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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **37 (1964)**

Heft III

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-113485

Gauge Invariance as a Consequence of Galilei-Invariance for Elementary Particles

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Abstract: If a distinction is made between kinematical and dynamical symmetries of a quantum mechanical localizable system it is possible to extend the principle of Galilei-invariance to elementary particles, subject to arbitrary external forces. This principle implies a severe restriction on the possible types of external forces which can act on a particle. In fact we show in this paper that for a scalar particle the only forces which are compatible with this principle are those which derive from a scalar and vectorpotential. The Hamiltonian takes on the usual gauge-invariant form of a charged particle in interaction with arbitrary electromagnetic forces.

1. Introduction

The non-relativistic Schrödinger equation for a single, charged, and spinless particle is determined by a Hamiltonian operator $\boldsymbol{H}$ of the form

$$H = \frac{1}{2} (\boldsymbol{P} - \boldsymbol{A})^2 + V \, , \tag{1}$$

where $\boldsymbol{A}$ is, apart from a numerical factor, the vectorpotential of the external field and $\boldsymbol{V}$ is the scalar potential. Both of these quantities may, in the most general case, be functions of the position operator $\boldsymbol{Q}$ and eventually of the time t. The operator $\boldsymbol{P}$ is the canonically conjugate momentum operator to $\boldsymbol{Q}$, and the pair satisfies the usual commutation rules.

This particular form (1) of the Hamiltonian is easy to derive from the classical Hamiltonian of a charged particle in an arbitrary external field. The classical equations of motion which result from this Hamiltonian imply that the particle is subject to the Lorentz force

$$\boldsymbol{F} = - V + \dot{\boldsymbol{Q}} \, x \, (\nabla \, x \, \boldsymbol{A}) \, . \tag{2}$$

The velocity dependend part of this force is of a very special kind. For instance it is linear in the velocity and it is always perpendicular to it.

In classical physics there are many other forces possible and it is not difficult to set up, with the usual transcription rules, a corresponding quantum mechanical equation of motion. Yet such forces do not seem to occur for elementary systems. It is therefore not unreasonable to attribute to the systems which are defined by a Hamiltonian of the form (1) a more fundamental significance which should be expressible in a simple and general invariance principle.

We shall show that this principle is a suitably generalized form of the principle of Galilei-invariance, and we shall show that it excludes all Hamiltonians except those of the form (1).

The Galilei transformations proper at a fixed instant of time are those transformations which change the velocity by an arbitrary constant amount, and which leave the position unchanged. They correspond to the observables used by an observer who is in constant relative motion with respect to the original observer.

Such transformations are not symmetry transformations in the usual sense of the word, since the Hamiltonian operator is not invariant, not even for a free particle. We shall therefore introduce the distinction between kinematic and dynamic symmetry transformations. This distinction is only significant for quantum mechanical systems. It will be discussed in the following section 2. In section 3 we postulate that Galilei transformations shall be a kinematic symmetry transformation, and in section 4 we show that this postulate restricts the possible Hamiltonians to those of the form (1). In section 5 finally we show that these and only these Hamiltonians are gauge invariant.

2. Kinematical and Dynamical Symmetries

Let A_i be a set of self adjoint operators representing all the observables of a system. A permutation $A_i \to A_i'$ of this set is called a *kinematic symmetry* transformation if there exists a unitary or antiunitary operator U such that

$$A_i' = U A_i U^{-1} . \tag{3}$$

Examples of such transformations are easily constructed. The one-dimensional motion of a spinless particle is described by two basic observables, the position operator Q and the momentum operator P and certain functions of them which need not be further specified. If we carry out the translation of the position $Q \to Q + a$ or of the momentum $P \to P + b$, or both we obtain a kinematic symmetry transformation since

$$Q + a = e^{iaP} Q e^{-iaP} , \tag{4}$$

$$P + b = e^{-ibQ} P e^{ibQ} . \tag{5}$$

It is clear from the above definition that kinematical symmetry transformations are canonical transformations, but the converse is not true.

