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Functional-Analytic Discussion of the Linearized Boltzmann Equation

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Summary. The existence theory for the linearized Boltzmann equation is discussed by means of semigroup methods. The infinite medium and systems with boundaries, are treated. It is shown that the Boltzmann operator is the infinitesimal generator of a contraction semigroup, which solves the initial value problem. The connection between this general solution and the Chapman-Enskog method is then considered. It is found that under suitable conditions, it is the 'hydrodynamic' part of the general solution, which is calculated by Chapman-Enskog. Concerning this, an expansion formula involving the normal solutions is obtained. Applications of the semigroup method to other kinetic equations are discussed.

I. Introduction

There exist various formal procedures for constructing solutions to the Boltzmann equation, for instance the Hilbert or the Chapman-Enskog expansion. Although these work very well in practice, one must ask how the calculated quantities are related to the actual solutions of the Boltzmann equation. To answer this question one has to construct these actual solutions, and then connect them with the approximate ones.

This problem was attacked mainly by H. GRAD in a series of papers [1, 2, 3] by means of classical analysis. In the present paper the problem is treated using semigroup methods. In this way it is straightforward to get a complete existence theory for the linearized Boltzmann equation.

In the following section the basic concepts are introduced and the infinite-space problem is considered. This is the simplest case, because it concerns only a pure initial value problem. We find that the solution of the Boltzmann equation is uniquely given from the initial distribution by a contraction semigroup. In the third section the case of a finite system is treated. This is a mixed problem; in addition to the initial condition, boundary conditions are imposed, which must be satisfied at all times. Also in this case the solution is given by a contraction semigroup, if the boundary condition is such that the stationary distribution is a strict Maxwellian (for example specular reflection or constant temperature along the walls). Otherwise the solution has to be altered due to the fact that we consider the Boltzmann equation linearized about a strict Maxwell distribution. In the fourth section we turn to the second problem mentioned above, namely the relation between these general solutions and the normal solutions of the Hilbert and Chapman-Enskog theory. This part falls into line with a work of MCLENNAN [4]. In contrast to the general existence theory two important restrictions are now necessary: the interaction potential must be of the class of hard potentials introduced by GRAD [2], and the initial distribution must be restricted in the spatial derivatives. Then considering for instance the infinite space problem the semigroup solution can be split into a hydrodynamic and a microscopic part. The former can be expanded in terms of the normal solutions. The last section is devoted to some soncluding remarks, and it is pointed out that an external electromagnetic field can be easily dealt within the framework of the linearized Boltzmann equation. Application of the semigroup method to kinetic equations with other collision operators, for instance occurring in neutron diffusion problems and solid state physics is possible as well.

II. The Boltzmann Equation for an Infinite Medium

We start with a few fundamental results of the theory of contraction semigroups [6, 7].

Let $\mathcal{H}$ be a Hilbert space. A one-parameter family T^t, $t \geqslant 0$ of bounded linear operators on $\mathcal{H}$ is called a contraction semigroup of class C^0, if

$$T^s\,T^t = T^{s+t} \qquad \text{(semigroup property)}$$

$$T^0 = \mathbf{1}$$

$$\| T^t \| \leqslant 1 \qquad \text{(contraction property)}$$

$$s - \lim_{t \to 0} T^t f = f, \quad \forall f \in \mathcal{H} \quad \text{(strong continuity)}\,.$$

The infinitesimal generator A of T^t is defined by

$$A f = s - \lim_{t \to 0} \frac{1}{t}\,(T^t - \mathbf{1})\,f \tag{1}$$

where the domain $\mathcal{D}(A)$ consists of those $f \in \mathcal{H}$, such that (1) exists. It is an important property of the infinitesimal generator, that it is dissipative: A is called dissipative in case it is densely defined and

$$Re\,(f, Af) \leqslant 0\,, \qquad \forall f \in \mathcal{D}(A)\,.$$

The following theorem gives a complete characterization of A.

Theorem 1 (HILLE, YOSHIDA)

A linear operator A is the infinitesimal generator of a contraction semigroup T^t if and only if one of the following conditions is satisfied:

1) A is closed and dissipative and the adjoint A^* is dissipative.

2) A is dissipative and the range of $\lambda - A$ coincides with $\mathcal{H}$ for some λ with $Re\,\lambda > 0$.

Then T^t can be represented as follows

$$T^t f = s - \lim_{\eta \to \infty} \exp\left(t\,\eta\,A\,(\eta - A)^{-1}\right) f\,.$$

For operators which generate a contraction semigroup of class C^0 the so-called abstract initial value problem can be solved.

Theorem 2

Let A be the infinitesimal generator of a semigroup of class C^0, $f_0 \in \mathcal{D}(A)$ and $h(t)$ continuously differentiable for $t \geqslant 0$. Then there exists one and only one function $f(t)$ from $[0, \infty)$ to $\mathcal{H}$ with

1) $f(t)$ is continuously differentiable, $f(t) \in \mathcal{D}(A)$, for each $t > 0$,
2) $d/dt\, f(t) = A\, f(t) + h(t)$,
3) $s - \lim_{t \to 0} f(t) = f_0$.

This function $f(t)$ is given by

$$f(t) = T^t f_0 + \int_0^t T^{t-s} h(s)\, ds\,.$$

After these preliminaries we turn to the linearized Boltzmann equation:

$$\frac{\partial f}{\partial t} = -\, \boldsymbol{v}\, \frac{\partial f}{\partial \boldsymbol{x}} - I\, f\,. \tag{2}$$

$f(\boldsymbol{x}, \boldsymbol{v}, t)$ determines the Boltzmann distribution function $F = f_0(1 + f)$, where

$$f_0 = n_0 \left(\frac{m}{2\, \pi\, k\, T_0}\right)^{3/2} \exp - \frac{m\, v^2}{2\, k\, T_0} \tag{3}$$

is the space-independent Maxwell distribution and f a small deviation from it. I denotes the linearized collision operator [2]

$$I\, f = \int (f + f_1 - f' - f_1')\, f_0(v_1)\, |\, \boldsymbol{v} - \boldsymbol{v}_1\, |\, \sigma\, d^2\Omega\, d^3v_1 \stackrel{\text{def}}{\equiv} \nu(v)\, f - K\, f\,. \tag{4}$$

Here $\sigma\, (|\, \boldsymbol{v} - \boldsymbol{v}_1\, |, \vartheta)$ is the differential scattering cross section ϑ the scattering angle and $\nu(v)$ the so-called collision frequency, the arguments $\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}', \boldsymbol{v}_1'$ for the f's are the velocities of two colliding molecules before and after the collision respectively.

We consider (2) within the Hilbert space $\mathcal{H} = \mathrm{L}^2(\boldsymbol{x}) \otimes \mathrm{L}^2_{f_0}(\boldsymbol{v})$ with the following scalar product

$$(f, g) = \int \bar{f}\, g\, f_0(v)\, d^3v\, d^3x\,.$$

The $\mathrm{L}^2(\boldsymbol{x})$ can be formed over the whole R^3 or over some compact part Ω of it, in which case it will be denoted by $\mathrm{L}^2(\Omega)$, according to the region covered by the medium. The L^2-spaces are taken complex since a Fourier transformation is to be used.

