# The baryon-meson vertex in SL(6, C)

Autor(en): **Gorgé, V. / Leutwyler, H.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **41 (1968)**

Heft 2

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-113880

# The Baryon-Meson Vertex in SL(6, C)[1]

by **V. Gorgé** and **H. Leutwyler**
Institut für theoretische Physik der Universität Bern, Bern (Switzerland)

(4. X. 67)

*Abstract.* The vertex describing the coupling of the baryon octet with the pseudoscalar and vector mesons is worked out under the assumption that the coupling is invariant with respect to the semidirect product of the Poincaré group with the group SL(6, C) as proposed by Budini and Fronsdal. The resulting integral representations for the form factors in the production channel $M \to B + \bar{B}$ are evaluated numerically.

## 1. Introduction

As is well known the static symmetry group SU(6) is applicable only to states at rest. Many authors have tried to reformulate the symmetry in such a fashion as to be valid for particles in motion as well [1]. If this restriction to a particular co-ordinate system, the rest frame, is dropped, relativistic invariance comes into play which interrelates the properties of particles moving with different velocities. Relativistic invariance means that the full symmetry group $G$ of the states of a system contains the Poincaré group as a subgroup. In this language, a relativistic generalization of SU(6) is a symmetry group $G$ that contains the Poincaré group and reduces to SU(6) for states at rest. Fortunately, Budini and Fronsdal [2] have been able to show that general requirements of physical interpretation allow only one relativistic extension of SU(6) in this sense. These authors show that $G$ must necessarily be the semi-direct product of the Poincaré group with a group $S$,

$$G = P \boxtimes S \tag{1.1}$$

where $S$ is normal subgroup. If furthermore $G$ is required to reduce to SU(6) for states at rest, the group $S$ must be isomorphic to SL(6, C). To specify the structure of $G$ more precisely, let us denote[2]) the generators of SL(6, C) by $S_r$, $r = 1, \ldots, 70$, in

[1]) Supported in part by the Swiss National Science Foundation.
[2]) Notation: Greek indices from the middle of the alphabet are used to denote components of Lorentz tensors; $\mu, \nu = 0, 1, 2, 3$. Greek indices from the beginning of the alphabet denote components of spinors; $\alpha, \beta = 1, 2$. Tensors under SU(3) are labelled with lower case latin indices $a, b = 1, 2, 3$ whereas we use capitals $A, B = 1, \ldots, 6$ for SU(6)-tensors. We make frequent use of the conventional correspondence $A \to (a, \alpha)$ that displays the reduction $SU(6) \supset SU(3) \otimes SU(2)$. We adopt the metric tensor $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and use the following antisymmetric tensors $\varepsilon^{\mu\nu\varrho\sigma}$, $\varepsilon^{0123} = -\varepsilon_{0123} = 1$; $\varepsilon^{abc}$, $\varepsilon^{123} = \varepsilon_{123} = 1$; $E^{\alpha\beta}$, $E^{12} = E_{12} = 1$. The generalized $2 \times 2$ Pauli matrices $\sigma_\mu$, $\tilde{\sigma}_\mu$ are defined by $\sigma_0 = \tilde{\sigma}_0 = 1$, $\sigma_i = -\tilde{\sigma}_i = \tau_i$, where $\tau_1$, $\tau_2$, $\tau_3$ are the conventional Pauli matrices associated with the rotation group. The matrices $\sigma_{\mu\nu}$ are defined by $\sigma_{\mu\nu} = 1/2\,\{\tilde{\sigma}_\mu\,\sigma_\nu - \tilde{\sigma}_\nu\,\sigma_\mu\}$.

particular those of the subgroup SL(2, C) $\subset$ SL(6, C) by $S_{\mu\nu}$ and let $(M_{\mu\nu}, P_\lambda)$ be the generators of the Poincaré group $P$. It is convenient to replace the operators of total angular momentum, $M_{\mu\nu}$ by those of the 'orbital angular momentum' $L_{\mu\nu}$ defined by

$$L_{\mu\nu} = M_{\mu\nu} - S_{\mu\nu} \tag{1.2}$$

and to consider the algebra generated by $(L_{\mu\nu}, P_\lambda, S_r)$ instead of $(M_{\mu\nu}, P_\lambda, S_r)$. Budini and Fronsdal show that the operators $(L_{\mu\nu}, P_\lambda)$ generate a group $P^*$ that is isomorphic to the Poincaré group. Furthermore these operators commute with the generators $S_r$ of SL(6, C). This implies that $G$ is isomorphic to the direct product of the Poincaré group with SL(6, C)

$$G = P^* \otimes \mathrm{SL}(6, C)\,. \tag{1.3}$$

This shows that the operator $P^\mu P_\mu$ is an invariant of $G$. All particles contained in an irreducible representation of $G$ therefore have the same mass, in accordance with a well-known theorem due to O'Raifeartaigh [3].

Since $G$ is to be applied to physical states we must look for unitary representations of this group. Unitary irreducible representations (UIR) of $G$ are direct products of UIR of $P^*$ and SL(6, C).

## 2. Representations of the Unphysical Poincaré Group P*

To find the appropriate UIR of the factor $P^*$ we note that the operators $S_{ik}$ $(i, k = 1, 2, 3)$ are generators of the subgroup SU(6) $\subset$ SL(6, C) which is identified with the static symmetry group for states at rest. The physical interpretation of these operators is therefore known: they represent the operators of spin angular momentum. On the other hand, the operators of total angular momentum are given by the generators $M_{ik}$ of the physical Poincaré group. Clearly, for one particle states at rest these quantities coincide with the operators $S_{ik}$, i.e. we must have

$$L_{ik}\,\psi = 0$$

if $\psi$ represents a particle at rest. Therefore, as far as one particle representations of the group $G$ are concerned, only spin zero representations of the factor $P^*$ are relevant.

The spin zero representations of $P^*$ can be defined in the conventional way on a space of functions $\phi(p)$ on the hyperboloid $p^2 = m^2$, $p^0 > 0$ by

$$u(\Lambda, a)\,\phi(p) = e^{i a \Lambda^{-1} p}\,\phi(\Lambda^{-1} p) \tag{2.1}$$

where $\Lambda$ denotes an element of the unphysical homogeneous Lorentz group contained in $P^*$ and $a$ is a translation. The above representation is unitary with respect to the inner product

$$(\phi, \psi) = \int d\Omega(p)\,\phi^*(p)\,\psi(p)\,; \quad d\Omega(p) = d^3p\,\frac{m}{p^0}\,. \tag{2.2}$$

## 3. Representations of SL(6, C)

Next let us consider the UIR of the second factor, SL(6, C). The UIR of this group have been determined by Gelfand and Naimark [4]. Except for the trivial representation they are all infinite dimensional and consist of an infinite ladder of irreducible representations of the maximal compact subgroup SU(6). The physical content of the corresponding UIR of the group $G$ is obtained most directly by considering those states which describe particles at rest, $\phi(p) = \delta^3(\mathbf{p})$. Clearly these states furnish an irreducible representation of SL(6, C) and we conclude that the theory contains infinitely many different physical particles which, in the exact symmetry limit, all have the same mass. The occurence of such infinite series of particles is the price one pays for the fact that the Budini-Fronsdal-theory, unlike its predecessors, is compatible with unitarity. Loosely speaking the older theories also contained production of an infinite series of additional multiplets, but since these states were not interpreted as physical particles, the conservation of probability was violated.

*Baryons*

In order to determine the appropriate representation of SL(6, C) that describes the baryonic states we recall that in the framework of the static symmetry group the lowest lying baryonic levels form a 56-dimensional irreducible representation of SU(6). In the Budini-Fronsdal-theory this particular multiplet is only one of an infinite ladder of SU(6) multiplets contained in a representation of SL(6, C) and there are in fact a multitude of representations of the group SL(6, C) that contain the multiplet *56* in the reduction SL(6, C) ⊃ SU(6). We shall in the following consider only the simplest case which is the only one discussed in the literature [2]. In the terminology of GELFAND and NAIMARK we restrict ourselves to the most degenerate representations of the principal series, characterized by the partition (5, 1).

In this work we shall make use of the reformulation of the theory of GELFAND and NAIMARK proposed in a preceding paper [5] referred to as *I*. This reformulation is based on an analysis of the representations found by GELFAND and NAIMARK in terms of homogeneous functions. The particular class of representations belonging to the partition (5, 1) is defined on a space of homogeneous functions $F(x)$ of a single contravariant vector $x$ with six complex components $x_A$, $A = 1, \dots, 6$. In this space of functions the representation $u(S)$ is defined by

$$u(S)\, F(x) = F(x\, S); \quad (x\, S)_A = x_B\, S^B_A\,. \tag{3.1}$$

The function $F(x)$ is homogeneous

$$F(\lambda\, x) = \lambda^{\alpha^+}\, \lambda^{*\,\alpha^-}\, F(x) \tag{3.2}$$

and the degrees of homogeneity $\alpha^+$, $\alpha^-$ may be used to characterize the representation in question. They are related to the invariants $m$ and $\varrho$ introduced by GELFAND and NAIMARK through[3])

$$\alpha^\pm = \mp \frac{m}{2} + i\,\frac{\varrho}{2} - 3\,. \tag{3.3}$$

The representation defined by (3.1) and (3.2) is unitary in the inner product

$$(F, G) = \int d\mu(x)\; F^*(x)\; G(x); \quad d\mu(x) = \delta\,(x_6 - 1)\; dx \tag{3.4}$$

[3]) We adhere to the normalization $m_1 = \varrho_1 = 0$ and put $m_2 = m$, $\varrho_2 = \varrho$.

where $dx$ denotes the product of the differentials of real and imaginary parts of the 6 complex components of the vector $x$. The representation $u(S)$ defined above contains the multiplet *56* of SU(6) if and only if $m = -3$, i.e.

$$\alpha^+ = -\frac{3}{2} + i\,\frac{\varrho}{2}\,, \quad \alpha^- = -\frac{9}{2} + i\,\frac{\varrho}{2}\,. \tag{3.5}$$

The subspace that transforms according to the representation *56* of SU(6) is spanned by the vectors

$$F_{ABC}(x) = x_A\, x_B\, x_C (x\, x^+)^{-9/2 + i\varrho/2} \tag{3.6}$$

We thus have a one-parameter family of suitable representations labelled by the real parameter $\varrho$. These representations contain the multiplets *56*, *700*,... of SU(6).