For instance, still with the above example, we may consider the transformation

$$\left.\begin{aligned} Q' &= \frac{1}{2} Q^2 , \\ P' &= \frac{1}{2} (P Q^{-1} + Q^{-1} P) . \end{aligned}\right\} \tag{6}$$

It is canonical, since the relation $(Q', P') = i$ is true on a dense linear manifold of the Hilbert space, but it is not a kinematical symmetry transformation. Indeed the latter leave the spectrum of the operators invariant. But the spectrum of Q is the entire real axis while that of Q' is the positive real axis only.

We see therefore that the distinction between canonical transformations and kinematical symmetry transformations is closely related to the non-uniqueness of the canonical commutation rules. This requires some comment since one often finds with reference to the well-known uniqueness proof of VON NEUMANN*) statements to the contrary in the literature.

This proof of VON NEUMANN refers to the commutation rules in the bounded form. If we define the operators $U_\alpha = e^{i\alpha P}$, $V_\beta = e^{i\beta Q}$, then one shows by formal manipulations that the canonical commutation rules

$$[Q, P] = i \,, \tag{7}$$

imply

$$U_\alpha V_\beta = e^{i\alpha\beta} V_\beta U_\alpha \,. \tag{8}$$

One can prove that all irreducible representations of (8) are unitarily equivalent. This does not imply that all irreducible representations of (7) are equivalent too. The representation theory of (7) is a much more difficult problem than that of (8). because Q and P are unbounded operators and they can be defined at most on a dense linear manifold of the Hilbert space. In order to make the representation problem well-defined one should require that there should exist a dense linear manifold D on which both Q and P are defined and which is invariant both under the operation of Q and P.

It is easy to show with examples that there exist many inequivalent representations of (7) which satisfy these conditions. We have given two with the above example.

One might therefore conclude that one should add a further condition in order to force uniqueness of the representation. One such condition, motivated by physical considerations, would be that the operators Q and P are essentially self adjoint on D. Whether this condition will imply uniqueness is not known.

After this digression we return to the notion of kinematic symmetry transformations. It is clear that the set of all such transformations forms a group which we shall call the *kinematic symmetry group* of the system.

Among all the possible kinematical symmetries we can consider the subgroup of those transformations which leave the Hamiltonian invariant too. We call such transformations *dynamical* symmetry transformations. They constitute a subgroup, which we shall call the *dynamical symmetry group*.

A given transformation $A_i \to A_i'$ determines the operator U up to a numerical factor of magnitude 1 if the system of observables is an irreducible system. If this is not the case, then there exist superselection rules and U is only determined up to an arbitrary supersymmetry**).

3. Galilei-Invariance for an Elementary Localizable System

We shall begin this section with a brief review of the notions of localizability as well as homogeneity and isotropy. Localizability expresses the fundamental

*) J. VON NEUMANN, Math. Ann. *104*, 570 (1931).

**) For this notion see J. M. JAUCH and B. MISRA, Helv. Phys. Acta *34*, 699 (1961).

property of a particle of being situated in a (usually Euclidean) configuration space. In our discussion this space is assumed to be the three-dimensional Euclidean space E_3. Physically 'being situated' means that there exists a series of possible measurements which determine the location of the particle in this space all of which are compatible with one another. Mathematically this property is expressed by a spectral measure which associates with every Borel subset Δ of E_3 a projection operator E_Δ such that

$$E_{\Delta_1 \cap \Delta_2} = E_{\Delta_1} E_{\Delta_2} \tag{9}$$

and

$$E_{\Delta'} = I - E_\Delta , \tag{10}$$

where Δ' is the complementary set of Δ within E_3.

The system is called an *elementary system* if these projection operators are a complete system of commuting observables*). This means that the abelian algebra of bounded operators $\mathfrak{A} = \{E\}''$ generated by them is maximal abelian: $\mathfrak{A} = \mathfrak{A}'$.

