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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **40 (1967)**

Heft 4

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-113772

Cuspidal Points on Landau Singularities[1])

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(17. XI. 66)

Abstract. A necessary and sufficient condition on a Feynman graph, whose dual diagram is normal, for its leading Landau singularity to have cuspidal points is given. It is shown how an equation determining the location of these points may be written down and the method is illustrated by some examples.

1. Introduction

The analytic properties of scattering amplitudes in perturbation theory have been studied by R. J. Eden and co-workers with the aim of establishing the Mandelstam representation to all orders in perturbation theory. They were able to show that the Mandelstam representation holds, under certain conditions related to the masses involved and determined by the fourth order graphs, for all graphs whose leading Landau curves did not have isolated real points (acnodes) lying in normal threshold cuts (for a report on this work see [7]). In a later paper [1] they made a detailed analysis of the crossed square diagram. The scattered particles were taken to have masses M, m and the particles exchanged, mass m. The leading Landau curve of this diagram has two acnodes lying in the normal threshold cuts and above the continuous branch of the curve in this region, provided that $M^2 > (4 + 2\sqrt{2})\, m^2$. It is therefore possible that the complex part of the Landau curve attached to the acnodes should be singular and the Mandelstam representation fail for these mass values. The authors of [1] show that this is indeed the case by considering what happens as M/m is further increased. One of the acnodes moves downwards and to the left until it meets the continuous arc of the Landau curve. The acnode then disappears and instead the continuous arc develops a pair of cusps. As M/m is further increased these cusps separate and for $M^2 > 4\sqrt{3}\, m^2$ part of the segment of the Landau curve between the cusps lies outside both normal threshold cuts. This segment corresponds to positive values of the Feynman parameters and is therefore singular on the physical sheet, and the piece of complex

[1]) The content of this paper forms part of a thesis submitted by the author to the University of Cambridge in August 1966 in partial fulfillment of the requirements for a PhD.

curve attached to it which joins this segment to the second acnode is therefore also singular. It then follows from the continuity theorem for singularities of functions of several complex variables, that the complex part of the Landau curve joining the acnodes is singular as soon as they appear, *i.e.* for $M^2 > (4 + 2\sqrt{2})\, m^2$. If it could be proved that the complex part of a Landau curve attached to an acnode is singular only when its singularity is guaranteed by this mechanism, it would follow that the Mandelstam representation holds for diagrams which do not have cusps.

Cuspidal points on Landau singularities have a general importance for the singularity structure of the corresponding Feynman amplitude. A general understanding of the behaviour of Feynman amplitudes may be obtained by analogy with the behaviour of a function F of a single complex variable z, defined as a contour integral with respect to a second variable t of a function g analytic in z and t. The singularities of $F(z)$ arise for values of z for which two singularities of $g(z, t)$ coincide in the t plane, trapping the contour of integration. A general point on a Landau singularity corresponds to the occurrence in the integration space of the analogue of this mechanism for functions of several complex variables. It may happen that there are values of z for which three singularities of $g(z, t)$ coincide in the t plane. A cuspidal point on a Landau singularity corresponds to the analogue of this situation for function of several complex variables. The more complicated nature of the 'pinch' is reflected in a more complicated behaviour of the function $f(z)$ for values of z in the neighbourhood of the pinch.

In this paper we give a necessary and sufficient condition on a Feynman graph, whose dual diagram is normal, for its leading Landau singularity to have cuspidal points. For two-particle scattering graphs this condition takes a particularly simple form: the Landau curve of a two-particle scattering graph has no cusps if and only if the graph belongs to the class C of graphs which are generated by two particle unitarity and crossing (graphs which arise in the strip approximation). The class C was introduced by T. Regge and G. Barruchi [2]; they prove that the Landau curves of graphs of this class have no cusps in the course of a detailed analysis of the properties of these curves. Our result thus completes theirs. If the leading Landau singularity of a graph has cuspidal points, then an equation determining the location of these points may be written down. This equation, like the Landau equations themselves, is implicit in the sense that it contains the loop momenta and Feynman parameters in addition to the external momenta. It is, in practice, more convenient to determine the location of cuspidal points by solving this equation, together with the Landau equations, than to obtain, from the Landau equations, by eliminating the loop momenta and Feynman parameters, an explicit equation $L = 0$ for the Landau singularity in the external momenta, whose cuspidal points can then be determined by writing down the necessary conditions on the derivatives of L with respect to the external momenta. For two-particle scattering graphs where one has a curve $L(s, t) = 0$ these conditions are