*Antibaryons*

As usual the antiparticles are associated with the complex conjugate representation. In particular the states belonging to the representation *56** of SU(6) are described by the homogeneous functions $F_{ABC}(x)^*$. It turns out however that a slightly different treatment [6] of the antiparticle states is more convenient. According to a theorem[4]) by GELFAND and NAIMARK the representations belonging to the partitions (5, 1) and (1, 5) are pairwise equivalent. In particular the complex conjugate of the baryon representation is equivalent to a representation belonging to the partition (1, 5) and we shall use this representation for the antibaryons. According to *I* the partition (1, 5) corresponds to homogeneous functions $F(y)$ of a single contravariant vector $y^A$ and the transformation rule reads

$$u(S)\, F(y) = F(S^{-1}\, y)\,; \quad (S^{-1}\, y)^A = S^{-1A}{}_B y^B\,. \tag{3.7}$$

In terms of this space of homogeneous functions the representatives of the antibaryon multiplet *56** are given by

$$G^{ABC}(y) = y^A\, y^B\, y^C (y^+\, y)^{-9/2 + i\varrho/2}\,. \tag{3.8}$$

This representation is unitary in the inner product

$$(F, G) = \int d\mu(y)\; F^*(y)\; G(y)\,; \quad d\mu(y) = \delta\,(y^1 - 1)\; dy\,. \tag{3.9}$$

Clearly we could also describe the baryons by means of homogeneous functions of the type $F(y)$. In this case the multiplet *56* is represented by the functions $G^{ABC}(y)^*$.

*Mesons*

We now turn to the representation that describes the mesonic states. The simplest representations that contain the multiplet *35* of SU(6) are again those belonging to the most degenerate partition (5, 1). Unfortunately, if the mesons are assigned to one of these representations, one arrives at the unrealistic conclusion that the baryon-meson coupling constants vanish. This is because these representations do not allow an invariant coupling to the direct product of the baryon and the antibaryon representation. The next simplest choice is the partition (1, 4, 1). It has been shown in *I* that this

[4]) The relevant theorem is cited in *I*.

partition corresponds to homogeneous functions $F(\xi, \eta)$ of a pair of vectors $\xi_A$, $\eta^B$ instead of functions of a single vector as in the case of the more degenerate partition (5, 1). The representation $u(S)$ is in this case defined by

$$u(S)\ F(\xi, \eta) = F(\xi\ S,\ S^{-1}\ \eta)\,. \tag{3.10}$$

Note that according to *I* the vectors $\xi$ and $\eta$ are orthogonal $\xi\,\eta = \xi_A\,\eta^A = 0$. One verifies that there are two different types of representations belonging to this partition that contain the multiplet *35*. The first one consists of a ladder of SU(6) multiplets starting with *35*, *280*, .... These representations have been proposed for the description of mesons by RÜHL [6]. The second type consists of a ladder starting with *1*, *35*, *35*,... and has been investigated in detail by FRONSDAL [2]. We shall here restrict ourselves to the first type of representations [7].

The 35 states that transform irreducibly under SU(6) are in this case represented by the functions

$$H^B_A(\xi, \eta) = \xi_A\ \eta^B (\xi\ \xi^+)^{-3+i\sigma/2}\ (\eta^+\ \eta)^{-3+i\tau/2}\,. \tag{3.11}$$

The invariants $\sigma$ and $\tau$ are again arbitrary real parameters that specify the particular representation under consideration. The representations introduced above are unitary with respect to the inner product

$$(F, G) = \int d\mu(\xi, \eta)\ F^*(\xi, \eta)\ G(\xi, \eta)$$
$$d\mu(\xi, \eta) = \delta\ (\xi_6 - 1)\ \delta\ (\eta^1 - 1)\ \delta(\xi\ \eta)\ d\xi\ d\eta\,. \tag{3.12}$$

## 4. Particles in Motion

In the last section we have specified the representations of the group $G$ relevant for the description of baryons and mesons. In order to find the states that describe physical particles of momentum $q$ we have to determine the subspaces transforming irreducibly under the physical Poincaré group $P$. Let us first consider the baryons. The particles belonging to the multiplet *56* of SU(6) at rest are described by the wave functions

$$F(p, x) = \delta(\boldsymbol{p})\ F_{ABC}(x)\ \chi^{ABC}\,. \tag{4.1}$$

Here $\chi^{ABC}$ denotes the SU(6) spinor of the particular state in question. The corresponding state with the same internal quantum numbers but with momentum $q$ is obtained in the conventional fashion by means of a pure Lorentz transformation

$$F(q; p, x) = U(\Lambda)\ F(p, x)\,. \tag{4.2}$$

We now decompose the physical Lorentz transformation $\Lambda$ into the direct product of an element of the unphysical Poincaré group $P^*$ and an element $S \in$ SL(6, C) and write

$$U(\Lambda) = u(\Lambda) \otimes u(S)\,. \tag{4.3}$$

Note that for infinitesimal transformations this decomposition simply reduces to $M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$. The operator $u(\Lambda)$ acts exclusively on the first factor in (4.1)

$$u(\Lambda)\ \delta(\boldsymbol{p}) = \frac{p^0}{M}\ \delta\ (\boldsymbol{p} - \boldsymbol{q}) \tag{4.4}$$

where $M$ denotes the baryon mass, whereas $u(S)$ acts on the second factor

$$u(S)\, F_{ABC}(x) = F_{ABC}(x\, S)\,. \tag{4.5}$$

Since $S$ is either one of the two elements of the subgroup SL(2, C) $\subset$ SL(6, C) that cover the homogeneous Lorentz transformation $\Lambda$ we have[2])

$$S\, S^{+} = \frac{\tilde{q}}{M}\,; \quad \tilde{q} = q^{\mu}\, \tilde{\sigma}_{\mu}\,. \tag{4.6}$$

Furthermore we note that the product $F(x\, S)\, \chi$ involves the BARGMANN-WIGNER spinors [8]

$$\chi^{ABC}(q) = S^{A}{}_{D} S^{B}{}_{E} S^{C}{}_{F} \chi^{DEF} \tag{4.7}$$

associated with particles in motion. We thus have

$$\begin{aligned} F(q; p, x) &= p^0\, \delta\, (\boldsymbol{p} - \boldsymbol{q})\, M^{7/2 - i\varrho/2}\, F_{ABC}(x; q)\, \chi^{ABC}(q) \\ F_{ABC}(x; q) &= x_A\, x_B\, x_C (x\, \tilde{q}\, x^{+})^{-9/2 + i\varrho/2}\,. \end{aligned} \tag{4.8}$$

For later use we note the form of the BARGMANN-WIGNER spinors for the states belonging to the spin 1/2 octet [9]

$$\chi^{ABC}(q) = \frac{1}{3\sqrt{2}} \{\varepsilon^{abd}\, B^c_d\, \varepsilon^{\alpha\beta}\, \chi^{\gamma}(q) + \varepsilon^{bcd}\, B^a_d\, \varepsilon^{\beta\gamma}\, \chi^{\alpha}(q) + \varepsilon^{cad}\, B^b_d\, \varepsilon^{\gamma\alpha}\, \chi^{\beta}(q)\}\,. \tag{4.9}$$

Note that the wave function $\chi^{\alpha}(q)$ coincides with the conventional two-component spinor for particles of spin 1/2 in motion. The matrices $B^a_b$ represent the traceless SU(3) tensors for the baryon octet.

Entirely analogous expressions are valid for the antibaryons. For the mesons one finds[5])

$$\begin{aligned} H(q; p, \xi, \eta) &= p^0\, \delta\, (\boldsymbol{p} - \boldsymbol{q})\, m^{5 - i/2(\sigma + \tau)}\, H^B_A(\xi, \eta; q)\, \chi^A_B(q) \\ H^B_A(\xi, \eta; q) &= \xi_A\, \eta^B (\xi\, \tilde{q}\, \xi^{+})^{-3 + i\sigma/2}\, (\eta^{+}\, q\, \eta)^{-3 + i\tau/2}\,. \end{aligned} \tag{4.10}$$

In the case of the pseudoscalar meson octet the BARGMANN-WIGNER spinor happens to be independent of momentum

$$\chi^A_B(q) = P^a_b\, \delta^{\alpha}_{\beta} \tag{4.11}$$

where $P^a_b$ is the traceless SU(3) tensor associated with the pseudoscalar octet, whereas for the vector mesons we have

$$\chi^A_B(q) = V^a_b\, \varepsilon^{\mu}\, \sigma_{\mu\nu}{}^{\alpha}_{\beta}\, q^{\nu}/m\,. \tag{4.12}$$

The vector $\varepsilon^{\mu}$ characterizes the polarization of the meson and the SU(3) tensor $V^a_b$ represents the nonet in the conventional fashion.