Apart from technical reasons, the choice of the Hilbert space $\mathcal{H}$ as above has physical meaning, since the norm in $\mathcal{H}$ is related to the relevant thermodynamic potential of the system. We linearize Boltzmann's H-function

$$H = \int F \log \left[F\, \frac{1}{n_0} \left(\frac{2\, \pi\, k\, T_0}{m}\right)^{3/2}\right] d^3v\, d^3x$$

considering a finite system and assuming that energy transfer through the boundaries of the system is possible, but the number of particles is constant, $N = \int F\, d^3v\, d^3x = \int f_0\, d^3v\, d^3x$.

We get

$$H = -\frac{E}{k\,T_0} + \frac{1}{2}\int f^2 f_0\, d^3v\, d^3x\,, \quad E = \text{energy}, \quad f \text{ real}\,.$$

If H is identified with the negative entropy by $-1/k\,S$, there results

$$\frac{1}{2}\,\|f\|^2 = \frac{E - T_0\,S}{k\,T_0} = \frac{F}{k\,T_0}\,.$$

Thus, in case of a closed system, with $E = \text{const.}$, $\|\;\|^2$ is a measure for the entropy S; otherwise for the free energy F.

In the following we introduce dimensionless variables by replacing

$$t \to \frac{t}{\tau} \qquad \boldsymbol{v} \to \frac{\boldsymbol{v}}{\sqrt{k\,T_0/m}} = \frac{\boldsymbol{v}}{v_0} \qquad \boldsymbol{x} \to \frac{\boldsymbol{x}}{v_0\,\tau} \text{ and } f_0 \to \varphi_0 = \frac{1}{(2\,\pi)^{3/2}} \exp -\frac{1}{2}\,v^2\,,$$

τ is some time of the order of magnitude of the mean collision time. In order to solve (2) in the whole space, we must study the properties of the Boltzmann operator

$$B = -\boldsymbol{v}\,\frac{\partial}{\partial \boldsymbol{x}} - I \text{ in } \mathcal{H} = \mathrm{L}^2(R^3) \otimes \mathrm{L}^2_{\varphi_0}(R^3)\,.$$

This can be done simply by means of a spatial Fourier transformation F_x, let

$$U = F_x \otimes \mathbf{1} \tag{5}$$

then

$$\hat{B} = U\,B\,U^{-1} = -\,i(\boldsymbol{k}\,\boldsymbol{v}) - I\,. \tag{6}$$

The collision operator I is in general unbounded, because of the multiplication by the collision frequency $\nu(v)$. However we suppose that the remaining integral operator K in (4) is a bounded operator in $\mathcal{H}$. Indeed this was shown by GRAD [2] under the weak assumption that the collision cross section $\sigma(\vartheta, \boldsymbol{w})$ satisfies

$$\sigma(\vartheta, \boldsymbol{w}) < b\left(1 + \frac{1}{w^{2-\varepsilon}}\right) \quad \begin{matrix} b > 0 \\ 0 < \varepsilon < 1\,. \end{matrix} \tag{7}$$

Then the domain of $\hat{B}$ can be easily described:

$$\mathcal{D}(\hat{B}):\ \hat{f} \in \hat{\mathcal{H}} = \mathrm{L}^2(\boldsymbol{k}) \otimes \mathrm{L}^2_{\varphi_0}(\boldsymbol{v})\,,\ [-\,i(\boldsymbol{k}\,\boldsymbol{v}) - \nu(v)]\ \hat{f} \in \hat{\mathcal{H}}; \tag{8}$$

since $\nu(v)$ is real, the latter is equivalent to

$$(\boldsymbol{k}\,\boldsymbol{v})\,\hat{f} \in \hat{\mathcal{H}} \quad \text{and} \quad \nu(v)\,\hat{f} \in \hat{\mathcal{H}} \tag{9}$$

separately. The operator $\hat{B}$ defined in this way is dissipative

$$Re\,(\hat{f}, \hat{B}\,\hat{f})_{\hat{\mathcal{H}}} = -\,(\hat{f}, I\,\hat{f})_{\hat{\mathcal{H}}} \leqslant 0$$

because I is symmetric and positive. Moreover, since K is bounded,

$$\hat{B}^* = i(\boldsymbol{k}\,\boldsymbol{v}) - \nu(v) + K^* = i(\boldsymbol{k}\,\boldsymbol{v}) - I \tag{10}$$

is dissipative too, and $\hat{B}^{**} = \hat{B}$ is closed. Therefore applying theorems 1 and 2 we get a unique solution of the Boltzmann equation given by a contraction semigroup T^t: Let

$$\hat{f}(0) = U f(0) \in \mathfrak{D}(\hat{B})$$

be the initial distribution. Then

$$f(t) = U^{-1} \hat{T}^t U f(0) = T^t f(0) . \tag{11}$$

If condition (7) is not satisfied, the Boltzmann operator (respectively the collision operator I) still can be densely defined and is then dissipative. But instead of (10) we in general only have

$$\hat{B}^* \supseteq i(\boldsymbol{k}\,\boldsymbol{v}) - I . \tag{12}$$

In this case the following more general argument can be used. The operator J defined by

$$J\,\hat{f}(\boldsymbol{k}, \boldsymbol{v}) = \overline{\hat{f}(\boldsymbol{k}, \boldsymbol{v})} \qquad \hat{f} \in \hat{\mathcal{H}}$$

is a conjugation in $\hat{\mathcal{H}}$:

$$(J\,\hat{f}, J\,\hat{g}) = \overline{(\hat{f}, \hat{g})} \qquad J^2 = 1 .$$

Since the collision operator I commutes with J, we conclude that

$$J\,\hat{B}\,J = i(\boldsymbol{k}\,\boldsymbol{v}) - I .$$

Because of (12) this shows that $\hat{B}$ is J-symmetric

$$\hat{B}^* \supseteq J\,\hat{B}\,J .$$

It is a general result [6], that a J-symmetric dissipative operator always posesses a J-selfadjoint extension, which preserves the dissipativity. Taking this extension $\hat{B}_1$, we have

$$\hat{B}_1^* = J\,\hat{B}_1\,J$$

and therefore $\hat{B}_1^*$ together with $\hat{B}_1$ are dissipative. Now we arrive at the same result as above. $\hat{B}_1$ generates a contraction semigroup, which gives the solution to the Boltzmann equation. The only difference to the foregoing case is that we do not have an explicit description of the domain of the Boltzmann operator.

III. Systems with Boundaries

Boundary conditions can be dealt with by the semigroup method in the following manner: One includes the boundary conditions in the definition of the domain of the infinitesimal generator A. Then if at $t = 0$ the boundary conditions are satisfied, they remain satisfied for all times $t > 0$ in view of theorem 2 (1).