It is natural to adopt the spectral representation of this spectral measure by defining the Hilbert space $\mathfrak{H} = L^2(E_3)$ as the space of all Lebesgue square integrable functions $\psi(\boldsymbol{x})$ on E_3. The projection operators operate then as follows on such functions:

$$(E_\Delta \psi)(\boldsymbol{x}) = 1_\Delta(x)\, \psi(\boldsymbol{x}) , \tag{11}$$

where $1_\Delta(\boldsymbol{x})$ is the characteristic function of the set Δ, defined by

$$1_\Delta(\boldsymbol{x}) = \begin{cases} 1 \text{ for } x \in \Delta , \\ 0 \text{ for } x \notin \Delta . \end{cases} \tag{12}$$

The notions of homogeneity and isotropy of the physical space E_3 is expressed by the statement that the spectral measure E is a 'system of imprimitivities' with respect to the group of Euclidean motions of E_3. This means the following: let $(\boldsymbol{\alpha}, R)$ denote a general element of this group. It can be considered as the transformation

$$\boldsymbol{x}' = R\,\boldsymbol{x} + \boldsymbol{\alpha} ,$$

where R is a real orthogonal matrix. The elements of this group satisfy the composition law

$$(\alpha_1, R)(\boldsymbol{\alpha}_2, R_2) = (\boldsymbol{\alpha}_1 + R\,\boldsymbol{\alpha}_2, R_1 R_2) . \tag{13}$$

Denote by $[\Delta](\boldsymbol{\alpha}, R)$ the set of points $R\,\boldsymbol{x} + \boldsymbol{\alpha}$ with $\boldsymbol{x} \in \Delta$, then E is a system of imprimitivities with respect to this group if there exist unitary operators $W(\boldsymbol{\alpha}, R)$ such that

$$E_{[\Delta](\boldsymbol{\alpha}, R)} = W^{-1}(\boldsymbol{\alpha}, R)\, E_\Delta\, W(\boldsymbol{\alpha}, R) . \tag{14}$$

This equation implies that the operators $W(\boldsymbol{\alpha}, R)$ are a projective representation of the 6-parameter group of Euclidean motions, that is they satisfy equations of the form

$$W(\boldsymbol{\alpha}_1, R)\, W(\boldsymbol{\alpha}_2, R_2) = \omega(\boldsymbol{\alpha}_1 R_1; \boldsymbol{\alpha}_2 R_2)\, W(\boldsymbol{\alpha}_1 + R_1 \boldsymbol{\alpha}_2, R_1 R_2) , \tag{15}$$

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*) J. M. Jauch, Helv. Phys. Acta *33*, 711 (1960).

where $\omega(\boldsymbol{\alpha}_1, R_1 \, \boldsymbol{\alpha}_2 R_2)$ is a numerical factor of magnitude 1. It was shown by BARGMANN*) that the arbitrary phase factors in the definition of the operators W can be adjusted so that $\omega = 1$. Every projective representation of this group is thus equivalent to a vectorrepresentation. We shall therefore omit ω in the following.

If one introduces the 'position operators' Q_r ($r = 1, 2, 3$) defined by

$$(Q_r \, \psi) \, (\boldsymbol{x}) = x_r \, \psi(\boldsymbol{x}) \, , \tag{16}$$

then one proves easily that these operators transform under the group $W(\boldsymbol{\alpha}, R)$ in the following way

$$Q'_r = W(\boldsymbol{\alpha}, R) \, Q_r \, W^{-1}(\boldsymbol{\alpha}, R) \, , \tag{17}$$

where

$$Q'_r = (R \, \boldsymbol{Q})_r + \alpha_r \, . \tag{18}$$

The translations proper are obtained by choosing for R the unit matrix I. Thus we denote by

$$U_{\boldsymbol{\alpha}} = W(\boldsymbol{\alpha}, I)$$

the operators with the property

$$Q_r + \alpha_r = U_{\boldsymbol{\alpha}} \, Q_r \, U_{\boldsymbol{\alpha}}^{-1} \, . \tag{19}$$

