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A New Proof of von Neumann's Theorem Concerning the Uniqueness of the Schrödinger Operators

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Abstract. We give a new proof of von Neumann's theorem, which states that the Schrödinger representation of the canonical commutation relations is (up to unitary equivalence) the only one, in the case of a finite number of canonical variables.

1. Introduction

In 1931 von Neumann [1] gave an elegant proof for the uniqueness (up to unitary equivalence) of the Schrödinger representation of the canonical commutation relations in the case of a finite number of canonical variables. In this paper we give a new proof of this important theorem of von Neumann which has its origin in the following idea: A representation of the canonical commutation relations in the formulation of Weyl [2] (see def. 1 below) can be considered as a special class of unitary representations of an abstract locally compact group. It is then natural to ask oneself whether the methods Wigner used in his analysis for the unitary representations of the inhomogeneous Lorentz group could be used here too. This is indeed the case. One gets in this way a very natural proof of the theorem of von Neumann, as well as an illustration of Wigner's methods.

2. Statement of von Neumann's theorem

We consider two canonical conjugate variables (the generalisation to an arbitrary finite number is obvious) P and Q with $[P, Q] = -i\,\mathbf{1}$. Following Weyl [2] we consider the unitary operators $U(a) = e^{iaP}$ and $V(b) = e^{ibQ}$. By formal manipulations one easily obtains [2]

$$U(a_1)\,U(a_2) = U\,(a_1 + a_2)\,; \quad V(b_1)\,V(b_2) = V\,(b_1 + b_2)$$
$$U(a)\,V(b) = e^{iab}\,V(b)U(a)\,. \tag{1}$$

We are thus led to the

Definition 1

By a representation of the canonical commutation relations we mean a mapping from R^2 into the set of unitary operators of a Hilbert space $\mathcal{H}$, in which to each pair $(a, b) \in R^2$ there correspond two unitary operators $U(a)$ and $V(b)$ satisfying (1) and which are weakly continuous in a and b.

It is well known (see for instance [3]) and easy to verify that a representation of the canonical commutation relations in the sense of Definition 1 can be considered as a special unitary representation of the following abstract locally compact group G:G is the set of triples

$$(a, b; e^{i\varphi}) \quad ; \quad a, b \in \mathsf{R} \quad ; \quad 0 \leq \varphi < 2\pi$$

with the multiplication law

$$(a_1, b_1; e^{i\varphi_1})\,(a_2, b_2; e^{i\varphi_2}) = (a_1 + a_2, b_1 + b_2; e^{-i b_1 a_2}\, e^{i(\varphi_1+\varphi_2)}) .$$

If one considers unitary representations $\mathcal{U}(a, b; e^{i\varphi})$ of G such that the subgroup $(1,1; e^{i\varphi})$ is represented by $\mathcal{U}(0,0; e^{i\varphi}) = e^{i\varphi} \cdot \mathbf{1}$ then the operators $U(a) \equiv \mathcal{U}(a, 0; 1)$ and $V(b) \equiv \mathcal{U}(0, b; 1)$ form a representation of the canonical commutation relations. Conversely: If $(a, b) \to (U(a), V(b))$ is a representation of the canonical commutation relations then $\mathcal{U}(a, b; e^{i\varphi}) = e^{i\varphi}\, U(a)\, V(b)$ is a unitary representation of G with $\mathcal{U}(0, 0; e^{i\varphi}) = e^{i\varphi}$. We will use below the methods of Wigner in order to analyze this special type of unitary representation. In this way we get (to our knowledge) a new proof of von Neumann's theorem.

3. Analysis of a special case

Before treating the general problem, we consider first (mainly for pedagogical reasons) the special case of a Hilbert space $\mathcal{H}$ which is cyclic with respect to the set of operators $\{U(a) \mid a \in R\}$. In this case it will turn out, that the representation is already irreducible and can be mapped isometrically on the Schrödinger representation. The analysis of the general problem will be very similar to this special case.