Let Ω be an open bounded region of R^3 with smooth boundary $\partial\Omega$. We will define the Boltzmann operator (4) in the Hilbert space

$$\mathcal{H}(\Omega) = \mathrm{L}^2(\Omega) \otimes \mathrm{L}^2_{\varphi_0}(R^3) .$$

In order to get explicit domains we assume again $I = \nu(v) - K$ with bounded K. Let us first introduce an operator A_1

$$A_1 f = -\boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}} - I f$$

$$\mathcal{D}(A_1): f \in C_x^\infty(\Omega) , f_{\partial\Omega} = 0$$

$$\boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}} \in \mathcal{H}(\Omega), \ \nu(v) f \in \mathcal{H}(\Omega) . \tag{13}$$

To determine it's adjoint operator A_1^* (respectively the adjoint of $A_2 = -\boldsymbol{v}\, \partial/\partial \boldsymbol{x} - \nu(v)$) we use again the Fourier operator (5), which now transforms $\mathcal{H}(\Omega)$ into a subspace $\hat{\mathcal{H}}(\Omega) \subset \hat{\mathcal{H}}$. Since every φ of the form

$$\varphi = \begin{cases} f \in \mathcal{D}(A_1) & \text{for } \boldsymbol{x} \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

belongs to the domain $\mathcal{D}(B)$ of the previous Boltzmann operator in $\mathcal{H}$, we have

$$(g, A_2 f)_{\mathcal{H}(\Omega)} = (\hat{g}, (-i(\boldsymbol{k}\,\boldsymbol{v})\,\hat{\varphi} - \nu(v))\hat{\varphi})_{\hat{\mathcal{H}}} = (i(\boldsymbol{k}\,\boldsymbol{v})\,\hat{g} - \nu(v)\,\hat{g}, \hat{f})_{\hat{\mathcal{H}}(\Omega)}$$

$$g \in \mathcal{H}(\Omega) , \quad f \in \mathcal{D}(A_1)$$

and this is continuous for $f \in \mathcal{D}(A_1)$ if and only if $(\boldsymbol{k}\,\boldsymbol{v})\,\hat{g} \in \hat{\mathcal{H}}(\Omega)$, $\nu(v)\,\hat{g} \in \hat{\mathcal{H}}(\Omega)$ (compare (9)). Therefore A_1^* is given by

$$A_1^* g = \boldsymbol{v} \frac{\partial g}{\partial \boldsymbol{x}} - I g \qquad \mathcal{D}(A_1^*): g \in \mathcal{H}(\Omega), \ \boldsymbol{v} \frac{\partial g}{\partial \boldsymbol{x}} \in \mathcal{H}(\Omega) , \ \nu(v)\, g \in \mathcal{H}(\Omega)$$

where $\partial/\partial x_i$ now denotes the distributional derivatives. Obviously A_1 is dissipative, but A_1^* is not. So we look for extensions A

$$A \supset A_1 \qquad A^* \subset A_1^*$$

of A_1 such that A^* together with A becomes dissipative.

Since I is already dissipative on $\mathcal{D}(A_1^*)$, we need only consider the convective term. Taking the closure $\bar{A}_1$ of A_1 the C^∞-functions in $\mathcal{D}(A_1)$ are replaced by functions $f \in \mathcal{H}(\Omega)$ with first distributional derivatives $\boldsymbol{v}\, \partial f/\partial \boldsymbol{x} \in \mathcal{H}(\Omega)$. For $f, g \in \mathcal{D}(A_1^*)$ we can integrate by parts

$$\left(f, \boldsymbol{v} \frac{\partial g}{\partial \boldsymbol{x}}\right) = -\left(\boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}}, g\right) + \int_{\partial\Omega}\!\!\int d^3v\, \varphi_0\, \bar{f}\, g\, \boldsymbol{v}\, d\boldsymbol{\sigma} \tag{14}$$

$$2\, Re \left(f, -\boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}}\right) = -\int_{\partial\Omega}\!\!\int d^3v\, \varphi_0\, |f|^2 \boldsymbol{v}\, d\boldsymbol{\sigma} . \tag{15}$$

The value on the right hand side in (15) depends on the boundary conditions. The physically interesting boundary conditions are local in the sense that they connect only the values of $f(\boldsymbol{x}, \boldsymbol{v})$ at the same point x of the surface. Otherwise there would be a correlation between different points of the wall caused by processes inside the wall. Supposing local boundary conditions we see from (15), that the convective term is dissipative if and only if

$$\int d^3v\, \varphi_0\, |f|^2\, (\boldsymbol{v}, \boldsymbol{n}) = \int_0^\infty dv\, v^3\, \varphi_0(v) \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta$$

$$\sin\vartheta\, \cos\vartheta\, [|f(\boldsymbol{x}, \boldsymbol{v}_+)|^2 - |f(\boldsymbol{x}, \boldsymbol{v}_-)|^2 \geqslant 0 \tag{16}$$

where $\boldsymbol{n}$ is the unit vector in the normal direction,

$$\boldsymbol{v}\,\boldsymbol{n} = v \cos\vartheta$$

$$\boldsymbol{v}_\pm = (v \sin\vartheta \cos\varphi,\, v \sin\vartheta \sin\varphi,\, \pm\, v \cos\vartheta)\,.$$

We introduce for every point $\boldsymbol{x}$ of the surface a Hilbert space $\mathcal{H}'_x$ with respect to the velocity $\boldsymbol{v} = (v \sin\vartheta \cos\varphi,\, v \sin\vartheta \sin\varphi,\, v \cos\vartheta)$ with the following scalar product

$$(f, g)' = \int_0^\infty dv\, v^3\, \varphi_0(v) \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta\, \frac{1}{2} \sin 2\vartheta\, \bar{f}(\boldsymbol{x}, \boldsymbol{v})\, g(\boldsymbol{x}, \boldsymbol{v})\,, \quad \boldsymbol{x} \text{ fixed.}$$

This is an ordinary $L^2_{v\varphi_0}(R^3)$, if one makes the identification

$$f'(v, \varphi, \vartheta') = f\left(v, \varphi, \vartheta = \frac{1}{2}\, \vartheta'\right)$$

hence

$$(f, g)' = \frac{1}{4} \int v\, \varphi_0(v)\, \bar{f}'\, g'\, d^3v \overset{\text{def}}{=} \frac{1}{4}\, (f', g')'\,.$$

Now we choose linear boundary conditions

$$f'_\pm(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{x}, \boldsymbol{v}_\pm) \qquad f'_- = R_x\, f'_+ \qquad \boldsymbol{x} \in \partial\Omega \tag{17}$$

connecting the incoming and emitted distributions; R_x is a linear operator in $\mathcal{H}'_x$. The boundary conditions must be homogeneous to define a linear set. With (17) we conclude from (16) that the convective term is dissipative if R_x is a contraction in $\mathcal{H}'_x$, $\| R_x \|' \leqslant 1$. Therefore we define the Boltzmann operator A on the following domain:

$$\mathcal{D}(A){:}\; f(\boldsymbol{x}, \boldsymbol{v}) \in \mathcal{D}(A_1^*)\,,\; f'_\pm \in \mathcal{H}'_x\,,\;\; f'_- = R_x\, f'_+$$

$$\| R_x \|' \leqslant 1\,, \qquad \boldsymbol{x} \in \partial\Omega\,. \tag{18}$$

