where such a representation is not possible for instance. This structure property is summarized by the statement that the propositions form an *ortho-complemented lattice*.

We shall briefly enumerate in the following the characteristic properties of such a lattice to the extent that they are used for the purpose of this paper. For a more detailed discussion of the structure theory of this lattice we refer to the thesis of one of us[9]) which will be published in the near future[10]).

Let $\mathcal{L}$ be the set of all propositions of a physical system. In such a set exists first of all a partial ordering. We say two propositions a and b are in the relation $a \subseteqq b$ if, whenever a is true, b is also true. This relation is the empirical analogue of the logical implication in the propositional calculus of ordinary logic. We must remember that it is ultimately an empirical statement and expresses the fundamental kinematic structure of the physical system under discussion. This relation satisfies the following formal properties

$$\left.\begin{array}{ll} 1) & a \subseteqq b \text{ and } b \subseteqq a \text{ is equivalent to } a = b, \\ 2) & a \subseteqq b \text{ and } b \subseteqq c \text{ implies } a \subseteqq c, \end{array}\right\} \quad \mathrm{I}$$

which characterize a partial ordering of the set of propositions $\mathcal{L}$. We shall reserve the notation $a \subset b$ for $a \subseteqq b$ but *not* $a = b$.

To any pair of propositions a and b from $\mathcal{L}$ we can associate two others in $\mathcal{L}$, which we may call the *greatest lower bound* or the *least upper bound* and which we denote respectively by $a \cap b$ and $a \cup b$. They are defined by the following properties

$$\left.\begin{array}{ll} 1) & x \subseteqq a \cap b \text{ is equivalent to } x \subseteqq a \text{ and } x \subseteqq b\,, \\ 2) & y \supseteqq a \cup b \text{ is equivalent to } y \supseteqq a \text{ and } y \supseteqq b\,. \end{array}\right\} \quad \text{II}$$

The proposition $a \cap b$ is nothing else than the proposition a 'and' b while $a \cup b$ is the proposition a 'or' b. They are obviously also propositions and in fact it is possible to give an operational description of their measurements if a and b are measurable[9]).

The formation of least upper bound and greatest lower bound can be extended to any subset $\{a_i\}$ $(i \in I)$ of propositions and we assert that it is meaningful to define propositions $\bigcap_i a_i$ and $\bigcup_i a_i$ in $\mathcal{L}$, with the properties

$$\left.\begin{array}{ll} 1) & x \subseteqq \bigcap_i a_i \text{ is equivalent to } x \subseteqq a_i \text{ for all } i \in I\,, \\ 2) & y \supseteqq \bigcup_i a_i \text{ is equivalent to } y \supseteqq a_i \text{ for all } i \in I\,. \end{array}\right\} \quad \text{II}'$$

This implies the existence of the *absurd* and the *trivial* propositions ϕ and I defined by

$$\phi = \bigcap_{a_i \in \mathcal{L}} a_i\,, \qquad I = \bigcup_{a_i \in \mathcal{L}} a_i\,.$$

There is a third operation needed to complete the propositional calculus of physical propositions and that is the *negation*. To every proposition a we can associate another, a', denoting the proposition 'non' a. This is the proposition which is false whenever a is true. It satisfies therefore the following characteristic properties

$$\left.\begin{array}{ll} 1) & (a')' = a\,, \\ 2) & a \cup a' = I\,, \quad a \cap a' = \phi\,, \\ 3) & (a \cup b)' = a' \cap b'\,. \end{array}\right\} \quad \text{III}$$

At this point we make a few remarks which will take the place of a fuller elaboration of the propositional calculus implied by the properties I, II, III.