## 5. General Remarks on Applications of the Budini-Fronsdal Theory

The first applications of the BUDINI-FRONSDAL theory concerned baryonic matrix elements of the electromagnetic current [10]. The electromagnetic form factors of the baryon octet have been worked out under the assumption that the current transforms

[5]) The matrix $q$ stands for $q^{\mu}\, \sigma_{\mu}$.

like a finite dimensional tensor operator under SL(6, C). Clearly this problem involves the coupling of only two infinite dimensional representations of the group $G$.

There is, however, a simple alternative to this assumption. The fact that the electromagnetic form factors seem to be dominated by the contributions from the vector meson poles suggests that we identify the current with certain components of a tensor operator which transforms like the infinite dimensional meson representation. This assumption requires the evaluation of the baryon-meson vertex and involves the coupling of three infinite dimensional representations of $G$. This vertex has been investigated by RÜHL [6]. Furthermore, the behaviour of the baryon-meson vertex at the thresholp of the decay channel has been analysed by FRONSDAL and WHITE [11].

In this connection the following fundamental difficulty arises. The symmetry group $G$ is in principle defined for on mass shell quantities only. It is applicable e.g. to a process like $N^* \to N + \pi$ with all momenta on the mass shell. The symmetry relates this process to processes like $N \to N + \pi$ which clearly take place only off mass shell. This difficulty of course reflects the fact that the symmetry is broken. The same problem arises for any broken symmetry e.g. it arises for SU(3). In that case the difficulty can however be solved at least in part[6]) by introducing off-shell amplitudes and by comparing the various amplitudes at the same kinematical configurations. In the case of the group $G$ the situation seems to be considerably worse. This is due to the fact that the BUDINI-FRONSDAL theory cannot be reformulated as a theory of local fields, subject to the conventional connection between spin and statistics [12]. This impossibility in turn is related to the fact that the theory involves particles of arbitrarily high spin and that it furthermore involves nontrivial form factors, i.e. nonlocal interactions ab initio. A local field theory would guarantee the conventional crossing relations that are of basic importance whenever analytic continuation is involved. In the absence of such a framework the simple recipe that works in the case of SU(3) must be expected to lead to inconsistencies.

We shall in the following ignore these problems and work out the baryon meson vertex in the channel $M \to B + \overline{B}$. We assume that the baryons and antibaryons all have the same mass $M$ and that the virtual meson that decays into these particles possesses the momentum $P = p_1 + p_2$ where $p_1$ and $p_2$ denote the momenta of $B$ and $\overline{B}$ respectively. The 'mass' of the virtual meson is given by $m = \sqrt{s}$ where $s$ denotes the square of the C.M. energy of the final state.

To conclude this section we mention that the predictions of the BUDINI-FRONSDAL theory for reactions that involve the invariant coupling of four infinite dimensional representations of SL(6, C) have been analysed by the Dubna group for the particular case of $B$–$\overline{B}$ annihilation at rest [13].

### 6. The Baryon-Meson-Vertex

In $I$ we have developped a technique for the analysis of degenerate representations of SL($n$, C) which allows us to carry out the necessary calculations for the coupling of the three infinite dimensional representations of $G$ involved in this problem to a point where numerical evaluation becomes possible. We mention that BAMBERG [7]

[6]) This recipe in general of course violates unitarity.

has proposed a quite different elegant method for the coupling of three representations of SL($n$, C) and is currently investigating the baryon-meson vertex for the case of the meson representation suggested by FRONSDAL.

We consider the baryon-meson vertex in the channel $M \to B + \bar{B}$. The virtual meson of momentum $P$ in the initial state is represented by the wave function $H(P; P', \xi, \eta)$ defined in Section 4. On the other hand the particles in the final state must be transformed according to the complex conjugate representation. As far as the symmetry group $G$ is concerned a baryon in the final state behaves exactly like an antibaryon in the initial state and we found it convenient to use the antibaryon wave function $G(p_1; p_1', y)$ for the description of the final state baryon of momentum $p_1$. Likewise the final state antibaryon is represented by $F(p_2; p_2', x)$. The vertex then takes the form

$$V(p_1, p_2 \mid P) = \int d\Omega(p)_1' \, d\Omega(p_2') \, d\Omega(P) \int d\mu(x) \, d\mu(y) \, d\mu(\xi, \eta) \, F(p_2; p_2', x) \times G(p_1; p_1', y) \, H(P; P', \xi, \eta) \, k(p_1', p_2' \mid P') \, K(x, y \mid \xi, \eta) \,. \tag{6.1}$$

The kernels $k$ and $K$ are the invariant coupling coefficients for the representations of the unphysical Poincaré group $P^*$ and for the group SL(6, C) respectively.

Consider first the kernel $k$. Invariance under $P^*$ implies that $k$ is of the form

$$k(p_1', p_2' \mid P') = \delta\,(p_1' + p_2' - P') \, F(s) \tag{6.2}$$

where we have indicated explicitly that the invariant multiplying the deltafunction need not be the same for all values of the virtual meson mass $m = \sqrt{s}$.

The SL(6, C)-kernel $K$ has been constructed by RÜHL [6]. It has been shown in $I$ that the construction of kernels of this type is a very simple matter if use is made of homogeneous variables. In the present case, $K$ must be a homogeneous function of the same degree as the product $F^* G^* H^*$, i.e.

$$K(l\,x, m\,y \mid \lambda\,\xi, \mu\,\eta) = l^{-9/2 - i\varrho/2} \, l^{*-3/2 - i\varrho/2} \, m^{-9/2 - i\varrho/2} \, m^{*-3/2 - i\varrho/2} \times \lambda^{-3 - i\sigma/2} \, \lambda^{*-2 - i\sigma/2} \, \mu^{-3 - i\tau/2} \, \mu^{*-2 - i\tau/2} \, K(x, y \mid \xi, \eta) \,. \tag{6.3}$$

Furthermore $K$ must be invariant under SL(6, C)

$$K(x\,S, S^{-1}\,y \mid \xi\,S, S^{-1}\,\eta) = K(x, y \mid \xi, \eta) \,. \tag{6.4}$$

The vectors $x_A$, $y^B$, $\xi_C$, $\eta^D$ possess the eight algebraically independent invariants $r, s, t, u, r^*, s^*, t^*, u^*$, where

$$r = x\,y \,, \quad s = x\,\eta \,, \quad t = y\,\xi \,, \quad u = \xi\,\eta \,. \tag{6.5}$$

The symbol $x\,y$ stands for the scalar product $x_A\,y^A$ etc. We note that since $\xi$ and $\eta$ are orthogonal the invariant $u$ vanishes. We now solve the homogeneity condition (6.3) by means of the ansatz

$$K(x, y \mid \xi, \eta) = r^{a+} \, s^{b+} \, t^{c+} \, r^{*a-} \, s^{*b-} \, t^{*c-} \,. \tag{6.6}$$

The exponents are to be determined in such a fashion that the kernel possesses the required degree of homogeneity. One finds that there does not exist a consistent solution, i.e. that there does not exist an invariant coupling of the three representations[7]), unless the parameters characterizing the meson representation satisfy $\sigma = \tau$ in which case the solution reads

$$\begin{aligned} a^+ &= -\frac{3}{2} - \frac{i}{2}(\varrho - \sigma)\,; \quad b^+ = c^+ = -3 - \frac{i}{2}\sigma \\ a^- &= \frac{1}{2} - \frac{i}{2}(\varrho - \sigma)\,; \quad b^- = c^- = -2 - \frac{i}{2}\sigma\,. \end{aligned} \tag{6.7}$$

We now insert the expression for the kernel $k$ in the vertex and carry out the trivial momentum integrations with the result

$$\begin{aligned} V(p_1, p_2 \mid P) &= \delta(p_1 + p_2 - P) \\ &\times F(s)\, M^{9-i\varrho}\, m^{6-i/2(\sigma+\tau)}\, J^{ABC\,G}_{DEF\,H}\, \chi_{ABC}(p_1)\, \overline{\chi}^{DEF}(p_2)\, \chi^H_G(P) \end{aligned} \tag{6.8}$$

where the SL(6, C) coupling coefficients are given by the integral

$$J^{ABC\,G}_{DEF\,H} = \int d\mu(x)\, d\mu(y)\, d\mu(\xi, \eta)\, F_{DEF}(x, p_2)\, G^{ABC}(y, p_1)\, H^G_H(\xi, \eta; P)\, K(x, y \mid \xi, \eta)\,. \tag{6.9}$$

The kernel $K$ is displayed explicitly in (6.5), (6.6) and (6.7).

## 7. Parity and Crossing

Before we start with the evaluation of the expression (6.9) for the coupling coefficients we briefly discuss the extension of the symmetry group $G$ to a group $\hat{G}$ that includes space reflections. The action of space reflection on the generators of the Poincaré group is well-known. Furthermore, the physical interpretation of the generators of SL(6, C) implies [15] that space reflection transforms the element $S \in$ SL(6, C) into the element $S^{+-1}$. This is easily checked for the subgroups SL(2, C) and SU(3); the group structure then determines a unique extension to the full group SL(6, C). If parity is to be a symmetry operation we must require that the unitary representation used for the description, say of the baryons, is not only a representation of the group $G$, but actually a representation of the full group $\hat{G}$. In particular we must have a unitary operator $P$ that obeys the usual multiplication rules with the generators of the Poincaré group and furthermore satisfies

$$P\, u(S)\, P^{-1} = u(S^{+-1})\,. \tag{7.1}$$

An operator $P$ with this property exists if and only if the representations $u(S)$ and $u(S^{+-1})$ are equivalent. In the case of the baryon representation they are inequivalent unless the parameter $\varrho$, which we have left unspecified, vanishes [15]. Similarly in the case of the mesons: parity maps the representation onto itself if and only if $\sigma = \tau = 0$ [15].