To determine the adjoint A^*, we see from (14), that for $f \in \mathcal{D}(A^*)$

$$\int_{\partial\Omega} \int d^3v\, \varphi_0\, \bar{f}\, g\, \boldsymbol{v}\, d\boldsymbol{\sigma} = 0\,, \qquad \forall\; g \in \mathcal{D}(A)$$

is necessary, that is

$$\frac{1}{4} \int d^3v\, v\, \varphi_0(v)\, [\bar{f}'_+(v)\, g'_+(v) - \bar{f}'_-(v)\, g'_-(v)] = 0$$

or with (18)

$$(f'_+, g'_+)' = (R_x^*\, f'_-, g'_+)'\,.$$

Hence the domain of the adjoint operator A^* is described by the adjoint boundary condition

$$\mathcal{D}(A^*): \quad f \in \mathcal{D}(A_1^*), \quad f'_+ = R_x^* f'_- .$$

If R_x is a contraction R_x^* is a contraction as well; consequently it follows from

$$2 \, Re \left(f, \boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}} - I f \right) = - 2 \, (f, I f) + \int_{\partial\Omega} |d\boldsymbol{\sigma}| \, [\| f'_+(v) \|'^2 - \| f'_-(v) \|'^2]$$

that A^* is dissipative too. Since

$$\mathcal{D}(A^{**}): \quad f \in \mathcal{D}(A_1^*) \qquad f'_- = R_x^{**} f'_+ ,$$

we see that $A^{**} = A$. By Theorem 1 and 2 we now arrive at the

Theorem 3

The Boltzmann operator A with $\mathcal{D}(A)$ given by (18) generates a contraction semigroup T^t, which gives the unique solution of the associated mixed initial value problem.

Actually the boundary conditions are imposed on the whole distribution function

$$F(\boldsymbol{x}, \boldsymbol{v}, t) = \varphi_0(v) \, (1 + \phi(\boldsymbol{x}, \boldsymbol{v}, t)) \tag{19}$$

and not on ϕ. For instance for specular reflection we have

$$F(\boldsymbol{x}, \boldsymbol{v}_-, t) = F(\boldsymbol{x}, \boldsymbol{v}_+, t) \qquad \boldsymbol{x} \in \partial\Omega .$$

But then

$$\phi'_- = \phi'_+ \qquad \text{i.e. } R = \mathbf{1}$$

and Theorem 3 applies. The stochastic boundary condition

$$F_-(\boldsymbol{v}) = \int_+ k(\boldsymbol{v}, \boldsymbol{u}) \, F(\boldsymbol{u}) \, d^3u = \int_+ k(\boldsymbol{v}, \boldsymbol{u}) \, \varphi_0(u) \, d^3u + \int_+ k(\boldsymbol{v}, \boldsymbol{u}) \, \varphi_0(u) \, \phi(\boldsymbol{u}) \, d^3u , \tag{20}$$

where the integrals are taken over the halfspace $(\boldsymbol{u}, \boldsymbol{n}) \geqslant 0$, leads to an inhomogeneous condition for ϕ. This cannot be handled within the above framework because it does not define a linear set. To avoid this difficulty, we consider

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = 1 + \phi(\boldsymbol{x}, \boldsymbol{v}, t) \qquad F = \varphi_0 f , \tag{21}$$

which also fulfils the Boltzmann equation but with homogeneous boundary conditions

$$R f'_+ = \frac{1}{\varphi_0(v)} \int k'(\boldsymbol{v}, \boldsymbol{u}) \, \varphi_0(u) \, f'_+(\boldsymbol{u}) \, d^3u \qquad f'_+ \in \mathcal{H}'$$

in our above convention. Now, R is a contraction if

$$\| R \|'^2 \leqslant \int d^3v \int d^3u \, | k'(\boldsymbol{v} \, \boldsymbol{u}) |^2 \, \frac{\varphi_0(u)}{\varphi_0(v)} \, \frac{v}{u} \leqslant 1 .$$

But this is only a poor result as we will explicitly verify for diffuse reflection. Here the emitted distribution is a Boltzmann distribution

$$\varphi = \frac{1}{(2\,\pi\,\tau)^{3/2}} \exp - \frac{v^2}{2\,\tau} \qquad \tau = \frac{T}{T_0} \tag{22}$$

with temperature T, and we have

$$k'(\boldsymbol{v}, \boldsymbol{u}) = \frac{u\,\varphi(v)}{\int d^3v\; v\,\varphi(v)} \tag{23}$$

$$R\,f' = \frac{1}{\int d^3v\; v\,\varphi(v)}\,\frac{\varphi(v)}{\varphi_0(v)} \int u\,\varphi_0(u)\,f'(\boldsymbol{u})\,d^3u = \frac{1}{\int d^3v\; v\,\varphi(v)}\,\frac{\varphi(v)}{\varphi_0(v)}\,(1, f')'$$

$$\|R\|'^2 = \frac{\|1\|'^2\,\|\varphi(v)/\varphi_0(v)\|'^2}{\left(\int d^3v\; v\,\varphi(v)\right)^2} = \frac{\int v\,\varphi_0(v)\,d^3v}{\left(\int d^3v\; v\,\varphi(v)\right)^2} \int \frac{\varphi^2(v)}{\varphi_0(v)}\,v\,d^3v$$

$$= \frac{1}{1-(\tau-1)^2} = \frac{T_0^2}{T_0^2-(T-T_0)^2} \geqslant 1\,. \tag{24}$$

Theorem 3 applies only if $T = T_0 =$ const. along the boundary. On the other hand there is only a quadratic deviation $\sim ((T - T_0)/T)^2$ from 1 in (24), bringing in mind that we have lost the linearity in this treatment of the boundary conditions (21).

To be consistent with linearity we must linearize the boundary conditions too. Instead of (22) we choose more generally

$$\varphi(\boldsymbol{x}, \boldsymbol{v}, t) = \frac{1}{(2\,\pi\,\tau)^{3/2}} \exp - \frac{1}{2\,\tau}\,(\boldsymbol{v} - \boldsymbol{w})^2\,, \qquad \boldsymbol{x} \in \partial\Omega \tag{25}$$

where $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, t)$ represents a possible movement of the wall at point $\boldsymbol{x} \in \partial\Omega$. This for instance occurs if sound is propagated from an oscillating wall into the medium. $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, t)$ is given and is supposed to be small. We write $\boldsymbol{w} = w_1\,\boldsymbol{n} + \boldsymbol{w}_2$, where w_1 is the component in the normal direction to the wall and w_2 the tangential component. Inserting (25) into (23), expanding with respect to the small quantities w, $1 - 1/\tau = (T - T_0)/T$ and keeping only the linear terms we get

$$k'(\boldsymbol{v}, \boldsymbol{u}) = \frac{u\,\varphi_0(v)}{\int d^3v\; v\,\varphi_0(v)} \left[1 + \left(\frac{v^2}{2} - 2\right) \frac{T-T_0}{T} + (\boldsymbol{w}, \boldsymbol{v}) - \sqrt{\frac{\pi}{2}}\;w_1\right]$$

and from (20)

$$\varphi_0\,(1 + \phi'_-(\boldsymbol{v})) = \int_+ k'(\boldsymbol{v}, \boldsymbol{u})\,\varphi_0(u)\,d^3u + \frac{\varphi_0(v)}{\int d^3v\; v\,\varphi_0(v)} \int_+ u\,\varphi_0(u)\,\phi(\boldsymbol{u})\,d^3u\,.$$