A lattice of propositions may contain minimal elements in the following sense: if $p \in \mathcal{L}$ and $x \subset p$ implies $x = \phi$ then we say p is a *minimal element* of $\mathcal{L}$. We shall refer to such elements also as *points*. If every element $a \in \mathcal{L}$ contains at least one point we shall call the lattice, *atomic*. All the usual treatments of quantum mechanics assume (usually implicitly) that the lattice of propositions is atomic. Actually the empirical justification for this is quite meager. It was von NEUMANN who first pointed out the existence of non-atomic lattices (called continuous geometries) and who suspected their possible physical significance in a generalized quantum mechanics.

We shall avoid in this paper the use of the assumption that the lattice is atomic.

We observe next that there is nothing in the properties I, II, and III which would distinguish classical from quantum kinematics. They are so general that they hold for any physical system which is accessible to observation. In order to distinguish more detailed features of these systems it is necessary to observe further structural properties of the lattice of propositions. In the paper by BIRKHOFF and VON NEUMANN already quoted in Reference [10]) it is shown that the classical systems are characterized by the *distributive law* which may be stated as follows in two equivalent dual forms.

$$\left.\begin{aligned} &1) \quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c)\,, \\ &2) \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)\,. \end{aligned}\right\} \quad \text{(D)}$$

It was further shown in the same paper that the distributive law is accessible to empirical verification and that it is in fact false for quantum systems. Further examples which demonstrate this fact are discussed in Reference [9]).

One of the main problems of general quantum mechanics is the discovery of the appropriate law which replaces the distributive law (D) of the classical systems. This problem has recently been solved [9]).

The new axiom can best be expressed by introducing first the concept of *compatibility*. Two propositions a and b are said to be compatible if they satisfy the symmetrical relation

$$(a \cap b') \cup b = (b \cap a') \cup a\,. \tag{1}$$

We shall use the shorter notation $a \leftrightarrow b$ for this relation. The detailed analysis of this relation shows that it has exactly the properties which one would associate with measurements which can be performed simultaneously without disturbing each other. For instance, if the propositions are represented by projection operators in a Hilbert space, as it is the case for ordinary quantum mechanics, then the relation (1) is equivalent with the property that the projections commute with one another.

It is a theorem that a lattice is distributive (that is, it is a lattice which satisfies (D)) if and only if any two propositions are compatible. Since (D) is empirically contradicted for quantum systems it is thus also empirically established that for such systems there always exist propositions which are not compatible. This important point will be essential in the argument to be presented establishing the impossibility of hidden variables.

The new axiom which replaces (D) for quantum systems can be expressed very concisely in the following form

$$a \subseteqq b \;\text{ implies }\; a \leftrightarrow b\,. \tag{P}$$

In this form it has an immediate physical interpretation which makes it very plausible indeed. We shall refer to a lattice which satisfies I, II′, III, and (P), a generalized *proposition system*.

The propositions of conventional quantum mechanics of simple systems do satisfy the axioms I, II′, III, and (P) but these axioms leave room for systems of greater generality. In particular, such a proposition system may, for instance, admit superselection rules. This is very important since such rules are known to exist [7]). In order to formulate this property it is convenient to introduce the notion of *coherent lattices*.

We shall say a proposition $x \in \mathcal{L}$ belongs to the centre $\mathcal{Z}$ of $\mathcal{L}$ if it is compatible with every other proposition in $\mathcal{L}$. The centre always contains the elements ϕ and I. A centre which contains no other elements is called trivial. We can now define: a lattice $\mathcal{L}$ is *coherent* if its centre is trivial.

If the proposition system $\mathcal{L}$ has a non-trivial centre we call it reducible and we say there exist superselection rules.

III. The "state" of a system

It is usually assumed that every physical system is in each instant in a definite "state". This notion belongs to the standard fare of classical mechanics and classical field theory. The state of a system is supposed to be an objectively given reality depending only on the history of preparation of the system.

If this notion of state expresses an objectively given condition of the system then one would expect that identical ways of preparing a state on one and the same system would give identical results for the outcome of the measurement of observables. Already in classical mechanics this is true only in a limited sense. For instance, if one has determined the volume, mass, and temperature of a quantity of an ideal gas then the state of this system is only determined in a thermodynamic sense. Considered as a mechanical system it is grossly underdetermined so that almost any quantity may have a wide range of values.

