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Spatial Cluster Decomposition Properties of the S-Matrix

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Although no nontrivial rigorous model of a local relativistic quantum field theory is known up to now, the general theory of quantized fields is believed to describe short-range interactions of relativistic particles. The physically reasonable properties of the Wightman framework¹⁾ supporting this are exemplified by the space-like cluster decomposition properties of the vacuum expectation values of products of field operators²⁾³⁾ and by the existence of asymptotic scattering states in strong convergence²⁾⁴⁾.

It is the purpose of this note to prove that the S-matrix in a local relativistic quantum field theory fulfills the spatial cluster decomposition properties, which have recently⁵⁾ been postulated to hold for any reasonable relativistic scattering theory. It will be demonstrated that the outcome of a scattering or production process between massive particles is asymptotically independent of the presence of other particles. This justifies an expansion of the S-matrix in cluster amplitudes⁵⁾ with some analogy to sums of connected Feynman diagrams in perturbation theory.

We consider a Wightman field theory of an at most countable family of local spinor fields $A_\mu^r(x)$. We assume that polynomials B_i^* in the smeared-out fields

$$B_i^* = \sum_{(r)(\mu)}^{\leq \infty} \int i\varphi_{(\mu)}^{(r)}(x_1, \dots, x_n) A_{\mu_1}^{r_1}(x_1) \dots A_{\mu_n}^{r_n}(x_n) dx_1 \dots dx_n \quad (1)$$

create from the vacuum Ω states belonging to irreducible representations $[m_i, s_i]$, $m_i > 0$, of the covering group of the inhomogeneous proper orthochronous Lorentz group:

$$B_i^* \Omega \in \mathfrak{H}_{[m_i, s_i]}, \quad B_i \Omega = 0. \quad (2)$$

For a smooth positive frequency solution $f \in \mathfrak{S}_+^{m_i}$ of the Klein Gordon equation to the mass m_i ($f_i(x) = (2\pi)^{-3/2} \int d^3p \theta(p) \delta(p^2 - m_i^2) \hat{f}_i(\mathbf{p}) e^{-ipx}$, $\hat{f}_i(\mathbf{p}) \in \mathfrak{S}(R^3)$) we define with $B_i(x) = U(x, 1) B_i U(x, 1)^{-1}$:

$$B_i(f_i, t) = i \int_{x^0=t} d^3x f_i^*(x) \overleftrightarrow{\partial}_0 B_i(x). \quad (3)$$

Under these assumptions HAAG⁴⁾ and RUELLE²⁾ have proved that

$$s - \lim_{t \rightarrow \pm \infty} B_1^*(f_1, t) \dots B_n^*(f_n, t) \Omega = |\bar{f}_1, \dots, \bar{f}_n^{ex}\rangle, \quad ex = in, out, \quad (4)$$

exists in the strong topology in $\mathfrak{H}$ and defines an asymptotic scattering state belonging to the wave functions $\bar{f}_i = \{\bar{f}_i(\mathbf{p})_\alpha, \alpha = -s_i, -s_i + 1, \dots, s_i\}$. The subspaces $\mathfrak{H}^{ex}$ of scattering states have the structure of Fock spaces over $\{\mathfrak{H}_{[m_i, s_i]}\}$. The scattering amplitudes $(\bar{f}_1, \dots, \bar{f}_m^{out} | \bar{g}_1 \dots \bar{g}_n^{in})$ are tensor-valued tempered distributions

$$S_{mn}^{(\alpha)(\beta)}(p_1, \dots, p_m; -q_1, \dots, -q_n) \in \mathfrak{S}'(R^{3(m+n)}) \quad (5)$$