Let us now suppose that we have chosen $\varrho = \sigma = \tau = 0$. The existence of a parity operator for these representations has the following immediate consequence. We have

[7]) In order to analyze the solutions of (6.3) and (6.4) in a more systematic manner one makes use of the so-called transitive domains. The form (6.6) for $K$ corresponds to the transitive domain of highest dimension.

noted that the coupling of the baryon, antibaryon and meson representations as representations of the group SL(6, C) is unique up to a constant. Since the parity operator maps these representations onto themselves, the parity transformed vertex must coincide with the vertex itself up to a constant, which, as an eigenvalue of $P$ must have the value $+1$ or $-1$. Therefore the vertex is either invariant under parity or changes sign; furthermore the group determines whether or not a sign change occurs. Explicit evaluation of the expression (6.9) will show that the eigenvalue is $+1$, provided the phase of the parity operator is chosen in the conventional way such that the lowest lying multiplets have the intrinsic parities $35^-$, $56^+$, $\overline{56}^-$, the multiplets denoting mesons, baryons and antibaryons respectively. The coupling $M \to B + \bar{B}$ is thus automatically invariant under space reflection.

This points to a serious difficulty related to the fact that the theory does not satisfy crossing. Consider the crossed channel, $B + M \to B$. The transformation properties of the corresponding matrix element under SL(6, C) are identical to those of the process $M \to B + \bar{B}$ considered previously. The vertex will be the same apart from the difference in the kinematical configuration. However, since we have replaced an antibaryon by a baryon, the product of the intrinsic parities changes sign. Therefore, the vertex describing the coupling $B + M \to B$ is not invariant under the parity operation, but changes sign, i.e. the process is forbidden if symmetry under the full group $\hat{G}$ is required. This is a particularly drastic example for the fact that the Budini-Fronsdal theory does not satisfy crossing: the analytic continuation of an invariant coupling to the crossed channel does not produce results that are invariant under the symmetry group in the crossed channel. For a detailed discussion of this point we refer the reader to the extensive literature on this subject [12].

Concerning the specific properties of the baryon meson vertex there is an obvious way out of the conclusion that the amplitude $B + M \to B$ vanishes. This difficulty arises only because we have chosen representations that are selfconjugate under the parity operation, $\varrho = \sigma = \tau = 0$. We can alternatively stay with nonzero values of these parameters, at the cost of doubling the representation spaces in the usual way, replacing the representation $u(S)$ by the direct sum $u(S) \oplus u(S^{+-1})$. In this representation each particle of positive intrinsic parity has its negative parity counterpart. If this alternative is adopted the symmetry group $\hat{G}$ allows for invariant couplings in both direct and crossed channels. However, the more general fact that these couplings are not related by analytic continuation remains unaffected.

We shall show in the next sections that, if necessary, the evaluation of the vertex can be carried out even in the case of arbitrary values of the parameters $\varrho$, $\sigma$ and $\tau$. Nevertheless, towards the end of the calculation we shall put $\varrho = \sigma = \tau = 0$ to simplify the final steps.

## 8. Evaluation of the Coupling Coefficients

It is the evaluation of the integral $J^{ABC\,G}_{DEF\,H}$ which extends over the four complex vectors $x_A$, $y^B$, $\xi_C$, $\eta^D$ that represents the core of our task, and it is here that the method based on homogeneous functions becomes effective. We have shown in $I$ that integrals of the type (6.9) can be written in a covariant fashion. A slight extension of the method described in $I$ and illustrated by means of an analogous coupling problem in the framework of SL(2, C) shows that the integral (6.9) can be rewritten as

$$J^{ABC\,G}_{DEF\,H} = N \int dx\, dy\, d\xi\, d\eta\, \delta(\xi\,\eta)\, x_D\, x_E\, x_F\, y^A\, y^B\, y^C\, \xi_H\, \eta^G \times e^{-x \tilde{p}_2 x^+ - y^+ p_1 y - \xi \tilde{P} \xi^+ - \eta^+ P \eta}\, K(x, y \mid \xi, \eta) \tag{8.1}$$

where $N$ is some normalization constant.

We treat this integral as follows. To get rid of the polynomial $x_D\, x_E\, x_F\, y^A\, y^B\, y^C\, \xi_H\, \eta^G$ we consider instead of $J^{ABC\,G}_{DEF\,H}$ the integral $J(A, B, C)$ defined by

$$J(A, B, C) = \int dx\, dy\, d\xi\, d\eta\, \delta(\xi\,\eta)\, e^{-x \tilde{p}_2 x^+ - y^+ p_1 y - \xi \tilde{P} \xi^+ - \eta^+ P \eta} \times [(x\,A\,y)\,(y^+\,A^+\,x^+)]^{a^-}\, [(x\,B\,\eta)\,(\eta^+\,B^+\,x^+)]^{b^-}\, [(\xi\,C\,y)\,(y^+\,C^+\,\xi^+)]^{c^-} \tag{8.2}$$

where we have introduced the auxiliary matrices $A$, $B$, $C$ in such a fashion that the coupling coefficients can be expressed as

$$J^{ABC\,G}_{DEF\,H} = N' \frac{\delta}{\delta A^D_A} \frac{\delta}{\delta A^E_B} \frac{\delta}{\delta B^F_G} \frac{\delta}{\delta C^H_C}\, J(A, B, C) \Big|_{A=B=C=1}. \tag{8.3}$$

*Integration over the Meson Variables*

The subintegration over the meson variables $\xi$ and $\eta$ in the expression for $J(A, B, C)$ concerns an integral of the form

$$J_1 = \int d\xi\, d\eta\, \delta(\xi\,\eta)\, e^{-\xi \tilde{P} \xi^+ - \eta^+ P \eta}\, [(u\,\eta)\,(\eta^+\,u^+)]^b\, [(\xi\,v)\,(v^+\,\xi^+)]^b \tag{8.4}$$

$$u_A = x_C\, B^C_A\,; \qquad v^A = C^A_B\, y^B$$

where we have introduced the vectors $u$, $v$ instead of $x\,B$ and $C\,y$ respectively. Furthermore, to simplify the notation we use the symbols $a$ and $b$ instead of $a^-$, $b^-$ or $c^-$

$$a \equiv a^-; \qquad b \equiv b^- = c^- = -2 - \frac{i}{2}\,\sigma\,. \tag{8.5}$$

As observed already by RÜHL [6] the integrand is singular at the points $u\,\eta = 0$, $\xi\,v = 0$ and the integral is not by itself a well-defined expression. We shall in the following define the integral by means of analytic continuation in the exponent $b$, a method used extensively by FRONSDAL [2]. The problems connected with the singularities of the kernels and the closely related question of whether or not the coupling is unique, clearly deserve a more sophisticated mathematical investigation, which is however outside the scope of this paper. In the following we suppose first that the exponent $b$ lies in a region where the integral is well defined; at the end of the calculation we shall continue the result analytically in $b$ to the particular value of interest here.

The integral $J_1$ can be determined by simply making use of the fact that it is an invariant homogeneous function of the matrix $P$ and the vectors $u_A$ and $v^A$. The matrix $P$ and its inverse $P^{-1} = m^{-2}\,\tilde{P}$ play the role of a metric in the space of the vectors $u$ and $v$. The integral $J_1$ must be a function of the invariants that can be constructed from these quantities, i.e. $J_1 = J_1\,(m, u\,P^{-1}\,u^+, v^+\,P\,v, u\,v, v^+\,u^+)$, where $m$ replaces the invariant $\det P = m^6$. Furthermore $J_1$ is a homogeneous function of its arguments and can be written in the form

$$J_1 = (u\,P^{-1}\,u^+)^b\,(v^+\,P\,v)^b\,m^{-10-2b}\,f(z) \qquad z = (u\,v)\,(v^+\,u^+)\,(u\,P^{-1}\,u^+)^{-1}\,(v^+\,P\,v)^{-1}. \tag{8.6}$$

This leaves only the function $f(z)$ unspecified. To determine this function we note that $J_1$ satisfies the differential equation

$$\frac{\partial}{\partial u_A} \frac{\partial}{\partial v^A} J_1 = 0 \tag{8.7}$$

which reflects the orthogonality of $\xi$ and $\eta$. Inserting the expression (8.6) for $J_1$ we obtain a differential equation for $f(z)$:

$$z(z-1) f'' + [(2b+4) + (1-2b) z] f' + b^2 f = 0 . \tag{8.8}$$

Note that $z$ varies in the range $0 \leqslant z \leqslant 1$; the end points of this interval are ordinary singular points of the differential equation. The behaviour of the two independent solutions at the singular point $z = 1$ is independent of the value of the exponent $b$: the regular solution behaves like $(z-1)^0$, the singular one like $(z-1)^{-4}$. Since the integral is well-defined at $z = 1$ for a suitable range of the exponent $b$ we must disregard the irregular solution. This determines the function $f(z)$ up to a constant. The value of this constant is irrelevant. We note the following explicit expression for the regular solution, valid in the range $0 > \mathrm{Re}\, b > -5$

$$f(z) = \int_0^\infty dm(\nu)\, (1 + \nu z)^b; \quad dm(\nu) = d\nu\, \nu^{-b-1} (1+\nu)^{-b-5} . \tag{8.9}$$