While thus individually measurable quantities are not entirely determined it makes sense to ask for the average values of these quantities under the conditions imposed by the preparation of the state. These averages can be measured and they are determined by the preparation of the system.

Of course, averages cannot be measured by an observation on a single system. It is necessary to consider an ensemble of identically prepared system. We can consider a state only a meaningful concept if it expresses a property of an ensemble of identically prepared systems. This formulation does not exclude the possibility of a state in the classical sense of the word, where the average of every observed quantity is identical with its individual value for each measurement. In classical mechanics of an n particle system such a state corresponds to a definite value for each of the momenta and coordinates for all the n particles. We shall call such a state a *dispersion-free* state, and we shall characterize it below by a precise mathematical property.

In classical systems the more general definition of a state is in many cases a practical necessity, simply because the necessary manipulations which would be needed to prepare a dispersion-free state are in general not feasible. But classical systems are such that states of this kind are at least possible *in principle*. This is the reason why we have become accustomed to attribute to the dispersion-free states an element of reality, which is independent of the actual observation of such states. Whether this is a general property of all physical systems is just the question under discussion and until it is decided it is better to use the general definition of the state, indicated above, which does not assume that it is dispersion-free.

The upshot of this preliminary discussion is thus the following: a state of a physical system is the result of a set of manipulations of the system which constitute the preparation of the state. A state can be measured by determining the probability distribution of a sufficiently large set of observables. The result of the measurement can be expressed with a certain function $w(a)$ defined on the set of all the propositions $a \in \mathcal{L}$. We can and will call this function the state of the system.

This function will have to satisfy certain properties which characterize it as a generalized probability function. We say generalized because an ordinary probability function is defined as a normalized, additive set function. The subsets of a set are always a Boolean lattice. The propositions of microsystems are, as we have pointed out in the preceding section, not a Boolean lattice and so we must generalize the notion of ordinary probability. This generalization must be done in such a way that on every Boolean sublattice of $\mathcal{L}$ the function $w(a)$ reduces to an ordinary probability function. In this way we arrive at the following definition:

a state is a functional $w(a)$ defined on the propositions $\mathcal{L}$ of a physical system with the following properties

$$0 \leq w(a) \leq 1 , \tag{1}$$

$$w(\phi) = 0 , \quad w(I) = 1 , \tag{2}$$

$$\text{if } a \leftrightarrow b \text{ then } w(a) + w(b) = w(a \cap b) + w(a \cup b) , \tag{3}$$

$$\text{if } w(a_i) = 1 \text{ then } w(\bigcap_i a_i) = 1 , \tag{4}$$

$$\text{if } a \neq \phi \text{ then there exists a state } w \text{ such that } w(a) \neq 0 . \tag{5}$$

Some comment may be appropriate about property (4). If a and b are two propositions, such that for a certain state $w(a) = w(b) = 1$, then this means that a measurement of a and of b will give with certainty the values 1. Axiom (4) says then that the proposition a 'and' b has in this same state also with certainty the value 1. If a and b are compatible then this is an easy consequence of (3), since then

$$w(a) + w(b) = 2 = w\,(a \cup b) + w\,(a \cap b)\ .$$

It follows that $w\,(a \cup b) = 1$. Thus $w\,(a \cap b) = 1$. Thus for an ordinary probability function on a Boolean lattice the relation

$$w(a) = w(b) = 1 \text{ implies } w\,(a \cap b) = 1 \tag{4^0}$$

is always satisfied. However, for the generalized probabilities such as they are needed for states (4)° is a separate postulate. If it is satisfied it can be generalized by induction to any finite system of propositions a_i. We require it to be valid for an arbitrary infinite system. With this requirement we have ruled out functions which would not have any reasonable physical interpretation. This postulate transcends a direct physical justification since physical observations can only refer to a finite number of proposition. We shall however show in the last section that this postulate can be replaced by (4)° if we only wish to derive corollary (3) of theorem I.