From this follows for the operators $V_{\boldsymbol{\beta}} = e^{i\boldsymbol{\beta}\cdot\boldsymbol{P}}$ the commutation rule in WEYL's form

$$U_{\boldsymbol{\alpha}} \, V_{\boldsymbol{\beta}} = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} \, V_{\boldsymbol{\beta}} \, U_{\boldsymbol{\alpha}} \, . \tag{20}$$

All these well-known things are presented here as a background for the main point of this section to be discussed now. We wish to extend the group of Euclidean motions by adding the Galilei transformations proper to be defined in the following.

By postulating the existence of position operators Q_r which do not depend explicitly on time we have chosen a particular picture in which the state vectors satisfy a Schrödinger equation of the form

$$i \, \dot{\psi} = H \, \psi \, . \tag{21}$$

Here H is a self adjoint operator representing the Hamiltonian of the system. We shall not assume that the Hamiltonian is independent of the time t.

For any observable A, not depending explicitly on time we have

$$\frac{d}{dt} (\psi, A \, \psi) = (\psi, \dot{A} \, \psi) \, , \tag{22}$$

where $\dot{A}$ is defined by

$$\dot{A} = i \, [H, A] \, . \tag{23}$$

In particular, if A is one of the position operators Q_r we may define the velocity operator

$$\dot{Q}_r = i \, [H, Q_r] \quad (r = 1, 2, 3) \, , \tag{24}$$

*) V. BARGMANN, Ann. Math. *59*, 1 (1952).

since H and Q_r are unbounded operators the operator $\dot{Q}_r$ is only defined on a dense linear manifold. We assume that $\dot{Q}_r$ is essentially self-adjoint on this domain of definition and we denote with the same symbol its unique self adjoint extension.

The $\dot{Q}_r$ thus defined is an observable. Its measurement can be effected with real experiments which need not be discussed here in detail.

We shall now define the system as *Galilei-invariant* if the transformation

$$\left.\begin{aligned} Q_r &\to Q_r\,, \\ \dot{Q}_r &\to \dot{Q}_r + v_r \end{aligned}\right\} \tag{25}$$

is a kinematic symmetry transformation.

It is clear that this condition implies certain restrictions on the nature of the operators $\dot{Q}_r$ and hence also on the operator H which is involved in the definition of $\dot{Q}_r$. For instance the spectrum of each $\dot{Q}_r$ is continuous and extends from $-\infty$ to $+\infty$. From this fact alone it can be inferred that the physical implication of this principle must be rather incisive. For instance relativistic systems are excluded as may be seen from the remark that velocity for relativistic systems is limited by the velocity of light.

Since the definition of $\dot{Q}_r$ involves the evolution operator H it is clear that the principle must also lead to certain restrictions on the operator. It is the main purpose of this paper to determine the effect of these restrictions on the operator H.

This we shall do in the following section.

4. Consequences of Galilei-Invariance

With the Galilei transformations of the last section the group of kinematical symmetry transformations comprises a nine-parameter group and we denote by $(\boldsymbol{v}, \boldsymbol{\alpha}, R)$ a general element of this group and by $W(\boldsymbol{v}, \boldsymbol{\alpha}, R)$ its unitary projective representation. The subgroup of unitary operators

$$G_{\boldsymbol{v}} = W(\boldsymbol{v}, o, I) \tag{26}$$

is the representation of the three-parameter subgroup of the Galilei transformations proper. It satisfies

$$G_{\boldsymbol{v}}\, \dot{Q}_r\, G_{\boldsymbol{v}}^{-1} = \dot{Q}_r + v_r \quad (r = 1, 2, 3)\,. \tag{27}$$