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial t} = 0 \qquad \left(\frac{\partial^2 L}{\partial s\, \partial t}\right)^2 = \frac{\partial^2 L}{\partial s^2}\, \frac{\partial^2 L}{\partial t^2}$$

where s, t are the Mandelstam variables. They appear to be rather restrictive, leading one to suppose that cusps are not to be expected for general mass values (see the discussion at the end of [1]). The argument is misleading because it ignores the specific geometric way in which the Landau singularities arise. This feature of the problem has been placed in its general mathematical setting in the work of F. PHAM [3] to which we refer the reader.

In section 2 we set out our main result. In order to show in what way the results are dependent in the dimensionality of space-time, we take our momentum vectors from an n-dimensional vector space. Since we wish to discuss the local form of the Landau singularity in the complex region, our vectors will be complex and we may take the scalar product to be the complex Euclidean scalar product. In order to avoid entering into a discussion of the Feynman integrals we define the Landau singularity as the set of solutions of the Landau equations. In section 3 we give some examples to illustrate the general discussion.

2. The Main Result

Denote by P an n-dimensional vector space over the complex numbers. We denote by ${}_s q$ the s^{th} component of q relative to a stated basis in P. The scalar product of two vectors a, b in P is defined by

$$a \cdot b = \sum_{s=1}^{n} ({}_s a)\,({}_s b)\,. \tag{1}$$

The scalar product has the same algebraic form as the Euclidean scaler product in a vector space of dimension n over the real numbers, so P may be called a complex Euclidean space of dimension n. To describe configurations of vectors in P we may therefore use the language of Euclidean geometry with the convention that the geometrical concepts are to be interpreted algebraically by means of the formula (1).

Let G be a connected graph having N vertices, E edges with one vertex (external lines) and L edges with two vertices (internal lines). Label the vertices, external lines and internal lines by the integers from 1 to N, 1 to E, 1 to L respectively, and denote by p, m, i, variables with these index sets as their replacement sets.

Choose an orientation for G, and denote by $\varepsilon(i, p)$, $\varepsilon(m, p)$ the incidence numbers of the lines i, m relative to the vertex p. We denote by T the $n\,(E-1)$ dimensional space whose points are sets of E vectors p_m, one associated with each external line of G, satisfying the condition

$$\sum_p \varepsilon(m, p) p_m = 0\,. \qquad (2)^{2)}$$

For fixed $t = (p_m)\ \varepsilon\ T$ consider the space Q of all sets of L vectors q_i, one associated with each internal line of G, subject to the condition

$$\varepsilon(i, p) q_i + \varepsilon(m, p) p_m = 0\,. \tag{3}$$

Note that (2) is implied by (3).

The space Q may be parametrised in the following way:

G has $l = N - L + 1$ linearly independent loops. Choose a basis for these loops and index the basic loops by an index j running from 1 to l. Adjoin to G one more vertex, to be called ∞. Each external line of G is joined to ∞. The resulting graph G^∞ has

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[2]) In (2), and in subsequent equations, we use the summation convention.

no external lines and an orientation determined by that of G. G^∞ has $E - 1$ more loops than G. We may obtain a basic set of loops for G^∞ by taking the l basic loops of G and, for each of the first $(E - 1)$ external lines of G, choosing a loop m' in G^∞ which contains this external line, the external line E and no other external lines. Choose an orientation for each of the l basic loops for G, and give each loop m' of G the orientation coherent with that of the external line m'. Then any set of N vectors q_i satisfying (3) may be uniquely expressed in the form

$$q_i = \eta(j, i) k_j + \eta(m', i) p_{m'} . \tag{4}$$

In (4) the index m' is summed from 1 to $(E - 1)$; $\eta(j, i)$, $\eta(m', i)$ denote the incidence numbers of the loops j, m' with respect to the line i and the k_j are l vectors of P. Conversely, if (k_j) is a set of l vectors of P, and vectors q_i are defined by (4), they satisfy (3).