The question arises, of course, whether such functions which satisfy conditions (1) to (4) exist. Our experience with ordinary quantum mechanics gives us plenty of examples of such functions. It suffices to verify that the quantum mechanical states do indeed satisfy all four properties.

Postulate (5) is no real restriction, since propositions which are false for every state are in a sense identical with the absurd proposition ϕ and they can be omitted from all physical questions about the system. Two states are different if there exists a proposition a such that $w_1(a) \neq w_2(a)$. If w_1 and w_2 are two different states then $w(a) = \lambda_1\, w_1(a) + \lambda_2\, w_2(a)$ with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$ defines a new state which is different from either one. A state w which can thus be represented with two different states is called a *mixture*. A state which is not a mixture is called *pure* (also called homogeneous by VON NEUMANN).

The quantity $\sigma(a) = w(a) - w^2(a)$ will be called the *dispersion* of the state w on the proposition a. A state is called dispersion-free if $\sigma(a) = 0$ for all $a \in \mathcal{L}$. For such a state $w(a)$ is either 0 or 1. This means every proposition is either true or false with certainty.

If w_1 and w_2 are two different states then there exists a proposition $a \in \mathcal{L}$ such that $w_1(a) \neq w_2(a)$. For this a we have then $0 < \lambda_1 w_1(a) + \lambda_2 w_2(a) < 1$ for any $\lambda_1 > 0$, $\lambda_2 > 0$, with $\lambda_1 + \lambda_2 = 1$. It follows that the mixture $w = \lambda_1 w_1 + \lambda_2 w_2$ must have dispersion. We have thus proved: a dispersion-free state is necessarily pure. The converse need not be true.

IV. Dispersion-free states

We shall now ask whether dispersion-free states can exist on a system $\mathcal{L}$ of propositions. The basic property to be proved is contained in the following

Theorem I: *If there exists a dispersion-free state w on a proposition system $\mathcal{L}$ then there exists a point p in the centre of $\mathcal{L}$ for which $w(p) = 1$.*

Proof: Let $\mathcal{L}_1$ be the subset of all the propositions $a_i (i \in I)$ in $\mathcal{L}$ such that $w(a_i) = 1$, and let $a_0 = \bigcap_{i \in I} a_i$. By property (4) we have $w(a_0) = 1$; thus $a_0 \in \mathcal{L}_1$.

Let $x \subset a_0$, then $w(x) = 0$, since otherwise $w(x) = 1$, and thus $a_0 \subseteqq x$. Since x is compatible with x' (for the proof see Reference [9]) we have by properties (3) and (2)

$$w(x) + w(x') = w\,(x \cup x') = w(I) = 1\,.$$

Thus $x' \in \mathcal{L}_1$; consequently $a_0 \subseteqq x'$. It follows

$$x = x \cap a_0 \subseteqq x \cap x' = \phi \quad \text{or} \quad x = \phi\,.$$

We have thus shown that $x \subset a_0$ implies $x = \phi$ which is the same thing as saying that a_0 is a point: $a_0 = p$.

Next we prove that the point p is compatible with every other proposition in $\mathcal{L}$. Let x be arbitrary. If $x \in \mathcal{L}_1$ then $p \subseteqq x$, hence by axiom (P) $p \leftrightarrow x$. If on the other hand $x \notin \mathcal{L}_1$ then $x' \in \mathcal{L}_1$ and so $p \leftrightarrow x'$.

Using a theorem proved in Reference [9]) it follows from this also $p \leftrightarrow x$. This proves everything.

If $\mathcal{L}$ is coherent then every proposition which is compatible with every other proposition is either ϕ or I. Thus there exist no non-trivial proposition with this property, let alone points. We have thus shown

Corollary 1: *There exist no dispersion-free states on a coherent proposition system.*

This is the old result of von Neumann, but proved under much weaker assumptions. When applied to quantum mechanics it says that, in the absence of superselection rules, every state has dispersion.