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and define under the additional assumption $\mathfrak{H}^{in} = \mathfrak{H}^{out}$ a unitary S -matrix by $(\bar{f}_1, \dots, \bar{f}_m^{out} | S | \bar{g}_1, \dots, \bar{g}_n^{out}) = (\bar{f}_1, \dots, \bar{f}_m^{out} | \bar{g}_1, \dots, \bar{g}_n^{in})$. For a 3-vector $\mathbf{a}$ and $f \in \mathfrak{S}^+$ we set $f^a(x) = f(x^0, \mathbf{x} - \mathbf{a})$. If $B^*(f, t) \Omega = |f\rangle$, then, due to the translation invariance of $B(x)$, one has $B^*(f^a, t) \Omega = |\bar{f}^a\rangle$ with $\bar{f}^a(\mathbf{p})_\alpha = e^{i\mathbf{p}\mathbf{a}} \bar{f}(\mathbf{p})_\alpha$. We are now prepared to prove the

Theorem 1: Under the above assumptions

$$|\langle \bar{f}_1 \dots \bar{f}_k, \bar{f}_{k+1}^a, \dots, \bar{f}_{k+l}^{out} | \bar{g}_1, \dots, \bar{g}_m, \bar{g}_{m+1}^a, \dots, \bar{g}_{m+n}^{in} \rangle - \langle \bar{f}_1 \dots \bar{f}_k^{out} | \bar{g}_1, \dots, \bar{g}_m^{in} \rangle \langle \bar{f}_{k+1} \dots \bar{f}_{k+l}^{out} | \bar{g}_{m+1}, \dots, \bar{g}_{m+n}^{in} \rangle| \leq c (1 + |\mathbf{a}|)^{-1/2}, \quad (6)$$

for all $\bar{f}_i(\mathbf{p})_\alpha, \bar{g}_j(\mathbf{q})_\beta \in \mathfrak{D}(R^3)$ and arbitrary integers $k, l, m, n \geq 0$, with $c < \infty$ independent of $\mathbf{a}$. In particular one has

$$|\langle \bar{f}_1 \dots \bar{f}_k, \bar{f}_{k+1}^a, \dots, \bar{f}_{k+l}^{out} | \bar{g}_1, \dots, \bar{g}_m^{in} \rangle| \leq c (1 + |\mathbf{a}|)^{-1/2}. \quad (7)$$

Proof: If $B^* \Omega \in \mathfrak{H}_{[m, s]}$, $B \Omega = 0$, then there exists²⁾ for every $\bar{f} = \{\bar{f}(\mathbf{p})_\alpha \in \mathfrak{D}(R^3), \alpha = -s, -s+1, \dots, +s\}$ a finite number of quasi-local fields $B_r(x)$, $1 \leq r \leq N$, and $f_r \in \mathfrak{S}_+^m$ such that

$$B_r^*(x) \Omega \in \mathfrak{H}_{[m, s]}, \quad B_r(x) \Omega = 0, \quad \sum_{r=1}^N B_r^*(f_r, t) \Omega = |\bar{f}\rangle. \quad (8)$$

Without loss of generality we assume for (6) that one has already

$$\text{The } C_\infty\text{-function} \quad |\bar{f}_i\rangle = B_i^*(f_i, t) \Omega, \quad |\bar{g}_j\rangle = {}'B_j^*(\bar{g}_j, t) \Omega.$$

$$F(s, t, \mathbf{a}) = (B_1^*(f_1, s) \dots B_{k+l}^*(f_{k+l}^a, s) \Omega, {}'B_1^*(g_1, t) \dots {}'B_{m+n}^*(g_{m+n}^a, t) \Omega) - (B_1^*(f_1, s) \dots \Omega, {}'B_1^*(g_1, t) \dots \Omega) (\dots B_{k+l}^*(f_{k+l}, s) \Omega, \dots {}'B_{m+n}^*(g_{m+n}, t) \Omega) \quad (9)$$

converges for $s, -t \rightarrow +\infty$ to

$$\langle \bar{f}_1 \dots \bar{f}_k, \bar{f}_{k+1}^a, \dots, \bar{f}_{k+l}^{out} | \bar{g}_1, \dots, \bar{g}_m, \bar{g}_{m+1}^a, \dots, \bar{g}_{m+n}^{in} \rangle - \langle \bar{f}_1, \dots, \bar{f}_k^{out} | \bar{g}_1, \dots, \bar{g}_m^{in} \rangle \langle \bar{f}_{k+1} \dots \bar{f}_{k+l}^{out} | \bar{g}_{m+1}, \dots, \bar{g}_{m+n}^{in} \rangle.$$