Denote by K the $n\,l$-dimensional space whose points are sets of l vectors k_j. In K define an analytic set $S(t)$ by

$$S(t) = \bigcap_{i=1}^{N} S_i(t) \qquad S_i(t) = \{k : q_i^2 - m_i^2 = 0\} .$$

We suppose n sufficiently large that $S(t)$ is nonempty for all $t \varepsilon T$. The Landau singularity of the graph G is the set of points $t \varepsilon T$ such that $S(t)$ is not a manifold. This is the case if and only if the Landau equations

$$\alpha_i\, \eta(j, i) q_i = 0 \tag{5}$$

$$\alpha_i\, (q_i^2 - m_i^2) = 0 \qquad \text{(no summation on } i) \tag{6}$$

are satisfied by some $k \varepsilon K$ and some set of L complex numbers α_i. The set of points $t \varepsilon T$ for which (5) and (6) have a solution for which no α_i is zero form an analytic set whose closure L_G in T is called the leading Landau singularity of G.

Let $t_0 = (p_{m'})$ be a point on L_G and suppose that for $t \varepsilon L_G$ in a neighbourhood V of t_0 the Landau equations have a unique solution $k = k(t)$ for k and a solution unique up to a factor for the α_i. Construct the space $Y = K \times T$, and denote by p the natural projection of Y onto T. Denote by $L_G(Y)$ the set of points $(k(t), t) \varepsilon Y, t \varepsilon V$. The projection p maps $L_G(Y)$ analytically onto $L_G \cap V$. Let $(dp_{m'})$ be a tangent vector to L_G at t_0 and let $(dk_j, dp_{m'}) = (p^{-1})^* (dp_{m'})$ be the tangent vector to $L_G(Y)$ at $(k(t_0), t_0)$. From (4) we have

$$\eta(j, i) q_i \cdot dk_j + \eta(m', i) q_i \cdot dp_{m'} = 0 . \tag{7}$$

$$\text{(no summation on } i)$$

From the first of the Landau equations it follows that

$$\alpha_i\, \eta(m', i) q_i \cdot dp_{m'} = 0 . \tag{8}$$

The normal to L_G is thus $(\alpha_i\, \eta(m', i) q_i)$. The normal is unique so L_G is either nonsingular at t_0, *i.e.* locally a manifold or cuspidal, *i.e.* singular but having a unique tangent primal in T.

Since the physical amplitudes are functions defined only on the mass shell M

$$p_m^2 - M_m^2 = 0 \tag{9}$$

one should consider $L_G \cap M$ rather than L_G to be the Landau singularity. Now L_G may be nonsingular in T at $t_0 \varepsilon M$ but $L_G \cap M$ may nevertheless be singular in M because the normals for L_G and M are linearly dependent. This is true if there exist complex numbers β_m such that

$$\alpha_i \, \eta(m, i) q_i + \beta_{m'} p_{m'} + \beta_m p_m = 0 \tag{10}$$

$$\beta_m \, (p_m^2 - M_m^2) = 0 \quad \text{for all } m \tag{11}$$

Equations (5), (6), (10), (11) are the Landau equations for the graph G^∞ which has no external lines. They are therefore compatible only if the internal and external masses of G satisfy some relation. We suppose that this relation is not satisfied. For a full discussion of this point see [3].

Denote by $n_G(P)$ the dimension of the subspace S_G of P spanned by the vectors $q_i \varepsilon P$ which satisfy the Landau equations for $t = t_0$. Denote by S_E the subspace of P spanned by p_m. The dimension n_E of S_E is given by $n_E = \min (E - 1, n)$. Without loss of generality we may suppose at most one external line incident to any vertex. Then $S_E \subset S_G$ so $n_E \leqslant n_G(P)$. If $n_E = n_G(P)$, (G, P) is said to be normal; if $n_E < n_G(P)$, (G, P) is anomalous. If n is sufficiently large $n_G(P) = n_G$ is independent of P.

Let dq_i be a tangent vector to $S(t_0)$ at $k(t_0)$ in K.

$$\begin{aligned} d\,(q_i^2 - m_i^2) &= 2\, q_i \cdot dq_i + (dq_i)^2 \\ &= 2\, \eta(j, i) q_i \cdot dk_j + (\eta(j, i) dk_j)^2 \,. \end{aligned} \tag{12}$$

Hence we must have

$$\eta(j, i) q_i \cdot dk_j = 0 \qquad \text{(no summation on } i) \,. \tag{13}$$

By (5) the N equations (13) are linearly dependent and since the $(\alpha_i)_0$ are unique there is just one linear dependence relation between them. The local form of $S(t)$ is determined by the rank of the Hessian

$$H = \alpha_i \big(\eta(j, i) dk_j\big)^2 \tag{14}$$

on the $n\,l - (N - 1)$ dimensional vector space of tangent vectors dk satisfying (13). If H has maximum rank then L_G is nonsingular at t_0 — this is established in the course of the proof of the Picard-Lefschetz theorem [4]. If H has rank less than maximum then we call $t_0 \varepsilon L_G$ a cuspidal point and we refer to the equation $\det H = 0$ which expresses this condition as the cusp equation. This terminology will be justified in the following paragraph.