If $E(\Delta)$ (Δ is a Borel set) is the spectral measure, guaranteed by Stone's theorem, which belongs to the group $U(a)$, then $\mu(\Delta) = (\Psi_0, E(\Delta)\, \Psi_0)$, where Ψ_0 is the cyclic vector, is a positive regular normalized measure on the σ-algebra of Borel sets. It is well known [4] that the Hilbert space $\mathcal{H}$ can be mapped isometrically onto the Hilbert space $\mathfrak{L}^2_\mu(\mathsf{R})$ in such a way that the operators $U(a)$ are simply given by[1])

$$(U(a)\, \Psi)\,(p) = e^{iap}\, \Psi(p) . \tag{2}$$

As in the representation theory for the inhomogeneous Lorentz group [5], we consider the effect of $V(b)$ on trigonometric polynomials

$$\Psi(p) = \Sigma\, c_k\, e^{ia_k p} = \Sigma\, c_k\, U(a_k)\, \Psi_0(p) \tag{3}$$

where

$$\Psi_0(p) \equiv 1 .$$

Defining

$$Q(p, b) = V(b)\, \Psi_0(p) \in \mathfrak{L}^2_\mu(\mathsf{R}) \tag{4}$$

and using (1), we get

$$V(b)\, \Psi(p) = Q(p, b)\, \Psi(p - b) . \tag{5}$$

[1]) The mapping is defined: To each vector $E(\Delta)\, \Psi_0$ there corresponds the function $\chi_\Delta(p)$ = characteristic function of Δ. The image of Ψ_0 is $\Psi_0(p) \equiv 1$.

But since the functions (3) are dense in $\mathfrak{L}^2_\mu(R)$ (5) holds true in the whole space. The group property for the $V(b)$ applied to (5) gives the functional equation

$$Q(p, b_1)\, Q(p - b_1, b_2) = Q\,(p, b_1 + b_2)\,. \tag{6}$$

The unitarity of $V(b)$ implies

$$\int |\, \Psi(p)\,|^2\, d\mu(p) = \int |\, Q(p, b)\,|^2\, |\, \Psi(p - b)\,|^2\, d\mu(p)$$

for all $\Psi \in \mathfrak{L}^2_\mu$. Written symbolically this means

$$d\mu\,(p - b) = |\, Q(p, b)\,|^2\, d\mu(p)\,. \tag{7}$$

From this we conclude that the measure μ is quasi invariant. This means $\mu(\Delta) = 0$ implies $\mu({}_b\Delta) = 0$ for all $b \in R$; ${}_b\Delta = \{p - b \mid p \in \Delta\}$.

With this knowledge of our representation we want to show next that we can map it isometrically on the Schrödinger representation. In order to do this, we need the following theorem from measure theory: A quasiinvariant regular σ-finite measure on R^n is equivalent to the Lebesgue measure (this means that the sets of measure 0 are the same for both measures). The converse is also true: If a σ-finite regular measure is equivalent to the Lebesgue measure, then it is quasiinvariant [6]. An application of the Radon-Nykodym theorem implies: If μ is a (regular σ-finite) measure on R which is equivalent to the Lebesgue measure, then μ has the form

$$\mu(\Delta) = \int_\Delta \varrho(p)\, dp$$

where $\varrho(p)$ is a strictly positive function, integrable over all bounded (L-measurable) sets. Further $\varrho(p)$ is uniquely determined except on a set of L-measure zero [7].

Using these facts we can now argue as follows: First we can write (symbolically)

$$d\mu(p) = \varrho(p)\, dp \quad ; \quad \varrho(p) \quad \text{strictly positive}$$

hence

$$d\mu\,(p - b) = \frac{\varrho\,(p - b)}{\varrho(p)}\, d\mu(p)\,.$$

Comparison of this with (7) gives

$$\frac{\varrho\,(p - b)}{\varrho(p)} = |\, Q(p, b)\,|^2$$

and by putting $p = 0$

$$\varrho(p) = N \cdot |\, Q\,(0, -p)\,|^2\,.$$

Thus we arrive at

$$d\mu(p) = N \cdot |\, Q\,(0, -p)\,|^2\, dp\,. \tag{8}$$

N is a normalisation constant determined by

$$N \int |\, Q\,(0, -p)\,|^2\, dp = 1\,. \tag{9}$$

Let now $\mathfrak{L}^2(R)$ be the Hilbert space of square-integrable functions with respect to the Lebesgue measure. We construct the following mapping between $\mathfrak{L}^2_\mu(R)$ and $\mathfrak{L}^2(R)$:

$$\Psi(p) \to \hat{\Psi}(p) = \sqrt{N}\, Q\,(0, -p)\, \Psi(p)\,. \tag{10}$$

By (8) this is an isometry. Let $\hat{U}(a)$ and $\hat{V}(b)$ be the image of the operators $U(a)$ and $V(b)$ under this mapping. Obviously

$$\hat{U}(a)\ \hat{\Psi}(p) = e^{iap}\ \hat{\Psi}(p) \tag{11}$$

furthermore

$$\hat{V}(b)\ \hat{\Psi}(p) = \sqrt{N}\ Q\ (0, -p)\ Q(p, b)\ \Psi\ (p - b)$$

but by (6)

$$Q\ (0, -p)\ Q(p, b) = Q\ (0, -(p - b))$$

hence

$$(\hat{V}(b)\ \hat{\Psi})\ (p) = \hat{\Psi}\ (p - b) \tag{12}$$

(11) and (12) are just the Schrödinger representation in momentum space. We thus have established the following

Theorem 1. Let $(a, b) \to (U(a), V(b))$ be a representation of the canonical commutation relations which is cyclic with respect to the group $\{U(a) \mid a \in R\}$. Then this representation is isomorphic to the Schrödinger representation (11) and (12).

4. Proof of von Neumann's theorem

We consider now an arbitrary representation of the canonical commutation relations in a separable Hilbert space $\mathcal{H}$. In the case where $\mathcal{H}$ is not cyclic with respect to the set $\{U(a) \mid a \in R\}$ we can represent $\mathcal{H}$ as a direct integral of Hilbert spaces $\mathcal{H}(p)$ with respect to a normalised measure μ.

$$\mathcal{H} = \int \oplus\ \mathcal{H}(p)\ d\mu(p) \tag{13}$$

in such a way that $U(a)$ operates as [8]

$$(U(a)\ \Psi)\ (p) = e^{ipa}\ \Psi(p)\ . \tag{14}$$

We consider the spaces $\mathcal{H}(p)$ as l_2-spaces of dimension $n(p)$. Each element $\Psi \in \mathcal{H}$ is then a vector valued function $\Psi = \Psi(p)$, where each $\Psi(p)$ is equal to a sequence $\{g_k(p);\ k = 1, 2, \ldots, n(p)\}$. The norm in $\mathcal{H}$ is

$$\|\ \Psi\ \|^2 = \int \sum_{k=1}^{n(p)} |\ g_k(p)\ |^2\ d\mu(p)\ . \tag{15}$$

In $\mathcal{H}(p)$ we consider the following special vectors

$$\Psi_0^i(p) = \{\delta_{ik}\} \tag{16}$$

$\Psi(p)$ can then be written as

$$\Psi(p) = \sum_j \Psi^j(p), \quad \text{where} \quad \Psi^j(p) = \{\delta_{kj}\ g_k(p)\}\ . \tag{17}$$

We consider again trigonometrical polynomials

$$g_k(p) = \sum_{l=1}^{n_k} c_{kl}\ e^{i a_{kl} p}\ . \tag{18}$$

Everything goes now analogously to the special case. We have

$$U(a)\,\{g_k(p)\} = e^{ipa}\{g_k(p)\}\,, \tag{19}$$

and

$$\Psi^j(p) = \Sigma\, c_{kl}\, U(a_{kl})\, \Psi_0^j\,. \tag{20}$$

We define

$$(V(b)\,\Psi_0^j)\,(p) = \{v_k^j(p, b)\}\,. \tag{21}$$

Using (20), (1) and (21) we get

$$\begin{aligned} V(b)\,\Psi^j(p) &= \sum_l c_{kl}\, V(b) U(a_{kl})\, \Psi_0^j \\ &= \Big\{\sum_l c_{kl}\, e^{i a_{kl}(p-b)}\, v_k^j(p, b)\Big\} \\ &= \{v_k^j(p, b)\, g_k\,(p-b)\}\,. \end{aligned}$$