In order to show the boundedness of $(1 + |\mathbf{a}|)^{1/2} \lim_{s, -t \rightarrow \infty} F(s, t, \mathbf{a})$, we write for an arbitrary $A > 0$:

$$\lim_{s, -t \rightarrow \infty} F(s, t, \mathbf{a}) = - \int_A^\infty ds \int_{-\infty}^{-A} dt \frac{d}{ds} \frac{d}{dt} F(s, t, \mathbf{a}) + \int_A^\infty ds \frac{d}{ds} F(s, -A, \mathbf{a}) + \int_{-\infty}^{-A} dt \frac{d}{dt} F(A, t, \mathbf{a}) - F(A, -A, \mathbf{a}) \equiv \sum_{i=1}^4 F_i(A, \mathbf{a}). \quad (10)$$

Using the Schwarz inequality $F_1(A, \mathbf{a})$ can be majorized by

$$\int_A^\infty ds \left\| \frac{d}{ds} B_1^*(f_1, s) \dots B_{k+l}^*(f_{k+l}^a, s) \Omega \right\| \int_{-\infty}^{-A} dt \left\| \frac{d}{dt} {}'B_1^*(g_1, t) \dots {}'B_{m+n}^*(g_{m+n}^a, t) \Omega \right\| \quad (11)$$

and by four terms of the form

$$\int_A^\infty ds \left\| \frac{d}{ds} B_1^*(f_1, s) \dots B_k^*(f_k, s) \Omega \right\| \left\| B_{k+1}^*(f_{k+1}, s) \dots B_{k+l}^*(f_{k+l}, s) \Omega \right\| \times \int_{-\infty}^{-A} dt \left\| \frac{d}{dt} {}'B_1^*(g_1, t) \dots {}'B_m^*(g_m, t) \Omega \right\| \left\| {}'B_{m+1}^*(g_{m+1}, t) \dots {}'B_{m+n}^*(g_{m+n}, t) \Omega \right\|. \quad (12)$$

We expand $\|d/ds B_1^*(f_1, s) \dots B_{k+l}^*(f_{k+l}^a, s) \Omega\|^2$ in a sum of products of truncated vacuum expectation values (TVEV)⁽²⁾⁴⁾, where the 1-point- and pure 2-point contributions vanish. The TVEV for $r \geq 3$ give rise to factors like:

$$\int d\mathbf{x}_1, \dots, d\mathbf{x}_r \langle B_{i_1}^{(*)}(\mathbf{x}, s) \dots B_{i_r}^{(*)}(\mathbf{x}_r, s) \rangle^T h_1^{(a)}(\mathbf{x}_1, s) \dots h_r^{(a)}(\mathbf{x}_r, s) \quad (13)$$

with $h_i^{(a)} \in \mathfrak{S}_+$ possibly translated by $\mathbf{a}$. (13) can be majorized⁽²⁾⁶⁾ uniformly in $\mathbf{a}$ by

$$\leq c \sup_{\mathbf{x}_1(-\mathbf{a})} |h_1(\mathbf{x}_1, s)| \dots \sup_{\mathbf{x}_{r-1}(-\mathbf{a})} |h_{r-1}(\mathbf{x}_{r-1}, s)| \dots \\ \times \int d\mathbf{x}_r |h_r(\mathbf{x}_r(-\mathbf{a}), s)| \leq c'(1 + |s|)^{-3/2(r-2)}. \quad (14)$$

Since a similar estimate holds for (12), one obtains after s - and t -integration $|F_1(A, \mathbf{a})| < c_1(1 + A)^{-1}$, independently of $\mathbf{a}$. By expansion in TVEV one sees that terms like $\|B_1^*(f_1, A) \dots B_{k+l}^*(f_{k+l}^a, A) \Omega\|^2$ are uniformly bounded for all A and $\mathbf{a}$. Therefore $|F_i(A, \mathbf{a})| \leq c_i(1 + A)^{-1/2}$, $i = 2, 3$, holds equally, uniformly in $\mathbf{a}$.