If H has rank 1 less than maximum then there is a unique tangent vector which is a null eigenvector of H. We may choose local coordinates (z) in the neighbourhood of $k(t_0)$ in K so that $S_1(t), \ldots, S_{N-1}(t)$ have local equations.

$$S_1 : z_1 = 0, \; \ldots, \; S_{N-1} : z_{N-1} = 0$$

Let $s_N = 0$ be a local equation for S_N and denote by f the restriction to $\bigcap_{i=1}^{N-1} S_i$ of s_N. The Hessian form H, which we have written down in coordinate-free notation, may be identified with the Hessian of f, *i.e.* the square matrix of second derivatives of f with respect to the coordinates z_k, $k > N - 1$. We may choose the local coordinate system so that the null eigenvector of H is in the z_N direction. If

$$\left.\frac{\partial^3 f}{\partial z_N^3}\right|_{z=0} \neq 0 ,$$

then by a lemma similar to that of Morse [5] we may choose local coordinates so that

$$f(z_N, \ldots, z_{nl}) = a(t)\, z_N + z_N^3 + z_{N+1}^2 + z_{N+2}^2 + \ldots + z_{nl}^2 - b(t)$$

where $a(t)$, $b(t)$ are holomorphic functions of t and $a(t_0) = b(t_0) = 0$. If the analytic sets $\{t : a(t) = 0\}$, $\{t : b(t) = 0\}$ are in general position in T for $t = t_0$ we may choose local coordinates in T so that

$$a(t) = u_1 \qquad b(t) = u_2$$

u_1, u_2 the first two local coordinates. In these local coordinates L_G has the local equation

$$4\, u_1^3 + 27\, u_2^2 = 0$$

and is therefore cuspidal. If t_0 satisfies the additional conditions set out in this paragraph, *viz.* that H have rank exactly 1 less than maximum, that

$$\left.\frac{\partial^3 f}{\partial z^3}\right|_{z=0} \neq 0$$

and that for $t = t_0$ the sets $\{t : a(t) = 0\}$, $\{t : b(t) = 0\}$ be in general position in T, then we call t_0 a cuspidal point in the strict sense. An analysis of the local behaviour of a Feynman amplitude in the neighbourhood of such a point may be made with the help of the Picard-Lefschetz theorem. This analysis shows that a cuspidal point in the strict sense may be regarded as a point of 'effective self-intersection' of the Landau singularity. The additional conditions which define a cuspidal point in the strict sense will be violated only if further polynomial equations, additional to the Landau equations and the cusp equation are satisfied by the external momenta for $t = t_0$. It is therefore to be expected that a cuspidal point on L_G is, in general, a cuspidal point in the strict sense. In [3] it is indicated how this assumption of 'sufficient generality' can be precisely formulated within the framework of *R*. Thom's theory of stable differentiable mappings [6]. This theory shows that our expectation is justified if for $t = t_0$ the transversality condition (see [6] for general definition and [3] for the form the condition takes in the present context) holds.

H is singular for the tangent vector dk if

$$C_{jj_1}\, dk_j = \lambda_i\, \eta(j_1, i) q_i \tag{15}$$

where $C_{jj_1} = \alpha_i\, \eta(j, i)\, \eta(j_1, i)$ is an $l \times l$ matrix and the λ_i are Lagrange multipliers for the constraints (13). Suppose that the subspace S_G and its orthogonal complement in P, S_G are disjoint so that P may be written as a direct sum

$$P = S_G \oplus S_G^{\perp} . \tag{16}$$

Denote by dk^s, $dk^\perp$ the components of dk with respect to the decomposition (16). Then (15) splits into the equations

$$C_{jj_1}\, dk_j^\perp = 0 \tag{17}$$

$$C_{jj_1}\, dk_j^s = \lambda_i\, \eta(j_1, i) q_i\,. \tag{18}$$

If we substitute for q_i in terms of k_j in the loop equations (1) we obtain them in the form