We write the result in a compact form

$$\big(V(b)\,\Psi\big)\,(p) = Q(p, b)\,\Psi\,(p-b)\,. \tag{22}$$

Where the operator $Q(p, b)$, which maps $\mathcal{H}(p)$ into $\mathcal{H}(p-b)$, is defined by

$$Q(p, b)\,\Psi\,(p-b) = \Big\{\sum_j v_k^j(p, b)\, g_k\,(p-b)\Big\}\,. \tag{23}$$

Since the vectors (18) are dense in $\mathcal{H}$, (22) and (23) hold true in the whole space. [Formally we had the equations (14) and (22) also in our special case]. We again show that the measure μ is quasiinvariant. For this purpose we choose $\Psi(p) = \{\chi_\Delta(p), 0, 0\, \ldots\}$ where $\chi_\Delta(p)$ is the characteristic function for the set Δ, and compute

$$\|\,\Psi\,\|^2 = \|\,V(b)\,\Psi\,\|^2 :$$

$$\begin{aligned} \|\,\Psi\,\|^2 &= \int \chi_\Delta(p)\, d\mu(p) = \mu(\Delta) = \|\,V(b)\,\Psi\,\|^2 \\ &= \int \sum_{j=1}^{\infty} |\, v_1^j(p, b)\,|^2\, \chi_\Delta\,(p-b)\, d\mu(p)\,. \end{aligned}$$

From this the quasiinvariance is obvious. By the measure theoretic theorems quoted in § 3., we can represent the measure μ as

$$d\mu(p) = \varrho(p)\, dp \tag{24}$$

where $\varrho(p)$ is strictly positive. The mapping

$$\Psi(p) \leftrightarrow \sqrt{\varrho(p)}\,\Psi(p)$$

sets up a unitary map from $\mathcal{H} = \int \oplus\, \mathcal{H}(p)\, d\mu(p)$ onto $\mathcal{H}' = \int \oplus\, \mathcal{H}(p)\, dp$. We imagine that this has been done beforehand. Our measure is now the translational invariant Lebesgue measure. The group property of the $V(b)$ gives, using (22)

$$Q(p, b)\, Q\,(p-b, b') = Q\,(p, b+b') \tag{25}$$

putting $b' = p - b$, we get

$$Q(p, b)\, Q(p - b, p - b) = Q(p, p)$$

or

$$Q^{-1}(p, p)\, Q(p, b)\, Q(p - b, p - b) = \mathbf{1}\,. \tag{26}$$

Use of (22) and the invariance of the Lebesgue measure under translations shows that $Q(p, b)$ is a unitary operator. From this follows, that almost all spaces $\mathcal{H}(p)$ have the same dimension. We now carry out the unitary transformation.

$$\hat{\Psi}(p) = Q^{-1}(p, p)\, \Psi(p) \tag{27}$$

obviously

$$(\hat{U}(a)\, \hat{\Psi})\,(p) = e^{i p a}\, \hat{\Psi}(p)\,. \tag{28}$$

Using (26), we get for $\hat{V}(b)$

$$(\hat{V}(b)\, \hat{\Psi})\,(p) = \mathbf{1} \cdot \hat{\Psi}\,(p - b) \tag{29}$$

(27) and (29) show that we have decomposed the whole Hilbert space into a direct sum of Hilbert spaces $\mathfrak{L}^2(R)$ in each of which we have the irreducible Schrödinger representation. We summarise our results in the

Theorem 2 (von Neumann). Let there be given a representation of the canonical commutation relations in the sense of def. 1 in a separable Hilbert space $\mathcal{H}$. Then this representation can be decomposed into a direct sum of irreducible representations, each of which is unitarily equivalent to the Schrödinger representation.