In order to majorize $F_4(A, \mathbf{a})$, we first assume that all the B_i, B_j in (9) are localized in a compact region $K = \{x_0^2 + \mathbf{x}^2 \leq R^2\}$, (i.e. $\text{supp } \varphi_{(\mu)}^{(*)} \subset K^{\times n}$ in (1)) and that $\text{supp } f_i|_{x_0=0}, \text{supp } g_j|_{x_0=0} \subset \{\mathbf{x}^2 \leq R^2\}$. Then $B(f^a, t)$ is localized in $\bigcup_{|\mathbf{y}| \leq R+t} \{(x_0 - t)^2 + (\mathbf{x} - \mathbf{a} - \mathbf{y})^2 \leq R^2\}$. For $A \leq |\mathbf{a}|/9$ the $B(f_i, A), B(g_j, -A)$ are localized space-like to the $B(f_{k+i}^a, A), B(g_{m+j}^a, -A)$ for sufficiently large $|\mathbf{a}|$. Then the majorization of ³⁾, theorem 1, applies and gives for sufficiently large $|\mathbf{a}|$ and $A \leq |\mathbf{a}|/9$ the estimate $|F_4(A, \mathbf{a})| \leq c(A) |\mathbf{a}|^{3/2} e^{-M|\mathbf{a}|/2}$ ($M > 0$: smallest mass in the theory). The constant $c(A)$ is essentially a sum of products of norms like

$$\|B_m(g_m, -A) \dots B_k^*(f_k, A) \Omega\|,$$

which for $A \leq |\mathbf{a}|/9$ are uniformly bounded by a polynomial in $|\mathbf{a}|$, due to the temperedness of the Wightman distributions.

For arbitrary B_i, B_j, f_i, g_j one uses the method of ³⁾, section 5, and obtains for any integer $N \geq 0$ a constant c_4^N independent of $\mathbf{a}, A$ such that for $A \leq |\mathbf{a}|/9$:

$$|F_4(A, \mathbf{a})| \leq c_4^N (1 + |\mathbf{a}|)^{-N}. \quad (15)$$

The theorem is proved by setting $A = |\mathbf{a}|/9$.

The theorem can be extended to wave-packets $\bar{f}_i, \bar{g}_j$ from $\mathfrak{S}$, where the bounds c_i of $F_i(A, \mathbf{a})$, $1 \leq i \leq 3$, and c_4^N can be chosen to stay bounded on bounded sets of the $\{\bar{f}_i, \bar{g}_j\}$ in $\mathfrak{S}$. Using this one shows as in ⁷⁾ that (6) entails the following distribution identity:

Corollary: For all integers $k, l, m, n \geq 0$ one has in the topology of $\mathfrak{S}'(R^{3(k+l+m+n)})$ for $\kappa < 1/2$:

$$\lim_{a \rightarrow \infty} |1 + |\mathbf{a}||^\kappa |S_{k+l, m+n}^{(\alpha, \alpha')(\beta, \beta')}(\mathbf{p}_1, \dots, \mathbf{p}_{k+l}, -\mathbf{q}_1, \dots, -\mathbf{q}_{m+n}) e^{ia \left\{ \sum_{i=1}^k p_i - \sum_{j=1}^m q_j \right\}} \\ - S_{k, m}^{(\alpha)(\beta)}(\mathbf{p}_1, \dots, \mathbf{p}_k, -\mathbf{q}_1, \dots, -\mathbf{q}_m) S_{l, n}^{(\alpha')(\beta')}(\mathbf{p}_{k+1}, \dots, \mathbf{p}_{k+l}, -\mathbf{q}_{m+1}, \dots, -\mathbf{q}_{m+n})| = 0. \quad (16)$$

The theorem can be trivially extended to the decomposition into several clusters along a space-like plane. Furthermore the limit (6) relies only in a rudimentary way on the covariance of the theory under the homogeneous Lorentz group and on the microscopic locality of the fields. Essential for the proof was the existence of quasi-local

field operators, which create eigenstates of the mass operator from the vacuum and whose TVEV decrease sufficiently strongly for large space-like separations of the arguments. Therefore the spatial cluster decomposition property of the S-matrix holds as well in the Araki-Haag framework of (macroscopically) localized observables⁶⁾.