This gives the following inhomogeneous condition for ϕ

$$\phi'_-(\boldsymbol{v}) = g(\boldsymbol{v}) + \frac{1}{\int d^3v\; v\,\varphi_0(v)} \int u\,\varphi_0(u)\,\phi'_+(\boldsymbol{u})\,d^3u = g + R\,\phi'_+ \tag{26}$$

where

$$g(\boldsymbol{x}, \boldsymbol{v}, t) = \left(\frac{v^2}{2} - 2\right) \frac{T-T_0}{T} + (\boldsymbol{w}, \boldsymbol{v}) - \sqrt{\frac{\pi}{2}}\;w_1$$

and R is the contraction belonging to *constant* temperature T_0. We transform the inhomogeneity from the boundary condition to the Boltzmann equation by setting

$$\phi(\boldsymbol{x}, \boldsymbol{v}, t) - \phi_1(\boldsymbol{x}, \boldsymbol{v}, t) = f(\boldsymbol{x}, \boldsymbol{v}, t) , \tag{27}$$

where ϕ_1 is any function satisfying the inhomogeneous condition (26) such that

$$- \boldsymbol{v} \frac{\partial \phi_1}{\partial \boldsymbol{x}} - I \phi_1 - \frac{\partial \phi_1}{\partial t} = h \tag{28}$$

is continuously differentiable with respect to t. Then we have to solve

$$\frac{\partial f}{\partial t} = - \boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{x}} - I f + h \tag{29}$$

$$f'_{\pm}(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{x}, \boldsymbol{v}_{\pm}) \qquad f'_- = R f'_+ \qquad f(\boldsymbol{x}, \boldsymbol{v}, 0) = \phi(\boldsymbol{x}, \boldsymbol{v}, 0) - \phi_1(\boldsymbol{x}, \boldsymbol{v}, 0) . \tag{30}$$

Now Theorem 2 and 3 apply:

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = T^t f(\boldsymbol{x}, \boldsymbol{v}, 0) + \int_0^t T^{t-s} h(s) \, ds . \tag{31}$$

In the case of time-independent boundary conditions ϕ_1 can be chosen equal to the given initial distribution $\phi_0(\boldsymbol{x}, \boldsymbol{v})$. Then

$$h = - \boldsymbol{v} \frac{\partial \phi_0}{\partial \boldsymbol{x}} - I \phi_0 = \left. \frac{\partial \phi(\boldsymbol{x}, \boldsymbol{v}, t)}{\partial t} \right|_{t=0}$$

and the first term in (31) drops out such that

$$\phi(\boldsymbol{x}, \boldsymbol{v}, t) = \phi_0(\boldsymbol{x}, \boldsymbol{v}) + \int_0^t T^{t-s} B \phi_0(\boldsymbol{x}, \boldsymbol{v}) \, ds . \tag{32}$$

The above argument obviously applies to a general stochastic kernel $k(\boldsymbol{v}, \boldsymbol{u})$ if this can be written as

$$k(\boldsymbol{v}, \boldsymbol{u}) = k_0(\boldsymbol{v}, \boldsymbol{u}) + k_1(\boldsymbol{v}, \boldsymbol{u}) ,$$

where k_0 corresponds to a contraction in $\mathcal{H}'$ and k_1 is small.

IV. Hydrodynamics

Let us return to the whole space problem of Section II; remarking that most of what follows can also be done for systems with boundaries. We will look at the behavior of the solution (11) for large t. If nothing more about the Boltzmann operator is known, one can only apply general results of ergodic theory [6], which say that the time average

$$\frac{1}{t} \int_0^t f(s) \, ds$$

or more sharply each Cesàro average

$$\frac{\alpha}{t^\alpha}\int_0^t (t-s)^{\alpha-1} f(s)\, ds \qquad \alpha > 0$$

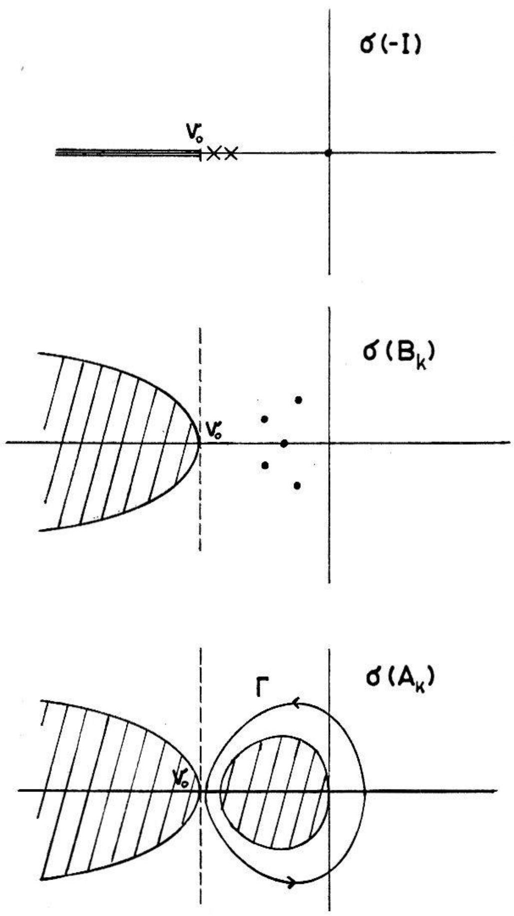
converges strongly for $t \to \infty$ to a stationary solution of the Boltzmann equation.

To get more detailed information, we suppose the interaction potential to be a hard potential in the terminology of GRAD [2], for instance a hard core potential in quantum mechanics, and also (7). This has the following consequences for the collision operator I: The spectrum of I in $L^2_{\varphi_0}$ consists of the five-fold point eigenvalue 0, which is separated from the rest of the spectrum, a continuum extending from a minimum ν_0 to ∞, and possibly further eigenvalues between 0 and ν_0. The collision frequency $\nu(v)$ is $\sim v$ for $v \to \infty$. The integral operator K in (4) is compact. In this situation the spectrum $\sigma(\hat{B}_k)$ of the operator $\hat{B}_k$

$$\hat{B}_k = -i(\boldsymbol{k}\,\boldsymbol{v}) - \nu(v) + K \qquad \boldsymbol{k} \text{ fixed} \tag{33}$$

in $L^2_{\varphi_0}(\boldsymbol{v})$ can be analysed by a method due to McLENNAN [4]. Since $\hat{B}_k$ is dissipative, the whole spectrum lies in the left half-plane. If $\boldsymbol{k}$ is not too large

$$\boldsymbol{k} \in S, \qquad S = \{\boldsymbol{k} \mid |\boldsymbol{k}| \leqslant \varkappa\} \tag{34}$$



Spectra of the various operators. The points in $\sigma(B_k)$ denote the hydrodynamic poles, the crosses in $\sigma(-I)$ are other possible point eigenvalues, which are suppressed in $\sigma(B_k)$ and $\sigma(A_\varkappa)$.

the point eigenvalues are perturbed analytically, in particular the perturbed eigenvalue 0, say $-i\,\omega_j(k)$ $(j = 1 \ldots 5)$, remains separated from the rest of the spectrum (see figure). The continuous spectrum extends to the region given by the values of the singular part $-i(\boldsymbol{k}\,\boldsymbol{v}) - \nu(v)$ in (33), because it is conserved under the compact perturbation K ([7], p. 244). We note that we can choose a constant $\gamma_\varkappa > 0$ such that the spectrum of $\hat{B}_k$ lies to the left of $-\gamma_\varkappa$ for all $|\boldsymbol{k}| \leqslant \varkappa$ apart from the $-i\,\omega_j(k)$. If I has no eigenvalues between 0 and ν_0 then $\gamma_\varkappa = \nu_0$ for all $\boldsymbol{k}$ of (34).