$$C_{jj_1}\, k_{j_1} + B_{jm'}\, p_{m'} = 0 \tag{19}$$

where $B_{jm'} = \alpha_i\, \eta(j, i)\; \eta(m', i)$ is an $l \times m'$ matrix. If $\det C = 0$ the equations (19) allow the internal momenta k_j to be expressed as linear combinations of the external momenta $p_{m'}$. Thus $\det C = 0$ for all points on L_G only if G is anomalous. We will suppose that G is normal. The condition that $S_G \cap S_G^\perp = \phi$ is then equivalent to the assertion that the Gram determinant of the external momenta $p_{m'}$ is non-zero. This condition is satisfied at a general point of L_G and we may assume in particular that it is satisfied for $t = t_0$. Similarly we may assume $\det C \neq 0$ for $t = t_0$. Then (17) implies $dk_j^\perp = 0$; in other words we need consider only tangent vectors $dk_j \,\varepsilon\, S_G = S_E$. The linear space of tangent vectors in S_G satisfying (13) is of dimension

$$n_G l - (N - 1) = \tau\,. \tag{20}$$

If $\tau > 0$ we can express the dk_j in terms of a basis for this space S_τ and, substituting these expressions into the equation (14) for H obtain a quadratic form H_τ on S_τ. H will be singular if and only if H_τ is singular. The cusp equation is thus

$$\det H_\tau = 0 \qquad \text{for } t = t_0\,. \tag{21}$$

If we exclude the unlikely possibility that $\det H_\tau = 0$ is incompatible with $t_0 \,\varepsilon\, L_G$ it follows that L_G has cuspidal points if and only if $\tau > 0$.

This possibility is realised in the following situation:

Suppose that G has one or more pairs of vertices joined by more than one line. Then L_G is reducible $L_G = \bigcup\limits_v (L_G)_v$. Each component $(L_G)_v$ is the leading Landau singularity of the graph G_1 obtained from G by replacing every set of lines joining the same pair of vertices by a single line and giving the internal mass², m^2, for this line one of the values $(\pm\, m_1 \pm \ldots \pm m_u)^2$ where $m_1, \ldots, m_u$ are the internal masses for the u lines. The components $(L_G)_v$ have no effective intersections. Hence if $\tau = 0$ for G_1, so that L_{G_1} has no cuspidal points, then L_G has no cuspidal points. Nonetheless the interpretation of the condition $\tau > 0$ given in the following paragraph shows that $\tau > 0$ for G.

Clearly there is no loss of generality in considering only irreducible graphs, *i.e.* graphs for which L_G is irreducible (as an algebraic variety in T for general values of the internal masses). The example given above, and others suggest that $\det H_\tau = 0$ for $t = t_0$ may be incompatible with $t_0 \,\varepsilon\, L_G$ only if L_G is reducible. We shall assume that this conjecture is correct.

To interpret the condition $\tau > 0$ consider the Landau equations which define L_G. These are algebraic equations in the components of the loop momenta k_j and of the external momenta p_m. Also L_G is of complex codimension 1 in T. It follows that the freedom d of loop momenta k_j satisfying the Landau equations for given $t \varepsilon T$ must be -1. For a normal graph the Landau equations (5) imply $k_j \varepsilon S_G$ for all $_j$. The Landau equations (6) impose N further independent conditions on the k_j. The freedom of vectors k_j satisfying these two conditions is

$$d' = n_G l - N = \tau - 1 \,. \tag{22}$$

If $\tau = 0$, $d' = d$, *i.e.* these conditions are equivalent to the full set of Landau equations. If $\tau > 0$, $d' > d$ and there must be $d' - d = \tau$ further conditions imposed on the k_j by the loop equations (5). In [7] it is shown that these additional conditions are of two kinds—tautening conditions and caplanarity conditions[3]). Hence we have the

Cusp criterion

The leading Landau singularity of a normal and irreducible graph G has no cuspidal points if and only if the dual diagram of G is determined by the internal mass conditions alone[4]).