In terms of this representation we finally have

$$J_1 = m^{-10-2b} \int_0^\infty dm(\nu)\, D^b$$

$$D = (x B P^{-1} B x^+)(y^+ C^+ P C y) + \nu (x B C y)(y^+ C^+ B^+ x^+) . \tag{8.10}$$

*Integration over the Antibaryon Variable*

We now turn to the next step, the integration over the vector $x_A$. The expression $J(A, B, C)$ can be written as

$$J(A, B, C) = \int dy\, e^{-y^+ p_1 y} J_2; \quad J_2 = \int dx\, e^{-x \tilde{p}_2 x^+} [(x A y)(y^+ A^+ x^+)]^a J_1 . \tag{8.11}$$

To evaluate the integral $J_2$ we insert the representation (8.10) for $J_1$ and furthermore make use of exponential representations for the factors $D^b$ and $[(x A y)(y^+ A^+ x^+)]^a$

$$D^b = n_1 \int_0^\infty dm(\varkappa)\, e^{-\varkappa D}; \quad dm(\varkappa) = d\varkappa\, \varkappa^{-1-b}$$

$$[(x A y)(y^+ A^+ x^+)]^a = n_2 \int^C dm(\lambda)\, e^{-\lambda^2 (x A y)(y^+ A^+ x^+)}; \quad dm(\lambda) = d\lambda\, \lambda^{-1-2a} . \tag{8.12}$$

The path $C$ runs along the real axis from $-\infty$ to $+\infty$ but avoids the singularity of $\lambda^{-1-2a}$ at $\lambda = 0$. The integral $J_2$ then takes the form

$$J_2 = n_1 n_2 m^{-10-2b} \int_0^\infty dm(\nu) \int_0^\infty dm(\varkappa) \int^C dm(\lambda) \int dx\, e^{-x M x^+}$$

$$M = \tilde{p}_2 + \varkappa (y^+ C^+ P C y) B P^{-1} B^+ + \varkappa \nu\, B C y y^+ C^+ B^+ + \lambda^2 A y y^+ A^+ . \tag{8.13}$$

The integration over $x$ can now be carried out in a trivial fashion by means of the well-known formula

$$\int dx\, e^{-xMx^+} = \frac{\pi^6}{\det M} \tag{8.14}$$

and we obtain

$$J_2 = m^{-10-2b} \int_0^\infty dm(\nu) \int_0^\infty dm(\varkappa) \int^C dm(\lambda)\, \frac{1}{\det M}\,. \tag{8.14}$$

In this expression we have again suppressed an overall normalization constant. The determinant of the matrix $M$ contains the remaining variable $y$ in a rather complicated way. This is due to the fact that the matrix $M$ contains the independent vectors $A\,y$ and $B\,C\,y$. It is therefore convenient at this point to carry out first the derivatives with respect to the matrices $A$, $B$ and $C$ required for the evaluation of the coupling coefficients $J^{ABC\,G}_{DEF\,H}$, then to set $A = B = C = 1$ (see Equation (8.3)) and to work out the remaining integral over $y$ only after these steps have been taken. In fact we shall postpone the integration over $y$ until the form factors have been projected out. The reason for this will become clear in the course of the calculation.

It is straightforward to work out the necessary derivatives with respect to the matrices $A$, $B$ and $C$. We quote the resulting integral representation for the coupling coefficients

$$J^{ABC}_{DEF\,\dot{G}H} = m^{3+i(\varrho+\sigma)} \int dy\, e^{-y^+ p_1 y}\, (y^+ P\, y)^{a+b} \int_0^\infty dm(\nu)$$
$$\times \int_0^\infty dm(\mu) \int_0^\infty dm(\omega)\, n^{-6}\, \Delta^{-5} \sum_{i=1}^{13} E_i\, Q^{ABC}_{iDEF\,\dot{G}H}\,. \tag{8.15}$$

Here we have introduced the symbol $J^{ABC}_{DEF\,\dot{G}H}$:

$$J^{ABC}_{DEF\,\dot{G}H} = P_{\dot{G}K}\, J^{ABC\,K}_{DEF\,H}$$

where $P_{\dot{G}K}$ is the matrix associated with the momentum of the meson, $P = P^\mu\,\sigma_\mu$. We have dropped an overall normalization constant. Furthermore a transformation of variables from $\varkappa$, $\lambda$ to $\mu$, $\omega$ has been performed

$$\varkappa = \mu\, m^2 (y^+ P\, y)^{-1}; \qquad dm(\mu) = d\mu\, \mu^{-b}$$
$$\lambda = \omega\, m (y^+ P\, y)^{-1/2}; \qquad dm(\omega) = d\omega\, \omega^{3-2a} \tag{8.16}$$

where $m$ denotes the meson mass[8]). To specify the rather involved expression for the integrand we introduce the vectors $n^\nu$ and $R^\nu$

$$n^\nu = p_2^\nu + \mu\, P^\nu; \qquad R^\nu = \frac{n^\nu}{n^2} \tag{8.17}$$

and the $6 \times 6$ matrices $R$ and $S$

$$R = R^\mu\, \sigma_\mu; \qquad S = R\, P^{-1}\, R\,. \tag{8.18}$$

[8]) Note that the singularity encountered in the Gaussian representation (8.12) is cancelled by a zero of the integrand that arises after the derivations have been carried out. We are therefore entitled to let path $C$ run through $\lambda = 0$ and to reduce it to an integral from 0 to $\infty$ by virtue of the symmetry in $\lambda$ and $-\lambda$.

In terms of these quantities the symbol $\Delta$ occuring in (8.15) reads

$$\Delta = (y^+ P y) + \gamma\, m^2 (y^+ R y)\,; \qquad \gamma = \mu\,\nu + \omega^2\,. \tag{8.19}$$

The scalars $E_1, \ldots, E_{13}$ are defined in Table I, together with the tensors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_{13}$. The quantities $\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{M}$ occuring in the table denote the following tensors

$$\begin{aligned} \boldsymbol{K}(A, B, C, D) &= y^A\, y^B\, y^C \underset{DEF}{\mathsf{S}}\, a_D\, a_E\, b_F\, c_{\dot G}\, d_H \\ \boldsymbol{L}(A, B, C, D) &= y^A\, y^B\, y^C \underset{DEF}{\mathsf{S}}\, A_{\dot G D}\, b_E\, c_F\, d_H \\ \boldsymbol{M}(A) &= y^A\, y^B\, y^C \underset{DEF}{\mathsf{S}}\, A_{\dot G H}\, r_D\, r_E\, r_F \end{aligned} \tag{8.20}$$

where $a, b, c, d, r$ stand for the vectors $y^+ A$, $y^+ B$, $C y$, $y^+ D$, $y^+ R$ respectively and where the operator $\underset{DEF}{\mathsf{S}}$ projects out the symmetric part of the tensor that follows.

Table I

| $i$ | $E_i$ | $\boldsymbol{Q}_i$ |
|---|---|---|
| 1 | $-6\,\mu\,\gamma\,\Delta\,(P\,R)\,m^2 + 4\,\mu\,\gamma^2\,(y^+ S\,y)\,m^6 + \gamma\,\Delta\,m^2$ | $\boldsymbol{K}(R, R, R, P)$ |
| 2 | $-\mu\,\gamma\,\Delta\,m^4$ | $\boldsymbol{K}(R, R, S, P)$ |
| 3 | $-3\,\mu\,\gamma\,\Delta\,m^4$ | $\boldsymbol{K}(R, S, R, P)$ |
| 4 | $6\,\mu\,\nu\,\Delta\,(P\,R) - 4\,\mu\,\nu\,\gamma\,(y^+ S\,y)\,m^4$ | $\boldsymbol{K}(R, R, P, P)$ |
| 5 | $3\,\mu\,\nu\,\Delta\,m^2$ | $\boldsymbol{K}(R, S, P, P)$ |
| 6 | $-4\,\mu\,\nu\,\gamma\,(y^+ P\,y)\,m^4$ | $\boldsymbol{K}(R, R, R, R)$ |
| 7 | $4\,\mu\,\nu^2\,(y^+ P\,y)\,m^2$ | $\boldsymbol{K}(R, R, P, R)$ |
| 8 | $6\,\mu\,\Delta^2\,(P\,R) - 3\,\mu\,\gamma\,\Delta\,(y^+ S\,y)\,m^4 - \Delta^2$ | $\boldsymbol{L}(R, R, R, P)$ |
| 9 | $\mu\,\Delta^2\,m^2$ | $\boldsymbol{L}(S, R, R, P)$ |
| 10 | $2\,\mu\,\Delta^2\,m^2$ | $\boldsymbol{L}(R, S, R, P)$ |
| 11 | $3\,\mu\,\nu\,\Delta\,(y^+ P\,y)\,m^2$ | $\boldsymbol{L}(R, R, R, R)$ |
| 12 | $\mu\,\nu\,\Delta\,(y^+ P\,y)\,m^2$ | $\boldsymbol{M}(R)$ |
| 13 | $-\nu\,\Delta\,(y^+ P\,y)$ | $\boldsymbol{M}(P)$ |

## 9. SU(6)-Coupling Invariants

From the general SL(6, C) invariant vertex $V(p_1\, p_2 \mid P)$ given by (6.8) the SU(3) ⊗ SU(2) coupling invariants follow by projection. We restrict the evaluation of these invariants to the baryon octet.