Now we turn to the Hilbert space $\hat{\mathcal{H}} = \mathrm{L}^2(\boldsymbol{k}) \otimes \mathrm{L}^2_{\varphi_0}(\boldsymbol{v})$ and introduce the projection operator $E_\varkappa$

$$E_\varkappa\, f(\boldsymbol{k}, \boldsymbol{v}) = \chi_\varkappa(\boldsymbol{k})\, f(\boldsymbol{k}, \boldsymbol{v}) = \begin{cases} f(\boldsymbol{k}, \boldsymbol{v}) & \text{if } |\boldsymbol{k}| \leqslant \varkappa \\ 0 & \text{otherwise.} \end{cases}$$

$E_\varkappa$ commutes with the Boltzmann operator $\hat{B}$ (6) in $\hat{\mathcal{H}}$, consequently the subspace

$$\hat{\mathcal{H}}_1 = E_\varkappa\, \hat{\mathcal{H}}$$

is invariant with respect to $\hat{T}^t$. From now we consider $\hat{T}^t$ in $\hat{\mathcal{H}}_1$ and T^t in $\mathcal{H}_1 = U^{-1}\,\hat{\mathcal{H}}_1$. This requires a restriction of the initial distribution f_0. Since $\varkappa$ must be supposed to be of the order of magnitude of the inverse mean free path, the restriction means that the initial distribution should vary slowly over a spatial distance of a mean free path, which is physically reasonable. The infinitesimal generator of $\hat{T}^t$ in $\hat{\mathcal{H}}_1$ has a spectrum which is a proper part of the spectrum of $\hat{B}$, namely the spectrum $\sigma(\hat{A}_\varkappa)$ of

$$\hat{A}_\varkappa = E_\varkappa\, \hat{B} = \hat{B}\, E_\varkappa = E_\varkappa\, \hat{B}\, E_\varkappa \quad \text{or of} \quad A_\varkappa = U^{-1}\,\hat{A}_\varkappa\, U\,.$$

The latter can be analysed by means of the following two lemmas.

Lemma 1

If λ is in the point spectrum $\sigma_P(\hat{B}_{k'})$ of $\hat{B}_{k'}$ for some $\boldsymbol{k}' \in S$, then $\lambda \in \sigma(A_\varkappa)$.

Proof. The spatial Fourier transformation

$$f(\boldsymbol{x}, \boldsymbol{v}) \rightsquigarrow \hat{f}_k(\boldsymbol{v}) = \frac{1}{(2\,\pi)^{3/2}} \int e^{-i k x}\, f(\boldsymbol{x}, \boldsymbol{v})\, d^3x = \frac{1}{(2\,\pi)^{3/2}}\, \langle e^{i k x}, f \rangle_x \in \mathrm{L}^2_{\varphi_0}(\boldsymbol{k})$$

defines an isometric mapping of $\mathcal{H}$ into a direct integral of Hilbert spaces $\mathrm{L}^2_{\varphi_0}$ [9]

$$\mathcal{H} \to \int \oplus\; \mathrm{L}^2_{\varphi_0}(\boldsymbol{k})\, d^3k\,. \tag{35}$$

Hereby the operators B and $A_\varkappa$ are transformed into

$$B \to \int \oplus\; \hat{B}_k\, d^3k \qquad A_\varkappa \to \int_S \oplus\; \hat{B}_k\, d^3k\,. \tag{36}$$

Since the embedding is isometric, we have

$$\|\lambda\, f - B\, f\|^2_{\mathcal{H}} = \int \|\lambda\, \hat{f}_k - \hat{B}_k\, \hat{f}_k\|^2_v\, d^3k\,.$$

Now let $\lambda \in \sigma_P(\hat{B}_{k'})$, $\boldsymbol{k}' \in S$ i.e. there exists $0 \neq \varphi_{k'} \in \mathcal{D}(\hat{B}_{k'})$ with

$$\lambda \varphi_{k'} - \hat{B}_{k'} \varphi_{k'} = 0 .$$

We must verify, that for any $\varepsilon > 0$ there exists an $f \in \mathcal{D}(A_\varkappa)$ such that

$$\| \lambda f - A_\varkappa f \|^2 \leqslant \varepsilon \| f \|^2 . \tag{37}$$

We choose a sphere $\Delta < S$ around $\boldsymbol{k}'$ and set

$$\psi(\boldsymbol{k}) = \begin{cases} 1 & \boldsymbol{k} \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for $f = U^{-1} \varphi_{k'}(\boldsymbol{v}) \psi(\boldsymbol{k})$

$$\begin{aligned} \| \lambda f - A_\varkappa f \|^2 &= \int_\Delta \| \lambda \varphi_{k'} - \hat{B}_{k'} \varphi_{k'} + i (\boldsymbol{k}' - \boldsymbol{k}, \boldsymbol{v}) \varphi_{k'} \|_v^2 \, d^3k \\ &\leqslant \int_\Delta d^3k \, | \boldsymbol{k}' - \boldsymbol{k} | \cdot \| v \varphi_{k'} \|_v^2 \\ &\leqslant \varepsilon^2(\Delta) \frac{\int d^3v \, \varphi_0 \, | v \varphi_{k'} |^2}{\int d^3v \, \varphi_0 \, | \varphi_{k'} |^2} \| \varphi_{k'} \psi \|_{\hat{\mathcal{H}}}^2 , \end{aligned} \tag{38}$$

where

$$\varepsilon^2(\Delta) = \frac{\int_\Delta d^3k \, | \boldsymbol{k} - \boldsymbol{k}' |^2}{\int_\Delta d^3k} \leqslant \max_{k \in \Delta} | \boldsymbol{k} - \boldsymbol{k}' |^2$$

can be made arbitrarily small for small Δ and the second factor in (38) is bounded independently of Δ because $\varphi_{k'} \in \mathcal{D}(\hat{B}_{k'})$. Therefore (37) is proved.

Lemma 2

If λ is in the resolvent set $\varrho(\hat{B}_k)$ of $\hat{B}_k$ for every $\boldsymbol{k} \in S$, then $\lambda \in \varrho(A_\varkappa)$.