A loop of G which has less than $n_G + 1$ lines we call deficient. The single loop subgraph of G formed by a deficient loop we call a subgraph of type A. The subgraph of G formed by the loops of G around a vertex of G for which there is a tautening condition we call a subgraph of type B. Note that if we have a set of tangent vectors dq_i satisfying the orthogonality conditions (13) for the lines of a subgraph G_1 of G then we may construct a set of tangent vectors satisfying (13) for all lines of G by defining dq_i to be zero for any line i not in G_1. To obtain a basis for S_τ choose τ independent caplanarity and tautening conditions, which together with the internal mass conditions define the dual diagram of G. Corresponding to each of these conditions is a subgraph of G, of type A or type B. Construct for each of these subgraphs a set of tangent vectors dq_i satisfying the orthogonality conditions for the lines of the subgraph and extend this set to a vector of S_τ. The vectors of S_τ constructed form a basis[5]).

Let G_1 be a subgraph of G. The restriction to the space of tangent vectors dq satisfying $dq_i = 0$ for $i \bar{\varepsilon} G_1$ of the Hessian form H of G is the Hessian form H_1 of G_1. However, a tangent vector singular for H_1 is not in general singular for H. This will be the case only if the subspace $S_{1\tau}$ of S_τ is an invariant subspace for H. Denote by G_c the subgraph of G formed by taking the union of all subgraphs of types A or B. From the method of construction a basis for S given in the preceding paragraph it follows that $S_{c\tau} = S_\tau$. Let $(G_c)_b \; {}_{b=1,\ldots,d}$ be the connected components of G_c. Then $S_\tau = \bigoplus_b S_{b\tau}$ and H_τ is completely reducible with respect to this decomposition. Thus the cuspidal points of L_G are the points on L_G which satisfy one of the conditions C_{G_b} : $\det H_{b\tau} = 0$.

[3]) If G has a loop with less than $n_G + 1$ lines, then the Landau equations require the momenta for these lines to be linearly dependent. This imposes one condition on the loop momenta independent of the two conditions above. We call this a caplanarity condition whether or not the loop has only three lines.

[4]) We refer to the configuration of vectors $q_i \varepsilon P$ satisfying the Landau equations for L_G as the dual diagram of G (irrespective of whether G is planar).

[5]) For a subgraph of type A the construction of the tangent vectors dq_i presents no difficulty. An example of the construction for a subgraph of type B is given below (Ex. 1).

If G is anomalous then $\det C = 0$ at all points on L_G. Moreover it is not true that for a general point of L_G the corresponding solution of the Landau equations is unique in k and unique up to a factor in the Feynman parameters α_i. The pinch in K corresponding to a general point of L_G for an anomalous graph with $n_G - n_E = s$ and $n_G \leqslant n$ corresponds to a non-simple pinch of dimension equal to that of the manifold of s-dimensional subspaces of P orthogonal to S_E. This is so because there is no way to single out a particular s-dimensional subspace of P orthogonal to S_E from which the components of the internal momenta orthogonal to S_E should be taken. H is now nowhere of maximum rank on L_G and cuspidal points on L_G are to be expected when the rank of H falls below its general value. The condition $\tau = 0$ remains sufficient to exclude cuspidal points since the rank of H remains constant on the space of tangent vectors taken from S_G. It should be pointed out that the analysis of the type of a Landau singularity given in [7] does not apply to the singularities of anomalous graphs.

3. Examples

Example 1. The Mercedes diagram (Fig. 1).
Loop momenta k_1, k_2, k_3 may be chosen so that

$$u_1 = k_3 - k_2 \qquad v_1 = k_1 + w_1$$

where $w_1 = 1/3\ (-\,p_3 + p_2)$, together with 6 further relations obtained by permuting the indices cyclically.

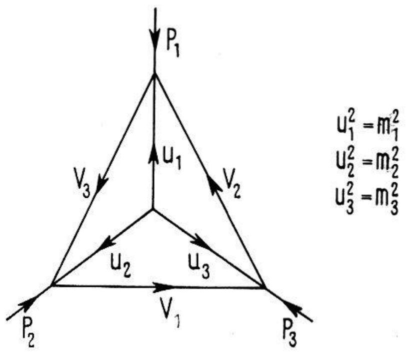

Fig. 1
Mercedes diagram

The Mercedes diagram is normal and irreducible and $n_G = 2$. There are no deficient loops and one tautening condition corresponding to the 3 loops around the center vertex. Choose loop momenta as indicated in the figure. We denote by b_1, b_2, b_3 the tangent vectors dk_1, dk_2, dk_3. Then the Hessian H is given by

$$H(b_1, b_2, b_3) = \alpha_1\, b_1^2 + \alpha_2\, b_2^2 + \alpha_3\, b_3^2 + \alpha_{12}\,(b_1 - b_2)^2 + \alpha_{23}\,(b_2 - b_3)^2 + \alpha_{31}\,(b_3 - b_1)^2 \tag{1}$$

and the orthogonality conditions are

$$b_1 \cdot v_1 = b_2 \cdot v_2 = b_3 \cdot v_3 = 0$$
$$(b_1 - b_2) \cdot u_3 = (b_2 - b_3) \cdot u_1 = (b_3 - b_1) \cdot u_2 = 0\,. \tag{2}$$