We next consider systems with superselection rules. According to corollary 1 they are the only candidates for dispersion-free states. But according to theorem 1 such proposition systems must be of a very special kind. We shall now establish the structure of such systems. Theorem 1 shows that a proposition system $\mathcal{L}$ which admits dispersion-free states always contains a point p in the centre $\mathcal{Z}$ of $\mathcal{L}$. The existence of a point in $\mathcal{Z}$ permits the separation of the lattice $\mathcal{L}$ into two classes by placing two elements x_1 and x_2 into the same class if they satisfy the equivalence relation $x_1 \cap p = x_2 \cap p$. The class which contains the element ϕ is itself a sublattice L_1 of $\mathcal{L}$ and one can verify that every element $x \in \mathcal{L}$ can be represented either in the form $x = \phi \cup x_1$ or in the form $x = p \cup x_1$ with $x_1 \in L$. This means the lattice $\mathcal{L}$ is a direct union of two lattices $\mathcal{L} = L_0 \cup L_1$ where L_0 consists of the two elements ϕ and p.

Let $\tilde{w}$ be another dispersion-free state on $\mathcal{L}$, and let $\tilde{a}_i$ $(i \in I)$ be the elements of $\mathcal{L}$ with the property $\tilde{w}(\tilde{a}_i) = 1$. Define $\tilde{a}_0 = \bigcap_{i \in I} \tilde{a}_i$, so that according to property (4) $\tilde{w}(\tilde{a}_0) = 1$. If $\tilde{w}(a_0) = 0$ then it would follow that $\tilde{w}(a_0) = 1$ or $\tilde{a}_0 \subseteqq a_0$ and since a_0 is a point $\tilde{a}_0 = a_0$. This would imply that $\tilde{w} = w$, contrary to the assumption. Thus $\tilde{a}_0 \neq a_0$, and consequently a_0 is of the form $\tilde{a}_0 = (\phi \cup p_1)$ with $p_1 \in L_1$. It follows that the centre of L_1 contains a point p_1, so that we can decompose L_1 into a direct union $L_1 = L_0' \cup L_2$, where L_0' consists of the two elements ϕ and p_1. Continuing in this fashion by a process of (possibly transfinite) induction we arrive at

Corollary 2: *Proposition systems which admit dispersion-free states are all of the form* $\mathcal{L} = \mathrm{K} \cup \mathrm{L}$, *where* K *is an atomic Boolean lattice. The dispersion-free states are zero on all propositions of the form* $\phi_K \cup \mathrm{x}$ $(\mathrm{x} \in \mathrm{L})$ *where* ϕ_K *is the zero of the lattice* K.

Let us now consider a system $\mathcal{L}$ which admits hidden variables. We define such a system as one with the property that every state is a mixture of dispersion-free states. Thus every state may be written in the form

$$w(a) = \sum_i \lambda_i\, w_i(a) \qquad \text{with} \qquad \lambda_i > 0\,, \quad \sum_i \lambda_i = 1\,,$$

and where $w_i(a)$ is dispersion-free.

Because $\mathcal{L}$ admits dispersion-free states it is, according to corollary 2, of the form $\mathcal{L} = K \cup L$, with K a Boolean sublattice. Let x be an element of $\mathcal{L}$ of the form $x = \phi_K \cup \xi$ with $\xi \in L$, and ϕ_K the zero of the lattice K. According to property (5) there exists a state w such that $w(x) \neq 0$. Since every state is assumed to be a mixture of dispersion-free states there exists even a dispersion-free state with this property. But this is in contradiction with corollary 2 which asserts that every dispersion-free states vanishes on elements of the form $\phi_K \cup \xi$. Thus we conclude that the lattice L reduces to zero and consequently $\mathcal{L} = K$. We have thus established the following

Corollary 3: *If a proposition system* $\mathcal{L}$ *admits hidden variables then every proposition of* $\mathcal{L}$ *is compatible with every other proposition of* $\mathcal{L}$.