Unlike the representation for the Euclidean motions the projective representations of the group with the elements $(\boldsymbol{v}, \boldsymbol{\alpha}, R)$ has a one-parameter family of inequivalent classes of representations (cf. note p. 288). Moreover it has been shown in the force-free case by INÖNÜ and WIGNER, that the class which contains the vectorrepresentation does not permit a reasonable physical interpretation*). We shall therefore

*) E. INÖNÜ and E. P. WIGNER, Nuovo Cim. *9*, 705 (1952).

assume that the operators W are one of the known projective representations of this group so that we have

$$W(\boldsymbol{v}_1, \boldsymbol{\alpha}_1, R_1)\, W(\boldsymbol{v}_2, \boldsymbol{\alpha}_2, R_2) = e^{i\xi}\, W(\boldsymbol{v}, \boldsymbol{\alpha}, R),$$

where $\boldsymbol{v}$, $\boldsymbol{\alpha}$, R are given by the composition law of the group

$$\boldsymbol{v} = \boldsymbol{v}_1 + R_2\, \boldsymbol{v}_1, \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + R_2\, \boldsymbol{\alpha}_1, \quad R = R_1\, R_2 .$$

The phase factor ξ can, by suitable choice of the phases, be brought into the form (cf. note p. 288)

$$\xi = \frac{\mu}{2}\, (\boldsymbol{v}_1\, \boldsymbol{\alpha}_2 - \boldsymbol{v}_2\, \boldsymbol{\alpha}_1),$$

where μ is a parameter which characterizes the projective representation. If we specialize this relation for the two three-parameter subgroups $G_{\boldsymbol{v}} = W(\boldsymbol{v}, o, I)$ and $U_{\boldsymbol{\alpha}} = W(o, \boldsymbol{\alpha}, I)$ we obtain

$$U_{\boldsymbol{\alpha}}\, G_{\boldsymbol{v}} = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{v}}\, G_{\boldsymbol{v}}\, U_{\boldsymbol{\alpha}} . \tag{20'}$$

Comparing this result with Equation (20) we see, by setting $\mu\, \boldsymbol{v} = \boldsymbol{\beta}$, that these are again the canonical commutation rules in WEYL's form.

At this point we make use of the fundamental uniqueness theorem concerning these commutation rules: The irreducible representations of the commutation rule (15) are all unitarily equivalent (cf. note p. 286).

In order to apply this theorem we need to know that both our representations (20) and (20)′ are irreducible. This is a consequence of the fact that the position operators Q_r form a complete set of commuting operators. To see this we verify first that every bounded operator X which commutes with both $U_{\boldsymbol{\alpha}}$ and $V_{\boldsymbol{\beta}}$ is a multiplum of the unit operator. Irreducibility follows then from SCHUR's lemma.

Thus let X be a bounded operator which commutes with $U_{\boldsymbol{\alpha}}$ and $V_{\boldsymbol{\beta}}$. Since the $V_{\boldsymbol{\beta}}$ generate a maximal abelian algebra, X must be some function of the Q_r, for instance $X = F(Q_r)$. Because it commutes also with $V_{\boldsymbol{\beta}}$, we find the relation

$$X = U_{\boldsymbol{\alpha}}^{-1}\, X\, U_{\boldsymbol{\alpha}} = F(Q_r + \alpha_r) = F(Q_r) .$$

Thus the function F is in fact a constant, X is a multiplum of the unit operator, and the representation is indeed irreducible. A corollary of this result is that the operators $U_{\boldsymbol{\alpha}}$ also generate a maximal abelian algebra.

The uniqueness theorem, applied to our two irreducible representations (20) and (20)′ gives now the following result: There exists a unitary operator S which commutes with $U_{\boldsymbol{\alpha}}$ and which satisfies

$$G_{\boldsymbol{v}} = S^{-1}\, e^{i\mu\boldsymbol{Q}\cdot\boldsymbol{v}}\, S . \tag{28}$$

Since S commutes with $\boldsymbol{P}$ it follows from this that

$$G_{\boldsymbol{v}}\, P_r\, G_{\boldsymbol{v}}^{-1} = P_r + \mu\, v_r . \tag{29}$$

By combining this result with Equation (27), we find that $\mu \dot{Q}_r - P_r$ commutes with G_v and must therefore be a function of the Q_r alone. Hence we find the important relation

$$\mu \dot{Q}_r = P_r - A_r , \tag{30}$$

where A_r ($r = 1, 2, 3$) are three functions of $\boldsymbol{Q}$, which may depend on the time t.