The reduction of the SU(6) tensor according to SU(3) ⊗ SU(2) is defined by

$$J^{k c g \gamma}_{f l h \;\dot\phi \zeta \eta} = \varepsilon_{\alpha\beta}\, \varepsilon^{\delta\varepsilon}\, \varepsilon_{a b l}\, \varepsilon^{d e k}\, J^{ABC}_{DEF\,\dot G H}\, M^{9-i\varrho}\, m^{6-i\sigma} \tag{9.1}$$

with $\gamma, \phi, \zeta, \eta$ denoting the SU(2) indices of $C, F, G, H$ respectively. This tensor is traceless in $k = f$ and $c = l$ as a consequence of the symmetry of $J^{\underline{ABC}}_{\overline{DEF}\,\dot G H}$. There are three SU(3)-invariant tensors of this type:

$$J^{k c g \gamma}_{f l h \;\dot\phi \zeta \eta} = \sum_{I = S, D, F} \Pi^{(I)\, k c g}_{\quad f l h}\, J^{(I)\,\gamma}_{\quad \dot\phi \zeta \eta} \tag{9.2}$$

where the three SU(3)-invariant projection operators $\Pi^{(S)}$, $\Pi^{(D)}$ and $\Pi^{(F)}$ are defined by

$$\Pi^{(I)}{}^{fcg}_{flh} = \Pi^{(I)}{}^{kcg}_{fch} = 0 \tag{9.3}$$

$$B^{+l}_{c}\, \bar{B}^{+f}_{k}\, M^{h}_{g}\, \Pi^{(I)}{}^{kcg}_{flh} = (B^+ M\, B)_I \tag{9.4}$$

with the conventional SU(3) coupling coefficients

$$(B^+ M\, B)_S = 0\,, \quad (B^+ M\, B)_D = \mathrm{tr}(B^+ [M, B]_+)\,,$$
$$(B^+ M\, B)_F = \mathrm{tr}(B^+ [M, B]_-)$$

if $M$ is the pseudoscalar or vector meson *octet* (tr $M = 0$) and

$$(B^+ M\, B)_S = \mathrm{tr}(B^+ B)\ \mathrm{tr}\, M\,, \quad (B^+ M\, B)_D = (B^+ M\, B)_F = 0$$

if $M$ is the meson singlet ($M^h_g = \varkappa\, \delta^h_g$).

The SU(3) invariants $J^{(I)}{}^{\gamma}_{\phi\dot{\zeta}\eta}$ may be further reduced by making use of invariance under the proper Lorentz group. For this purpose it is convenient to replace the SU(2) indices by vector indices:

$$J^{(I)}{}^{\gamma}_{\phi\dot{\zeta}\eta} = \frac{1}{M^2}\, \Omega^{(I)}{}^{\nu}_{\mu}\, (\sigma^\mu)_{\dot{\zeta}\eta}\, (\tilde{p}_1\, \sigma_\nu)^{\gamma}_{\phi}\,. \tag{9.5}$$

Invariance under the proper homogeneous Lorentz group then implies that the field $\Omega^{(I)}{}^{\mu}_{\nu}$ be an invariant tensor function of the two linearly independent 4-vectors $P^\mu = (p_1 + p_2)^\mu$ and $k^\mu = (p_1 - p_2)^\mu$:

$$\Omega^{(I)}{}^{\nu}_{\mu} = \frac{m}{2\, M} \Big\{ W^{(I)}_1(s)\, (s\, \delta^\nu_\mu - P_\mu\, P^\nu) + W^{(I)}_2(s)\, k_\mu\, k^\nu + W^{(I)}_3(s)\, k_\mu\, P^\nu$$
$$+ i\, W^{(I)}_4(s)\, \varepsilon^{\nu}_{\mu\varrho\sigma}\, P^\varrho\, k^\sigma + \frac{2\, M^2}{s}\, W^{(I)}_5(s)\, P_\mu\, P^\nu + \frac{2\, M^2}{s}\, W^{(I)}_6(s)\, P_\mu\, k^\nu \Big\}\,. \tag{9.6}$$

The $3\times 6$ functions of the kinematical invariant $s = m^2$, $W^{(I)}_1(s)$ to $W^{(I)}_6(s)$, are essentially the form factors describing the vertex.

Using the definition of the vertex function (6.8), the definition of the spinors (4.9), (4.11), (4.12) and the projections (9.2), (9.5) the vertex finally takes the form

$$V(p_1, p_2 \mid P) = F(s) \sum_{I = S, D, F} (B^+ V\, B)_I\, J^{(I)}{}^{\nu}_{\mu}\, \varepsilon^\mu\, (\chi^+(p_1)\, \sigma_\nu\, \Phi(p_2))$$
$$+ F(s) \sum_{I = S, D, F} (B^+ P\, B)_I\, J^{(I)\nu} (\chi^+(p_1)\, \sigma_\nu\, \Phi(p_2)) \tag{9.7}$$

$$J^{(I)}{}^{\nu}_{\mu} = \frac{1}{m^3\, M}\, (s\, \delta^\varrho_\mu - P_\mu\, P^\varrho)\, \Omega^{(I)}{}^{\nu}_{\varrho}\,, \quad J^{(I)\nu} = \frac{1}{m^2\, M}\, P^\varrho\, \Omega^{(I)}{}^{\nu}_{\varrho}$$

where $(B^+ V\, B)_I$ and $(B^+ P\, B)_I$ are the vector meson and pseudoscalar coupling coefficients respectively.

## 10. Form Factors

The conserved vector current[9])

$$j_\mu = F(s)\, J^\nu_\mu\, (\chi^+(p_1)\, \sigma_\nu\, \Phi(p_2))\,, \quad P^\mu\, j_\mu = 0 \tag{10.1}$$

[9]) In order to avoid unnecessary crowding of indices we omit in this section the SU(3) label distinguishing the currents and form factors.

and the scalar current

$$j = F(s)\ J^{\nu}(\chi^{+}(p_1)\ \sigma_{\nu}\ \Phi(p_2)) \tag{10.2}$$

have now to be identified with the usual expressions for these currents in terms of Dirac spinors in order to relate the invariants $W_1$ to $W_6$ with the more familiar form factors of the Dirac theory. The most general conserved vector and scalar under proper Lorentz transformations which can be constructed out of the Dirac spinors $\bar{u}(p_1)$ and $v(p_2)$ are[10])

$$j_{\mu} = \bar{u}(p_1)\Big[F_1(s)\ \gamma_{\mu} - F_2(s)\ \frac{1}{2M}\ \gamma_{\mu\nu}\ P^{\nu} + F_3(s)$$
$$\times \frac{1}{2M}\Big(P_{\mu} - \frac{s}{2M}\ \gamma_{\mu}\Big)\ \gamma_5 + F_4\ \frac{1}{2M}\ k_{\mu}\gamma_5\Big]\ v(p_2) \tag{10.3}$$

$$j = \frac{2M}{m}\ \bar{u}(p_1)\ [F_5(s)\ \gamma_5 + F_6]\ v(p_2)\,. \tag{10.4}$$

As we have pointed out in Section 5 the quantities $F_1$ and $F_2$ may be interpreted as the electric and magnetic form factors of the baryon in question if the quantum numbers of the vector meson are chosen to be those of the photon. The quantities $F_3$ and $F_4$ are the analogous form factors of the pseudovector current. As mentioned in Section 7 these form factors must vanish if $\varrho = \sigma = \tau = 0$ and this provides us with an important check on our calculations. Similarly $F_5$ corresponds to the so-called pseudovector coupling constant of the $N\pi$-vertex and can be normalized with the experimental value $F_5(m_\pi^2) = \sqrt{4\pi}\,f$, $f^2 = .08$. $F_6$ is then the analogous parity conjugate object. With this normalization the symmetry group determines all strong coupling invariants of the baryon-meson vertex up to the overall function $F(s)$.

The Dirac spinors $\bar{u}(p_1)$ and $v(p_2)$ are related to the 2-component spinors $\chi^{+}(p_1)$ and $\Phi(p_2)$ in the following way (Majorana representation):

$$u(p_1) = \begin{pmatrix} \chi^{\alpha}(p_1) \\ (1/M)\ (p_1)_{\dot\alpha\beta}\ \chi^{\beta}(p_1) \end{pmatrix} \qquad v(p_2) = \begin{pmatrix} \Phi^{\alpha}(p_2) \\ -\ (1/M)\ (p_2)_{\dot\alpha\beta}\ \Phi^{\beta}(p_2) \end{pmatrix}. \tag{10.5}$$

By straightforward algebra the Dirac currents (10.3), (10.4) are transformed to the form (10.1), (10.2) and comparison of the coefficients gives the form factors $F_1$ to $F_6$ in terms of the invariants $W_1$ to $W_6$:

$$F_1 = [(s/4M^2)\ (W_1 - W_4) + W_2]\ F(s)\,, \qquad F_4 = W_3\ F(s)$$
$$F_2 = (W_4 - W_2)\ F(s)\,, \qquad F_5 = 1/2\ W_5\ F(s)$$
$$F_3 = (W_4 - W_1)\ F(s)\,, \qquad F_6 = 1/2\ W_6\ F(s)\,. \tag{10.6}$$

With respect to later applications to the electromagnetic form factors we prefer to use instead of $F_1$ and $F_2$ the Sachs type form factors

$$G_M = F_1 + F_2 = \frac{1}{4M^2}\ (s\ W_1 + k^2\ W_4)\ F(s)$$

$$G_E = F_1 + \frac{s}{4M^2}\ F_2 = \frac{1}{4M^2}\ (s\ W_1 + k^2\ W_2)\ F(s)\,. \tag{10.7}$$

---

[10]) The $\gamma$-matrices are defined in terms of the $\sigma$-matrices:

$$\gamma_{\mu} = \begin{pmatrix} 0 & \tilde\sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix};\quad \gamma_{\mu\nu} = \frac{1}{2}\ (\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}) = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde\sigma_{\mu\nu} \end{pmatrix};\quad \gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore we shall write

$$G_{PS} = F_5$$

for the form factor describing the coupling of the pseudoscalar mesons to the baryon octet.