The dual diagram defining L_G may be constructed by the rules given in [7] for planar graphs G. (Fig. 2).

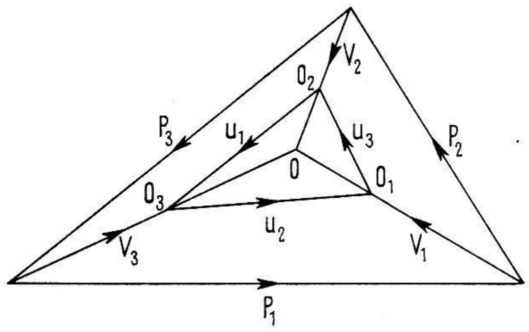

Fig. 2

Dual of Mercedes diagram

The tautening condition implies that the lines of the vectors v_1, v_2, v_3 are concurrent in 0. Denote by $a^\perp$ the vector in S_G perpendicular to a given vector a in S_G, obtained from a by a positive rotation throught $\pi/2$. From the dual diagram (Fig. 2) it is clear that the orthogonality relations (2) are satisfied by

$$b_1 = \overrightarrow{00_1}^\perp \qquad b_2 = \overrightarrow{00_2}^\perp \qquad b_3 = \overrightarrow{00_3}^\perp . \tag{3}$$

Then $H(b_1, b_2, b_3) = H(b_1^\perp, b_2^\perp, b_3^\perp)$

$$H(b_1, b_2, b_3) = \lambda^2 \{\alpha_1 \overrightarrow{00}_1^2 + \alpha_2 \overrightarrow{00}_2^2 + \alpha_3 \overrightarrow{00}_3^2 + \alpha_{12} \overrightarrow{0_10}_2^2 + \alpha_{23} \overrightarrow{0_20}_3^2 + \alpha_{31} \overrightarrow{0_30}_1^2\} . \tag{4}$$

Thus cuspidal points on L_G are given by

$$C_G : \alpha_1 \overrightarrow{00}_1^2 + \alpha_2 \overrightarrow{00}_2^2 + \alpha_3 \overrightarrow{00}_3^2 + \alpha_{23} m_1^2 + \alpha_{31} m_2^2 + \alpha_{12} m_3^2 = 0 .$$

Example 2. The double crossed square. (Fig. 3). The internal momenta are related to the loop and external momenta by the equations

$$u_1 = k_3 - k_2 + w \qquad v_1 = k_1 + w_1$$

where $w = -1/3 (p_1 + p_2 + p_3)$, $w_1 = -1/3 (-p_3 + p_2)$, together with the equations obtained by cyclically permuting 1, 2, 3. The double crossed square diagram is normal and irreducible and $n_G = 3$. It has 3 linearly independent loops and 6 lines so $\tau = 4$. There are 4 loops of deficiency 1. For each loop of deficiency 1 we construct a vector of S. These vectors are

$$dk = (s_1, 0, 0) \qquad dk = (0, s_2, 0) \qquad dk = (0, 0, s_3) \qquad dk = (s, s, s) . \tag{5}$$

Here s_1, s_2, s_3, s are vectors in S_G perpendicular to the planes determined by v_1, u_2, u_3; v_2, u_3, u_1; v_3, u_2, u_1; v_1, v_2, v_3.

$$H(b_1, b_2, b_3) = \alpha_1 b_1^2 + \alpha_2 b_2^2 + \alpha_3 b_3^2 + \alpha_{23} b_{23}^2 + \alpha_{31} b_{31}^2 + \alpha_{12} b_{12}^2 \tag{6}$$

$$\begin{aligned} H = {} & \alpha_1 (\lambda_1 s_1 + \lambda s)^2 + \alpha_2 (\lambda_2 s_2 + \lambda s)^2 + \alpha_3 (\lambda_3 s_3 + \lambda s)^2 \\ & + \alpha_{23} (\lambda_2 s_2 - \lambda_3 s_3)^2 + \alpha_{31} (\lambda_3 s_3 - \lambda_1 s_1)^2 + \alpha_{12} (\lambda_1 s_1 - \lambda_2 s_2)^2 . \end{aligned} \tag{7}$$