From relation (30) we obtain the commutation rules

$$\mu [Q_r , \dot{Q}_s] = i \, \delta_{rs} .$$

Thus if we define $H_0 = \dot{\boldsymbol{Q}}^2 \, \mu/2$ we find

$$i [H_0 , Q_s] = \dot{Q}_s .$$

It follows from this that the operator $H - H_0$ commutes with Q_s and, since the Q_s are a complete set of commuting observables, it must be a function V of the Q_r, which may depend on the time t.

Thus we have shown that the Hamiltonian H must have the form

$$H = \frac{1}{2 \mu} (\boldsymbol{P} - \boldsymbol{A})^2 + V , \tag{1}$$

where $\boldsymbol{A}$ and V are both functions of $\boldsymbol{Q}$ and possibly of the time t.

We have thus succeeded in deriving the special form (1) of the Hamiltonian by assuming only the principle of Galilei-invariance, stated in the preceding section.

5. Gauge-Invariance

In classical electrodynamics one shows that the electromagnetic field determines the potentials $\boldsymbol{A}$ and V only up to a gauge transformation

$$\left.\begin{aligned} \boldsymbol{A} &\to \boldsymbol{A} + \nabla\phi , \\ V &\to V - \frac{\partial \phi}{dt} . \end{aligned}\right\} \tag{31}$$

We shall now show that to this property of the potential corresponds a certain invariance property of the system characterized by the Hamiltonian (1).

Let Ω be the unitary operator defined by

$$(\Omega \, \psi) \, (\boldsymbol{x}) = e^{i\phi(\boldsymbol{x})} \, \psi(\boldsymbol{x}) . \tag{32}$$

Here $\phi(\boldsymbol{x})$ is an arbitrary differentiable function of $\boldsymbol{x}$ which may also depend explicitly on time. Under this transformation the operator $\boldsymbol{P}$ transforms according to

$$\Omega \, \boldsymbol{P} \, \Omega^{-1} = \boldsymbol{P} - \nabla\phi , \tag{33}$$

and the Schrödinger equation is changed into a new equation for the state vector $\varphi_t \equiv \Omega \, \psi_t$ which we may write as

$$i \, \dot{\varphi}_t = G \, \varphi_t .$$

The new operator G is obtained from the old one by the substitution of $\psi_t = \Omega^{-1} \varphi_t$ into Equation (21), with the result

$$G = \Omega H \Omega^{-1} + i \frac{d}{dt} \Omega^{-1} .$$

The explicit evaluation of this with (32) and (33) gives

$$G = \frac{1}{2 \mu} (\boldsymbol{P} - (\boldsymbol{A} + \nabla \phi))^2 + V - \frac{\partial \phi}{\partial t} . \tag{34}$$

Thus we see that the 'phase-transformations' (32) are equivalent with the gauge transformations (31). This we call the gauge invariance of the theory. The only interaction which has this property is that given by the Hamiltonian (1). We have thus established that gauge-invariance is a consequence of the Galilei-invariance as defined in section 3.

6. Concluding Remarks

The foregoing result establishes a connection between two at first sight entirely different principles namely Galilei-invariance and gauge-invariance. Which one of these is more fundamental is perhaps a matter of taste. The only thing that is certain, according to our result, is that one cannot violate one of them without violating the other too. This is the essential point of the result.

In spite of some effort, it has not been possible to extend these considerations to the relativistic case. This is rather strange, since gauge invariance is easily transferred to relativistic field theories, but of course not Galilei-invariance. Thus we do not yet know the relativistic analogue of this result.