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Appendix

In this appendix we give a shorter version of the proof in § 4, using however less elementary results from representation theory. We need the following

Theorem 3.

Every continuous unitary representation of the group R^n is unitary aequivalent to one of the following form:

$$(U(a)\Psi)\,(p) = e^{i p \cdot a}\, \Psi(p) \tag{A1}$$

where $(\Psi(p))$ is an element of a direct integral

$$\mathcal{H} = \int_{R^n} \oplus\, \mathcal{H}(p)\, d\mu(p)$$

with measure μ and dimension $n(p)$ of $\mathcal{H}(p)$. Any bounded operator A, which commutes with all $U(a)$ has the form

$$(A\Psi)\,(p) = A(p)\, \Psi(p)\,, \tag{A2}$$

$A(p)$ is a bounded operator in $\mathcal{H}(p)$, such that for all Ψ_1 and Ψ_2 in $\mathcal{H}$, $(\Psi_1(p), A(p)\Psi_2(p))$ is measurable. Two such representations with measures μ_1 and μ_2 and dimension functions $n_1(p)$ and $n_2(p)$, respectively are unitary aequivalent iff

1) μ_1 and μ_2 are aequivalent
2) $n_1(p) = n_2(p)$ except on a set of μ_1-measure 0.

Consider now (for fixed b) the unitary representation $W(a)$ of R^n defined by

$$W(a) = V(b)\, U(a)\, V^{-1}(b) = e^{-iab}\, U(a) \tag{A3}$$

We have

$$\big(W(a)\Psi\big)\,(p) = e^{i(p-p)\,a}\, \Psi(p)$$

Consider the unitary mapping

$$T_b\colon \quad \int \oplus\, \mathcal{H}(p)\, d\mu(p) \to \int \oplus\, \mathcal{H}(p-b)\, d\mu_b(p) \;:\; \Psi(p) \to (T_b\Psi)(p) = \Psi\,(p-b)$$

where $d\mu_b(p) = d\mu\,(p-b)$.

By this mapping $W(a)$ has the standard representation (A1). From theorem 3 we conclude, that μ is quasiinvariant and $n(p)$ is constant, except on a set of μ-measure 0. Hence we can identify all the spaces $\mathcal{H}(p)$ and take as earlier for μ the Lebesgue measure.

We now define

$$V(b) = Q_1(b)\; T(b)\,,$$

where $T(b)$ is the unitary operator defined by

$$\big(T(b)\Psi\big)\,(p) = \Psi\,(p-b)\,.$$

From

$$T(b)\; U(a) T^{-1}(b) = e^{-iab}\,(Ua)$$

follows (compare A3), that $Q_1(b)$ commutes with $U(a)$. Theorem 3 shows, that $Q_1(b)$ can be written as

$$\big(Q_1(b)\Psi\big)\,(p) = Q_1(p,\, b)\Psi(p).$$

Since $Q_1(b)$ is unitary, we conclude that $Q_1(p, b)$ is unitary for almost all p. The further conclusions are the same as in § 4 (Q_1 satisfies equation (25) etc.).

References

[1] von Neumann, J., Math. Ann. *104*, 570–578 [Vol. II, No. 7].
[2] Weyl, H., Z. Phys. *46*, 1–46 (1928).
[3] Gelfand, I. M. and Wilenkin, N. J., *Verallgemeinerte Funktionen IV* (Deutscher Verlag der Wissenschaften, Berlin 1964), p. 334.
[4] Gelfand, I. M. and Wilenkin, N. J., op. cit. p. 123.
[5] Bargmann, V., *Gruppentheoretische Analyse der Lorentzinvarianz* (Vorlesung an der ETH im S.S. 1963, ausgearbeitet von W. Schneider).
[6] Gelfand, I. M. and Wilenkin, N. J., op. cit. §5, p. 322.
[7] See [6].
[8] Gelfand, I. M. and Wilenkin, N. J., op. cit. p. 125.