Proof. From

$$\| \lambda \hat{f}_k - \hat{B}_k \hat{f}_k \|_v^2 \geqslant \delta \| \hat{f}_k \|^2 , \qquad \forall \; \boldsymbol{k} \in S , \quad \hat{f}_k \in \mathcal{D}(\hat{B}_k)$$

it follows with (36)

$$\| \lambda f - A_\varkappa f \|_{\mathcal{H}}^2 \geqslant \delta \int_S \| \hat{f}_k \|_v^2 \, d^3k = \delta \| f \|_{\mathcal{H}}^2 .$$

Hence the resolvent $R(\lambda; A_\varkappa) = (\lambda - A_\varkappa)^{-1}$ is a bounded operator on the range of $\lambda - A_\varkappa$. Since $A_\varkappa$ is J-self-adjoint on $\mathcal{H}_1$ (see Section II), its residual spectrum is empty and therefore $\lambda \in \varrho(A_\varkappa)$.

These two lemmas enable us to conclude the following about the spectrum of $A_\varkappa$ (see figure)

$$\bigcap_{k \in S} \varrho(B_k) \subseteq \varrho(A_\varkappa) , \qquad \bigcup_{k \in S} \sigma_P(B_k) \subseteq \sigma(A_\varkappa) . \tag{39}$$

It now follows from (39) and the choice of $\varkappa$ that the part

$$s = \bigcup_{k \in S} \sigma_0(B_k)$$

of $\sigma(A_\varkappa)$ arising from the disturbed eigenvalue 0 (the 'hydrodynamic part' of the spectrum) is separated from the rest by a regular region. It is a spectral set in the sense of DUNFORD [6]. We can form the contour integral

$$J = \frac{1}{2\pi i} \oint_\Gamma R(\lambda; A_\varkappa)\, d\lambda \tag{40}$$

around s and get a bounded indempotent operator J,

$$J^2 = J \,,$$

which commutes with $A_\varkappa$ [6]. J is a non-rectangular projection, which projects on the 'hydrodynamic' subspace. To define this we use the direct decomposition $\mathcal{H}_1 = \mathcal{H}_2 \oplus \mathcal{H}_3$: each $f \in \mathcal{H}_1$ can be uniquely written as

$$f = f_2 + f_3 \qquad f_2 = J\, f \in \mathcal{H}_2 \qquad f_3 = (1 - J)\, f \in \mathcal{H}_3 \,.$$

Let us introduce the operators

$$A_2 = J\, A_\varkappa = J\, A_\varkappa\, J$$
$$A_3 = (1 - J)\, A_\varkappa = (1 - J)\, A_\varkappa\, (1 - J) \,.$$

A_2 and A_3 both generate contraction semigroups T_2^t, T_3^t in $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively. Then the original semigroup T^t splits in $\mathcal{H}_1$ into a 'hydrodynamic' and a 'microscopic part':

$$T^t f = T_2^t f_2 + T_3^t f_3 = T^t f_2 + T^t f_3 \qquad f \in \mathcal{H}_1 \,.$$

Since the spectrum of A_3 lies to the left of $-\gamma_\varkappa < 0$, one may expect that the microscopic part decreases rapidly in time and at large times one is left with the hydrodynamic part alone. But this requires a closer investigation. It is exactly the hydrodynamic part, which is calculated by the Chapman-Enskog method.

Let us return to the direct integral (35). Applying this to the hydrodynamic subspace $\mathcal{H}_2 = J\, \mathcal{H}_1$ we have

$$J\, \mathcal{H}_1 \to \int_S \oplus\, P_k\, \mathrm{L}^2_{\varphi_0}(\boldsymbol{k})\, d^3k \,. \tag{41}$$

Here P_k is the projection onto the five dimensional space spanned by the eigenfunctions $g_j(\boldsymbol{k}, \boldsymbol{v})$ corresponding to the point eigenvalues $-i\,\omega_j(\boldsymbol{k})$, $j = 1 \dots 5$ of B_k. The $g_j(\boldsymbol{k}, \boldsymbol{v})$ are just the normal solutions of the Hilbert or Chapman-Enskog theory [4]. Now we consider the hydrodynamic part $T_2^t = J\, T^t\, J$ with respect to the representation (41). From (36) we see, that

$$A_2 = J\, A_\varkappa\, J \to \int_S \oplus\, P_k\, \hat{B}_k\, P_k\, d^3k \quad \text{and}$$

$$J\, T^t\, J \to \int_S \oplus\, \exp\,(t\, P_k\, \hat{B}_k\, P_k)\, d^3k \,. \tag{42}$$

Note that $P_k \hat{B} P_k$ is a bounded operator in a five dimensional space and therefore the exp is easily defined, for instance by the power series. The projection P_k can be expressed by means of the normal solutions

$$P_k = \sum_{j=1}^{5} (h_j(\boldsymbol{k}, \boldsymbol{v}) \, , \, \cdot)_v \, g_j(\boldsymbol{k}, \boldsymbol{v}) \, .$$

$h_j(\boldsymbol{k}, \boldsymbol{v})$ is the biorthogonal basis with respect to $g_j(\boldsymbol{k}, \boldsymbol{v})$ in $P_k \, \mathrm{L}^2_{\varphi_0}$ which is defined by

$$(g_j \, , \, h_e) = \delta_{j e} \, . \tag{43}$$

This must be used, because the g_j are not orthogonal for finite $\boldsymbol{k}$. Now every $f \in \mathcal{H}_2$, which for instance decreases rapidly for $|\boldsymbol{x}| \to \infty$ can be expanded in terms of the normal solutions

$$f = \frac{1}{(2\pi)^{3/2}} \sum_{j=1}^{5} \int f_j(\boldsymbol{k}) \, g_j(\boldsymbol{k}, \boldsymbol{v}) \, e^{i k x} \, d^3k \tag{44}$$

where

$$f_j(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \langle e^{i k x} \, h_j(\boldsymbol{k}, \boldsymbol{v}), f \rangle_{x,v} = \frac{1}{(2\pi)^{3/2}} \int e^{-i k x} \, \overline{h}_j(\boldsymbol{k}, \boldsymbol{v}) \, f(\boldsymbol{x}, \boldsymbol{v}) \, \varphi_0(v) \, d^3v \, d^3x \, . \tag{45}$$

Since, according to (42), the semigroup operates on g_j simply by multiplication with $e^{-i\omega_j t}$ we arrive at the following expansion formula for the hydrodynamic part

$$T^t f_0 = \frac{1}{2(\pi)^{3/2}} \sum_{j=1}^{5} \int f_j(\boldsymbol{k}) \, g_j(\boldsymbol{k}, \boldsymbol{v}) \, e^{i k x - i \omega_j(k) t} \, d^3k \quad f_0 \in \mathcal{H}_2 \, . \tag{46}$$

We see that a solution of this part is completely determined by the five quantities $f_j(\boldsymbol{k})$. These can be calculated from the five first moments. With $w_l = 1, m \, \boldsymbol{v}, 1/2 \, m \, v^2$ (m is the particle mass), it follows from (44) that

$$\frac{1}{(2\pi)^{3/2}} \langle e^{i k x} \, w_l, f_0 \rangle_{x,v} = \sum_{j=1}^{5} f_j(\boldsymbol{k}) \, (w_l, g_j(\boldsymbol{k}, \boldsymbol{v}))_v \qquad l = 1 \ldots 5 \, . \tag{47}$$

On the left-hand side are the Fourier-transformed hydrodynamic variables n, $n \, m \, \boldsymbol{w}$, $3/2 \, n \, k_B \, T + 1/2 \, n \, m \, w^2$, which then determine the solution (46). This property characterizes the Hilbert class of solutions of the Boltzmann equation. If $f(t)$ is determined by the hydrodynamic variables, then so are all higher moments, in particular the heat flow vector $\boldsymbol{j}_i$ and the stress tensor τ_{ij}. Inserting these into the conversation equations, one gets the hydrodynamic equations.