V. Hidden variables

The corollary 3 of the preceding section reduces the quest for hidden variables to the property of compatibility for every pair of propositions. It is somewhat unsatis factory that this result could only be established by using the property (4) of states which only for a finite number of propositions has an immediate physical interpretation. In this section we shall show that the statement of corollary 3 holds also for proposition systems and states which satisfy only the weaker condition (4)°. We prove first the

Lemma 1: If a proposition system $\mathcal{L}$ admits hidden variables and if $w(a) = w(b)$ for all states then $a = b$.

Proof: Consider the proposition $x = a \cap (a \cap b)'$. If $x \neq \varnothing$ then according to property (5) there exists a state such that $w(x) \neq 0$. Since every state is a mixture of dispersion-free states in a system which admits hidden variables there must even exist a state such that $w(x) = 1$. Since $a \cap (a \cap b)' \subseteqq a$ it follows $w(a) = 1 = w(b)$. Thus by property

$(4)^\circ$ $w(a \cap b) = 1$ and $w((a \cap b)') = 0$. Since $a \cap (a \cap b)' \subseteqq (a \cap b)'$ it follows $w(x) = 0$ which is a contradiction. Thus $a \cap (a \cap b)' = \phi$. Now using the axiom (P) we find

$$a = (a \cap (a \cap b)') \cup (a \cap b) = a \cap b$$

therefore $a \subseteqq b$. By interchanging the rôle of a and b one proves similarly $b \subseteqq a$. The last two relations imply $a = b$. This proves the Lemma 1.

Lemma 2: If a proposition system $\mathcal{L}$ admits hidden variables then for any pair of propositions $a, b \in \mathcal{L}$ and any state w

$$w(a) + w(b) = w(a \cup b) + w(a \cap b) .$$

Proof: Since $\mathcal{L}$ admits hidden variables every state w is of the form

$$w(a) = \sum_i \lambda_i w_i(a) \qquad \lambda_i > 0 \qquad \sum_i \lambda_i = 1$$

such that all states $w_i(a)$ are dispersion-free. From this follows that it suffices to establish the relation in question for dispersion-free states. Let $w(a)$ be dispersion-free, so that $w(a)$ is either 0 or 1. There are thus four cases possible

$$w(a) = w(b) = 0 ; \quad w(a) = 1 , \ w(b) = 0 ; \quad w(a) = 0 , \ w(b) = 1 ; \quad w(a) = w(b) = 1 .$$

The last two can be reduced to the first two by replacing a by a' and b by b'. It suffices to establish the relation for the first two cases.

If $w(a) = w(b) = 0$ then $w(a \cap b) \leq w(a)$ and therefore $w(a \cap b) = 0$ also. On the other hand $w(a') = 1 - w(a) = 1$ and $w(b') = 1$. Therefore by axiom $(4)^\circ$ $w(a' \cap b') = 1$. It follows that $w(a \cup b) = 1 - w((a \cup b)') = 1 - w(a' \cap b') = 0$. With this the relation is established for case 1.

Next consider $w(a) = 1$, $w(b) = 0$. It follows that $w(a \cup b) \geq w(a) = 1$; therefore $w(a \cup b) = 1$. Furthermore $w(a \cap b) \leq w(b) = 0$; therefore $w(a \cap b) = 0$. This establishes the relation in case 2, and the Lemma 2 is proved.

Theorem II: If $\mathcal{L}$ is a proposition system which admits hidden variables then $a \in \mathcal{L}$ and $b \in \mathcal{L}$ implies $a \leftrightarrow b$.

Remark: This theorem differs from corollary 3 of the preceding section insofar as for the proof of theorem I we have used condition (4) while for theorem II we require only the weaker condition $(4)^\circ$.