Altogether we thus have 9 form factors $G_M^{(I)}$, $G_E^{(I)}$, $G_{PS}^{(I)}$ of even parity and 9 form factos $F_3^{(I)}$, $F_4^{(I)}$, $F_6^{(I)}$ of odd parity. Actually the odd parity form factors will be shown to vanish and furthermore we will show that the 9 even parity form factors are expressible as linear combinations of only 3 basic functions which we shall tabulate numerically.

## 11. Integral Representation of the Coupling Invariants

It turns out to be a considerable advantage to extract first the form factors from the tensor $J^{ABC}_{DEF\dot{G}H}$ before attacking the remaining integrations over the baryon variable $y$. Starting with the integral representation (8.15) of the tensor $J^{ABC}_{DEF\dot{G}H}$ in its explicit form, given by the definitions (8.16) to (8.20) and by Table I, we can now work out the corresponding integral representations for the 18 coupling invariants $W_j^{(I)}$ ($I = S, D, F$; $j = 1, \dots, 6$) by taking traces over the SU(3) and SU(2) indices according to the projections (9.1), (9.2) and (9.5). The computation of the traces is considerably simplified if one works in a special coordinate system, the rest system of the meson.

$$(p_1 + p_2)^\mu = P^\mu = (m, 0, 0, 0)\,; \quad (p_1 - p_2)^\mu = k^\mu = (0, 0, 0, \varkappa)$$
$$\varkappa^2 = m^2 - 4\,M^2\,. \qquad (11.1)$$

In order to give an idea of the structure of the resulting expressions we give as an example the invariant $W_4^{(S)}$.

$$W_4^{(S)} = \frac{1}{24}\,F(s)\,M^{9-i\varrho}\,m^{9+i\varrho}\int dy\, e^{-(y^+ p_1 y)}\,(y^+\,P\,y)^{a+b}$$
$$\times \int_0^\infty dm(\nu) \int_0^\infty dm(\mu) \int_0^\infty dm(\omega)\; n^{-6}\,\Delta^{-5} \sum_{i=1}^{13} E_i\, Q_4^{(S)}\,(A, B, C, D, y)_i\,. \qquad (11.2)$$

$Q_4^{(S)}(A, B, C, D, y)_i$ are the traces corresponding to the $S$-invariant with respect to SU(3) labels and to the $W_4$-invariant with respect to the SU(2) labels of the 13 tensors $\boldsymbol{Q}_i$ (see (8.20)). The result can be expressed in terms of the three 4-vectors

$$P^\mu,\ R^\mu,\ S^\mu = \frac{1}{m^2}\,[2(R^\nu\,P_\nu)\,R^\mu - (R^\nu\,R_\nu)\,P^\mu] \qquad (11.3)$$

and the 4-vector $y^\mu$ defined by

$$y^\mu(\sigma_\mu)_{\dot{\alpha}\beta} = \sum_a y^*(\dot{a}, \dot{\alpha})\;y(a, \beta)\,; \quad y^\mu = y^+\,\sigma^\mu\,y\,. \qquad (11.4)$$

The arguments of the $\boldsymbol{Q}$'s, the matrices $A$, $B$, $C$, $D$, have to be replaced by the four-vectors (11.3) according to the sequencies given in Table I. Using the notation of this table we have to give in our example the three types o fquantities multiplying the $E$'s, i.e.

$$K_4^{(S)} = \frac{8\,M}{m^2\,\varkappa}\,(y^\mu\,y_\mu)\,[(y^0)^2 - (y^3)^2 - y^\mu\,y_\mu]\,\{(A\,A)\,[(p_1\,C)\,(D^*\,B) + (p_1{}^*\,C)\,(B\,D)] + 2(A\,B)\,[(p_1\,C)\,(D^*\,A) + (p_1{}^*\,C)\,(A\,D)]\}$$

$$L_4^{(S)} = \frac{8\,M}{m^2\,\varkappa}\,(y^\mu\,y_\mu)\,(B\,C)\,\{(p_1\,y)\,(D^*\,A) + (p_1{}^*\,y)\,(D\,A)\}$$

$$M_4^{(S)} = 0$$

where we have introduced the notation

$$(A\,B) = A^\mu\,B_\mu = A^0\,B^0 - A^3\,B^3\,, \quad (A^*\,B) = A^0\,B^3 - A^3\,B^0$$

(note that in the special coordinate system the vectors $P^\mu$, $R^\mu$, $S^\mu$, substituting the general arguments $A$, $B$, $C$, $D$, have only 0- and 3-components.)

The virtue of carrying out these projections first is obvious: the baryon variable appears now exclusively in terms of the Lorentz vector $y^\mu$ and this makes part of the $y$-integration trivial.

### 12. Integration over the Baryon Variable

The dependence on the variable $y$ in the form factors is contained in the four-vector $y^\mu$ and the integral is of the form

$$W = \int dy\, f(y^\mu)\,.$$

It is straight forward to reduce the integration over the 6 complex numbers $y_A$ to an integration over the real vector $y^\mu$ with the result

$$\int dy\, f(y^\mu) = \varkappa \int d^4y\;\theta(y^2)\;\theta(y^0)\;y^2\,f(y^\mu) = \varkappa \int_0^\infty dy\;y^5 \int d\Omega(\hat{y})\;f(y\,\hat{y}^\mu)$$

where $\varkappa$ is some constant and where we have defined $y$ as the length of the vector $y^\mu$ and $\hat{y}^\mu$ as its direction.

$$\begin{aligned} y^\mu &= y\,\hat{y}^\mu\,, \quad \hat{y}^\mu\,\hat{y}_\mu = 1 \\ \hat{y}^0 &= \cosh\chi \\ \hat{y}^1 &= \sinh\chi\,\sin\theta\,\cos\phi \\ \hat{y}^2 &= \sinh\chi\,\sin\theta\,\sin\phi \\ \hat{y}^3 &= \sinh\chi\,\cos\theta\,. \end{aligned} \tag{12.1}$$

Since the integrand is homogeneous in $y_A$ the variable $y$ multiplies the integrand $f(y\,\hat{y}^\mu)$ homogeneously in all terms of the sum and the integral for the form factors becomes

$$W_j^{(I)} = \text{const.}\;F(s)\;M^{9-i\varrho}\;m^{9+i\varrho}\int_0^\infty dy\;y^{a+b+5}\;d\Omega(\hat{y})\;e^{-y(p_1\hat{y})}\;(P\,\hat{y})^{a+b}$$
$$\times \int_0^\infty dm(\nu) \int_0^\infty dm(\mu) \int_0^\infty dm(\omega)\;n^{-6}\;\Gamma^{-5}\sum_{i=1}^{13} E_i'\;Q_j^{(I)}\;(A,\,B,\,C,\,D;\,\hat{y})_i\,. \tag{12.2}$$

Thereby we used

$$(y^+ \not{p}_1 y) = 2(p_1^\mu y_\mu) = 2\, y(p_1 \hat{y})$$

and defined the quantities

$$\Delta = 2\, y\, \Gamma; \quad \Gamma = (P\,\hat{y}) + s\,\gamma(R\,\hat{y})$$

$$E_i = \begin{cases} 2\, y\, E_i' & (i = 1, \ldots, 7) \\ 2\, y^2\, E_i' & (i = 8, \ldots, 13) \end{cases}$$

$$Q_j^{(I)}(A, B, C, D; y)_i = \begin{cases} y^4\, Q_j^{(I)}(A, B, C, D; \hat{y})_i & (i = 1, \ldots, 7) \\ y^3\, Q_j^{(I)}(A, B, C, D; \hat{y})_i & (i = 8, \ldots, 13)\,. \end{cases}$$

The integration over the length of the vector $y$ can be carried out immediately and at the same time the integration over the angle $\phi$ is trivial since $Q_{j,i}^{(I)}$ is independent of $\phi$ in the coordinate system (11.1). We are left with a 5-fold integral for the coupling invariants and get – dropping overall constants and putting $\cos\theta = z$ –

$$W_j^{(I)} = F(s)\, m^{9+i\varrho} \int_0^\infty d\chi\, \sinh^2\chi \int_{-1}^{+1} dz \int_0^\infty dm(\mu) \int_0^\infty dm(\nu) \int_0^\infty dm(\omega)$$

$$\times\, (p_1\,\hat{y})^{-(a+b+6)}\, (P\,\hat{y})^{a+b}\, n^{-6}\, \Gamma^{-5} \sum_{i=1}^{13} E_i'\, Q_j^{(I)}(A, B, C, D; \hat{y})_i\,. \qquad (12.3)$$

## 13. Some More Integrations

For the further evaluation of the form factors we specialize the representations for baryons and mesons by taking $a$ and $b$ real[11]) ($\varrho = \sigma = 0$) and hence $a + b = -3/2$. With this specialization three additional integrations can be done explicitly.