These conclusions are only possible, if the initial f_0 lies in the hydrodynamic subspace $\mathcal{H}_2$. If a general $f \in \mathcal{H}_1$ is given initially, then after an 'aging period', where the microscopic part is decreased, the solution is expected to be of the form (46). But the expansion coefficients $f_j(\boldsymbol{k})$ must now be calculated with (45) instead of (47). In (45) all higher moments and their spatial derivatives contribute for finite $\boldsymbol{k}$, not only the five hydrodynamic ones. If the $f_j(\boldsymbol{k})$ calculated from (45) are inserted in (47), one gets hydrodynamic variables belonging to a normal initial distribution $f_0 \in \mathcal{H}_2$, which gives the same hydrodynamic part as f. These hydrodynamic variables are the right initial values for the hydrodynamic equations, they correct the so-called initial layer.

Because the $g_j(\boldsymbol{k}, \boldsymbol{v})$ can be calculated by the Chapman-Enskog procedure or by perturbation theory, this problem can be solved explicitly. This shall be done in a subsequent paper.

V. Concluding Remarks

Our discussion of finite systems in Section III was complicated by the inhomogeneous boundary conditions. By this we mean conditions which vary along the boundaries in such a way that Theorem 3 does not directly apply, for example diffuse reflection with spatially varying temperature. The physical reason for this complication can be found in the occurence of a steady state. In fact, if the existence of a stationary solution is assumed, one can subtract it, and has then only to solve the Boltzmann equation with zero boundary conditions. This is immediately done by means of a contraction semigroup. Concerning the existence of the steady state f_∞ we have the following answer: Equation (29) shows that

$$- B\, f_\infty = \boldsymbol{v}\, \frac{\partial f_\infty}{\partial \boldsymbol{x}} + I\, f_\infty = h$$

must hold, where the domain $\mathcal{D}(B)$ is defined with homogeneous (time independent) boundary conditions (30). Then a steady state exists if 0 is not a point eigenvalue of B. This must be verified for individual cases, because it is not satisfied in all situations. A counterexample is given by the rigid rotation in a spherical symmetric domain.

A second remark is devoted to the consideration of an electromagnetic field in the Boltzmann equation. We start assuming a magnetic field $\boldsymbol{B}$ to be present, which may be space dependent but is time-independent. This causes a term

$$\boldsymbol{v} \times \boldsymbol{B}\, \frac{\partial}{\partial \boldsymbol{v}}\, (\varphi_0 + \varphi_0\, f) = \left(\boldsymbol{v} \times \boldsymbol{B}\, \frac{\partial f}{\partial \boldsymbol{v}}\right) \varphi_0\,,$$

because the contribution from the Maxwellian vanishes. The corresponding operator is skew-symmetric in our Hilbert space

$$\int \boldsymbol{v} \times \boldsymbol{B}\, \frac{\partial \bar{f}}{\partial \boldsymbol{v}}\, g\, \varphi_0\, d^3v = \int \sum_{j=1}^{3} (\boldsymbol{v} \times \boldsymbol{B})_j\, \frac{\partial \bar{f}}{\partial v_j}\, g\, \varphi_0\, d^3v$$

$$= - \int \sum_{j=1}^{3} \bar{f}\, \frac{\partial}{\partial v_j}\, [(\boldsymbol{v} \times \boldsymbol{B})_j\, g\, \varphi_0]\, d^3v = - \int \bar{f}\, \boldsymbol{v} \times \boldsymbol{B}\, \frac{\partial g}{\partial \boldsymbol{v}}\, \varphi_0\, d^3v$$

and therefore dissipative without any further assumptions. Hence our results remain unchanged in case of a magnetic field of arbitrary strength.

An electric field E on the other hand gives the following contribution

$$\boldsymbol{E}\, \frac{\partial}{\partial \boldsymbol{v}}\, (\varphi_0 + \varphi_0\, f) = - \boldsymbol{E}\, \boldsymbol{v}\, \varphi_0 + \boldsymbol{E}\, \frac{\partial}{\partial \boldsymbol{v}}\, (\varphi_0\, f)\,. \tag{48}$$

But here the second term is not dissipative. Furthermore the external regularity field of the operator $\boldsymbol{E}\; \partial/\partial \boldsymbol{v}$ is zero. This situation is similar to that of the inhomogeneous boundary conditions in Section III. Again it is possible to derive an existence theorem in the linear framework. If $\boldsymbol{E}$ is assumed to be small, only the first term on the right hand side of (48) has to be taken into account. This is simply an inhomogeneous term in the Boltzmann equation, and hence the solution takes the form (31).

An other direction of extending the method is the consideration of different collision operators in the Boltzmann equation. As we have found the existence theory can be established if the collision operator is dissipative and J-symmetric in an appropriate Hilbert space. Let us give two further examples of this kind. The first one is the simple collision operator for self-diffusion

$$(I f)(\boldsymbol{x}, \boldsymbol{v}) = \int (f(\boldsymbol{v}) - f(\boldsymbol{v}'))\, f_0(v_1)\, |\boldsymbol{v} - \boldsymbol{v}_1|\, \sigma(\vartheta, |\boldsymbol{v} - \boldsymbol{v}_1|)\, d^2\Omega\, d^3v_1$$

which occurs in neutron diffusion problems and in solid state physics describing scattering of electrons by static impurities. Here the same Hilbert space as above can be used. The second one is the more complicated collision operator for phonon scattering

$$\begin{aligned}(I f)(\boldsymbol{x}, \boldsymbol{k}, s) = \sum_{s', s''} \int \Big\{ & (f(\boldsymbol{k}, s) + f(\boldsymbol{k}', s') - f(\boldsymbol{k}'', s''))\,(N+1)\,(N'+1)\,N'' \\ & \cdot \omega(k)\,\omega(k')\,\omega(k'')\,\delta(\omega + \omega' - \omega'')\,p(\boldsymbol{k}, s; \boldsymbol{k}', s'; s'') \\ & + \frac{1}{2}\,(f(\boldsymbol{k}, s) - f(\boldsymbol{k}', s) - f(\boldsymbol{k}'', s))\,(N+1)\,N'\,N'' \\ & \cdot \omega(k)\,\omega(k')\,\omega(k'')\,\delta(\omega - \omega' - \omega'')\,p(\boldsymbol{k}, s; \boldsymbol{k}', s'; s'')\Big\}\, d^3k'\, d^3k''\end{aligned}$$

where

$$N(k) = \frac{1}{\exp(\hbar\,\omega(k)/k_B\,T) - 1}\,.$$

In this case the following scalar product must be used

$$(f, g) = \sum_S \int d^3x \int d^3k\, \bar{f}(\boldsymbol{x}, \boldsymbol{k}, s)\, g(\boldsymbol{x}, \boldsymbol{k}, s)\,.$$

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