Proof: For every state w we have

$$w\big((a \cap b') \cup b\big) = w(a \cap b') + w(b) = w(a) + w(b') - w(a \cup b') + w(b)$$
$$= w(a) + 1 - w(a \cup b') = w(a) + w(a' \cap b) = w\big(a \cup (a' \cap b)\big) .$$

With the preceding lemma 1 this leads to

$$(a \cap b') \cup b = a \cup (a' \cap b) \quad \text{or} \quad a \leftrightarrow b .$$

This proves the theorem.

This theorem permits the reduction of the question concerning hidden variables to an empirical one, viz., whether there exist propositions which are not compatible. Since the lattice operations have a physical interpretation which is accessible to an empirical verification we can decide the question by examining the actual behaviour of specific propositions under observations. To rule out hidden variables it suffices to exhibit two propositions of a physical system which are not compatible.

It turns out that this is quite easy. In fact, the occurrence of incompatible propositions leads to gross macroscopic effects which can easily be verified[11]). With this result the possible existence of hidden variables is decided in the negative.

In conclusion we add two remarks. In a recent publication one of the authors[12]) has raised the question whether hidden variables are perhaps possible in a system which admits non-commuting supersymmetries. Such a system (mentioned by VON NEUMANN) are the projection operators in a factor of type II. According to the foregoing result the answer is negative, since there are non-commuting projections in such rings.

It is however possible, and this is our second remark, that there exist proposition systems which admit *approximate dispersion-free states*[13]). We could, for instance, have the following situation. Let $\sigma(a) = w(a) - w^2(a)$ be the dispersion function and define the over-all dispersion by

$$\sigma = \sup_{a \in \alpha \mathcal{L}} \sigma(a) .$$

We shall then say the system has approximate dispersion-free states if there exists a sequence of states w_n, such that the corresponding $\sigma_n \to 0$ for $n \to \infty$.

It is fairly easy to show that such states do not exist in ordinary quantum mechanics. But nothing is known to us for more general proposition systems. This point merits further investigation.

References

1) J. V. NEUMANN, *Mathematische Grundlagen der Quantenmechanik* (Julius Springer, Berlin 1932). A recent review of this proof was given by J. ALBERTSON, Am. J. Phys. *29*, 478 (1961).

2) L. DE BROGLIE, *La théorie de la mesure en mécanique quantique* (Gauthier-Villars 1957).

3) In Reference [2]), page 27, one finds the following passage: «En réalité, la démonstration belle mais un peu lourde de M. VON NEUMANN ne nous apprend rien de bien nouveau dès que l'on connaît les relations d'incertitude».

4) D. BOHM, *Causality and chance in modern physics* (Routledge and Kegam Paul, London 1958).

5) In the preface to the book by BOHM (Reference [4])) DE BROGLIE writes: "It is possible that, looking into the future to a deeper level of physical reality we will be able to interpret the laws of probability and quantum physics as being the statistical results of the development of completely determined values of variables, which are at present hidden from us".

6) D. BOHM, Reference [4]), page 69: "There is good reason to assume the existence of a subquantum mechanical level that is more fundamental than that at which the present quantum theory holds."

7) G. C. WICK, A. S. WIGHTMAN, and E. P. WIGNER, Phys. Rev. *88*, 101 (1952).

8) A. M. GLEASON, J. Math. and Mech. *6*, 885 (1957).

9) C. PIRON, thesis, University of Lausanne, 1963.

10) The propositional calculus of quantum mechanics was first developed by G. BIRKHOFF and J. VON NEUMANN, Ann. Math. *37*, 823 (1936). See also D. FINKELSTEIN, D. SPEISER, and J. M. JAUCH, CERN-report 59-7.

11) Such an example was first given by G. BIRKHOFF and J. VON NEUMANN in Reference [10]). More are discussed in Reference [9]), to which we refer for further details.

12) J. M. JAUCH, "Continuous geometry and superselection rules", CERN-report 61-14.

13) This notion is due to G. MACKEY from whom we have learnt a great deal on several aspects of these questions. We take this opportunity to thank him here for discussions and correspondence.