In (6), $b_i = dq_i$. In (7) $\lambda_1, \lambda_2, \lambda_3, \lambda$ denote the components of dk relative to the basis (5) of S. Cuspidal points on L_G are given by the vanishing of the determinant of the quadratic form $H(\lambda, \lambda_1, \lambda_2, \lambda_3)$.

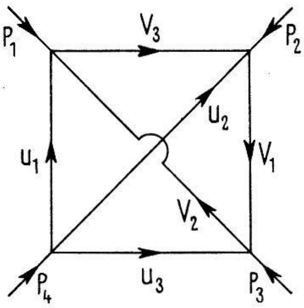

Fig. 3

Square with non-intersecting diagonals

Example 3. The crossed square diagram (Fig. 4).

The crossed square diagram is normal and irreducible and $n_G = 3$. It has two independent loops which can be chosen as in Fig. 4. Each of these loops has deficiency 1. Denote by $a \times b$ the vector product of two vectors a, b in the 3 dimensional space S_G. The space S may be parametrised by writing

$$dk_1 = \lambda_1 k_1 \times p_1 \qquad dk_2 = \lambda_2 k_2 \times p_2 \tag{8}$$

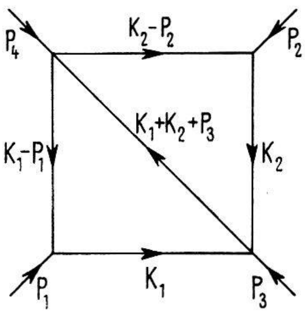

Fig. 4

Crossed square

Then the Hessian form

$$H = \alpha_1 (dk_1)^2 + \beta_1 (dk_1)^2 + (dk_1 + dk_2)^2 + \alpha_2 (dk_2)^2 + \beta_2 (dk_2)^2 \tag{9}$$

has the restriction to S

$$H = \lambda_1^2 (\delta_1 + 1)\, (k_1 \times p_1)^2 + \lambda_2^2 (\delta_2 + 1)\, (k_2 \times p_2)^2 + 2\, \lambda_1 \lambda_2\, (k_1 \times p_1) \cdot (k_2 \times p_2) \tag{10}$$

in (10) we have for convenience chosen a normalisation of the α's in which the parameter for the diagonal line is set equal to 1. Here

$$\delta_1 = \alpha_1 + \beta_1 \qquad \delta_2 = \alpha_2 + \beta_2 \tag{11}$$

$$H = \lambda_1^2 (\delta_1 + 1)\, G(k_1, p_1) + \lambda_2^2 (\delta_2 + 1)\, G(k_2, p_2) + 2\, \lambda_1 \lambda_2\, G(k_1, p_1; k_2, p_2) \tag{12}$$

where $G(k_1, p_1) = (k_1^2)(p_1^2) - (k_1 \cdot p_1)^2$ is the Gram determinant of k_1 and p_1 and $G(k_1, p_1; k_2, p_2) = (k_1 \cdot k_2)(p_1 \cdot p_2) - (k_1 \cdot p_2)(k_2 \cdot p_1)$. From (12) we obtain the condition satisfied by cuspidal points on L_G

$$C_G\colon [G(k_1, p_1; k_2, p_2)]^2 - (\delta_1 + 1)(\delta_2 + 1)[G(k_1, p_1)][G(k_2, p_2)] = 0 \qquad (13)$$

A detailed analysis of the Landau curve for this diagram for general values of the internal masses has been carried out by G. Barruchi [8] and independently by the author (unpublished). The Landau curve has total degree 12 and class 16. It has 24 cusps and 10 finite nodes. The genus is 9. In the case considered in [1] the internal masses are so specialized that $L(s, t) = 0$ splits into two components, one of which corresponds to points whose dual diagrams have a certain symmetry. The points on $L(s, t) = 0$ at which the cusp condition is satisfied are then six points on the symmetric component at which the symmetric component has cusps and six further points which are points in which the two components touch and the unsymmetric component has cusps. These points are all the intersections of the two components which are rational curves of order 6.

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