The approach of the limit as 0 ($|\mathbf{a}|^{-1/2}$) is governed by the decay of the wave-packets f_i, g_j . For non-overlapping wave-packets in velocity-space:

$$\left\{ \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m_i^2}} : \mathbf{p} \in \text{supp } \hat{f}_i \right\} \cap \left\{ \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m_j^2}} : \mathbf{p} \in \text{supp } \hat{f}_j \right\} = \emptyset, \quad i \neq j, \quad (17)$$

the limit (4) is attained faster than any power of $(1 + |t|)^{-1}$ (see also H. ARAKI, R. HAAG, and D. W. ROBINSON, to be published). Then the scattering amplitudes show stronger cluster decomposition properties:

Theorem 2: For non-overlapping $\{f_i\}$ and, separately, $\{g_j\}$ (5) can be majorized by $c_N(1 + |\mathbf{a}|)^{-N}$ ($c_N < \infty$ for all N) in all directions $\mathbf{e}$ satisfying for all $0 \leq a < \infty$:

$$\mathbf{a} = a \mathbf{e} \neq \frac{\mathbf{p}_i}{\sqrt{\mathbf{p}_i^2 + m_i^2}} - \frac{\mathbf{p}_j}{\sqrt{\mathbf{p}_j^2 + m_j^2}} \quad (18)$$

for $\mathbf{p}_i \in \text{supp } \hat{f}_i$ ($1 \leq i \leq k$), $\mathbf{p}_j \in \text{supp } \hat{f}_j$ ($k+1 \leq j \leq k+l$),
or $\mathbf{p}_i \in \text{supp } \hat{g}_i$ ($1 \leq i \leq m$), $\mathbf{p}_j \in \text{supp } \hat{g}_j$ ($m+1 \leq j \leq m+n$).

Proof: Using the notations of theorem 1 we shall show under (18) for $\mathbf{e}$ that $|F_i(a, \mathbf{e}, A)|$, $1 \leq i \leq 3$, can be majorized by $c_N^i(1 + A)^{-N}$, with $c_N^i < \infty$ for all N uniformly in a . Then theorem 2 follows by setting $A = a/9$.

Let $\varepsilon > 0$ be so small that the ε -neighbourhoods of the supports of $\{\hat{f}_i\}$ and, separately, of $\{\hat{g}_j\}$ still fulfill (17) and (18). Let $S(h, s) \equiv \{ \mathbf{x} = (\mathbf{p}/\sqrt{\mathbf{p}^2 + m^2}) t, \mathbf{p} \in U_\varepsilon(\text{supp } \hat{h}) \}$. Then the integral (13) over the region $S(h_1, s) \times \dots \times S(h_r, s)$ is strongly decreasing in s uniformly in a , due to the strong decrease of the supremum of $|\langle B_{i_1}^{(*)}(\mathbf{x}_1, s) \dots B_{i_r}^{(*)}(\mathbf{x}_r, s) \rangle^T|$ in this region. For, a purely geometric argument shows that because of (18) the infimum of $|\mathbf{x}_i(-\mathbf{a}) - \mathbf{x}_{i+1}(-\mathbf{a})|$ for $\mathbf{x}_k \in S(h_k, s)$ increases linearly in s uniformly in a , at least for one $1 \leq i \leq r-1$. On the other hand the strong decrease of $|h_k(\mathbf{x}_k, s)|$ in s uniformly for $\mathbf{x}_k \in R^3 - S(h_k, s)^2$ guarantees a similar majorization of (13) integrated over $R^3 - S(h_k, s)$.

Again, the results can be generalized to more complicated configurations along a space-like plane. No non-trivial temporal cluster decomposition properties of the S-matrix have been obtained.

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