a) The variables $\nu$ and $\omega$ are only involved in the terms $E_i'$ and $\Gamma$. Specializing (8.9) and (8.16) the integrals over $\nu$ and $\omega$ read

$$d_i = \int_0^\infty \frac{\nu\, d\nu}{(1+\nu)^3} \int_0^\infty d\omega\, \omega^2\, \Gamma^{-5}\, E_i'\,. \qquad (13.1)$$

Decomposition in partial fractions leads to three standard integrals, denoted as $I_1$, $I_2$ and $I_3$.

$$I_n = \int_0^\infty \frac{\nu\, d\nu}{(1+\nu)^3}\, \frac{1}{(\nu+\varepsilon)^{n+1/2}} \quad \text{with} \quad \varepsilon = \frac{(P\,\hat{y})}{\mu\, s(R\,\hat{y})} \qquad (13.2)$$

The $d_i$ are then linear combinations of these integrals; as an example we give one of the simpler terms:

$$d_2 = -2^{-4}\, \pi\, \mu^{5/2}\, \varepsilon^3\, (\cosh\chi)^{-4} \left[\frac{\mu\,\varepsilon}{2}\, (I_1 + I_2)\right].$$

---

[11]) With $\varrho = \sigma = 0$ our description of the $\overline{B}\, B\, M$-vertex coincides with the assumptions made by RÜHL, Ref. [6], and his general results are applicable in our case.

b) Since the integrand depends only on $\hat{y}^0 = \cosh\chi$ and $\hat{y}^3 = z \sinh\chi$ the following substitution of variables is indicated

$$u = \operatorname{tgh}\chi\,; \quad d\chi = \frac{du}{1-u^2}\,.$$

Extracting the factor $\cosh\chi$ in every scalar product and cross product each term of the integrand has the form

$$\int_0^\infty d\chi \int_{-1}^{+1} dz \sinh^2\chi\,(\cosh\chi)^{-p} f(u\,z) = \int_0^1 du \int_{-1}^{+1} dz\, u^2\,(1-u^2)^{(p-2)/2} f(u\,z)$$

with $p$ an even integer $\geqslant 2$. Substituting $u\,z = \zeta$ the integrals in the angle variables reduce then to a simple integral of the type

$$\int_0^1 du\, u^q \int_{-1}^{+1} dz\, f(u\,z) = \frac{1}{q}\int_{-1}^{+1} d\zeta\,(1-\zeta^q)\, f(\zeta)\,.$$

The final result of the complete expressions will be shown in Section 15. We have not been able to solve the remaining integrals in the variables $\mu$ and $\zeta$ analytically; the expressions are extremely involved and especially the logarithmic term arising in the integrals $I_n$ contains the variables under square roots and is therefore nearly prohibitive for a further analytic treatment. This statement does however not exclude the possibility that the integrated result would be expressible by means of well known transcendental functions as is indicated by the explicit results of Bamberg [7] and Bebié [12]).

## 14. Consistency Checks and Relations Among the Form Factors

After these long computational processes with complicated expressions one would feel happy to have some general consistency checks. There are such checks in terms of linear relations among the form factors and certain conditions following from subgroup properties. These relations are by no means explicit at every stage of the calculation. At the same time such relations drastically reduce the number of independent form factors which have to be evaluated.

a) Rühl [6] has shown that the collinear subgroup of SL(6, C) implies the following relations among the form factors, valid for all values of $s$:

$$G_M^{(S)} : G_M^{(D)} : G_M^{(F)} = 1:3:2 \tag{14.1}$$

$$G_{PS}^{(D)} = \frac{2}{3}\, G_E^{(D)} - G_E^{(F)} \tag{14.2}$$

$$G_{PS}^{(F)} = -\frac{5}{9}\, G_E^{(D)} - \frac{2}{3}\, G_E^{(F)}\,. \tag{14.3}$$

These 4 relations are identically satisfied by the integrands after the three final integrations have been performed.

---

[12]) H. Bebié (to be published) gives an explicit result for the form factors in the corresponding SL(2, C) model.

b) According to the arguments presented in Section 7 the vertex is either even or odd under space reflection. This statement provides us with a very deep consistency check since neither the integrands representing the even parity form factors nor the integrands associated with the pseudo form factors vanish. If our framework is consistent we should find zero for the numerical values of either of the two sets of form factors after the numerical integrations have been carried out. In fact we shall show that the pseudo form factors vanish numerically. Actually it is not necessary to evaluate all 9 pseudo form factors, since the integrands representing these quantities satisfy linear relations analogous to (14.1, 2. 3):

$$W_4^{(S)} : W_4^{(D)} : W_4^{(F)} = 1:2:3 \,; \quad \frac{s}{4\,M^2}\, F_3^{(I)} = W_4^{(I)} - G_M^{(I)} \tag{14.4}$$

$$\frac{4\,M^2}{s}\, F_6^{(D)} = \frac{2}{3}\, F_4^{(D)} - F_4^{(F)} \tag{14.5}$$

$$\frac{4\,M^2}{s}\, F_6^{(F)} = -\frac{5}{9}\, F_4^{(D)} - \frac{2}{3}\, F_4^{(F)} \,. \tag{14.6}$$

c) The $S$-type coupling of the scalar and pseudoscalar current must vanish since the pseudoscalar mesons form a pure octet

$$G_{PS}^{(S)} \equiv 0 \,; \qquad W_6^{(S)} \equiv 0 \,. \tag{14.7}$$

In the final stage the *integrand* does not have this property. But it can be shown that the *integrals* vanish also in their final form and this fact is a very strong check for the correctness of the methods used and the details of the computation.

d) Finally we found an additional linear relation among the charge form factors, also satisfied by the integrand,

$$G_E^{(S)} = 3\, G_E^{(F)} - \frac{5}{3}\, G_E^{(D)} \tag{14.8}$$

An analogous relation holds for $W_3$. To our knowledge this relation has not yet been stated in the literature and we have not been able to trace it back to any subgroup property of the symmetry.

With these 12 relations there are only 6 linearly independent invariants, three of even parity and three of odd parity.

## 15. Final Integral Representation for the Form Factors

The final result is obtained after all the integrations of Section 13 are done and after extensive use of the relations stated in Section 14. We give a set of three independent form factors, i.e. $G_M^{(D)}$, $G_E^{(D)}$, $G_E^{(F)}$ (this choice is of course arbitrary), and a corresponding set of invariants, i.e. $W_4^{(D)}$, $W_3^{(D)}$, $W_3^{(F)}$, which are sufficient to check the vanishing of the pseudo form factors $F_3$, $F_4$ and $F_6$ ($F_3 = 0$ if $W_4^{(D)} = G_M^D$, $F_4 = F_6 = 0$ if $W_3^{(D)} = W_3^{(F)} = 0$). All other form factors follow then from the relations (14.1) to (14.8). In the following result the mass scale is normalized by putting the baryon mass $M = 1$.

Every form factor or invariant $G_j^{(I)}$ ($j = M, E, PS$) or $W_j^{(I)}$ ($j = 1, \ldots, 6$) has the structure

$$G_j^{(I)}(s) = \pi \sqrt{2}\, F(s)\, s \int_{-1}^{+1} d\zeta \int_0^\infty d\mu \, \frac{\mu^{7/2}\, \varepsilon^3}{n^6\,(1-\alpha\,\zeta)^{9/2}} \sum_{i=1}^{13} e_i\, q_{j,i}^{(I)} \tag{15.1}$$

where

$$\alpha = ((s-4)/s)^{1/2}; \quad n^2 = 1 + s\,\mu\,(\mu+1); \quad \varepsilon = 2\,n^2/\mu\, s\,(1 + 2\,\mu + \alpha\,\zeta);$$

the $e_i$ are given in Table II and the $q_{j,i}^{(I)}$ are constructed with the expressions (15.2 to 5) according to Table III.

Table II

| $i$ | $e_i$ | $q_{j,i}^{(I)}$ |
|---|---|---|
| 1 | $s\,(1-\zeta^2)^4\,(-\mu\, s\,\tau\, J_1 + s\,\varrho\, J_2 + J_1)$ | $K_j^{(I)}(R, R, R, P)$ |
| 2 | $s\,(1-\zeta^2)^4\,(-\mu\, s\, J_1)$ | $K_j^{(I)}(R, R, S, P)$ |
| 3 | $s\,(1-\zeta^2)^4\,(-3\,\mu\, s\, J_1)$ | $K_j^{(I)}(R, S, R, P)$ |
| 4 | $s\,(1-\zeta^2)^4\,(\mu\,\tau\, J_4 - \varrho\, J_3)$ | $K_j^{(I)}(R, R, P, P)$ |
| 5 | $s\,(1-\zeta^2)^4\; 3\,\mu\, J_4$ | $K_j^{(I)}(R, S, P, P)$ |
| 6 | $s\,(1-\zeta^2)^4\,(-s\, J_3)$ | $K_j^{(I)}(R, R, R, R)$ |
| 7 | $s\,(1-\zeta^2)^4\, J_5$ | $K_j^{(I)}(R, R, P, R)$ |
| 8 | $2\,(1-\zeta^2)^3\,(\mu\, s\,\tau\, J_6 - 3\,\mu\, s\,\varrho\, J_1 - J_6)$ | $L_j^{(I)}(R, R, R, P)$ |
| 9 | $2\,(1-\zeta^2)^3\,\mu\, s\, J_6$ | $L_j^{(I)}(S, R, R, P)$ |
| 10 | $2\,(1-\zeta^2)^3\; 2\,\mu\, s\, J_6$ | $L_j^{(I)}(R, S, R, P)$ |
| 11 | $2\,(1-\zeta^2)^3\; 3\,\mu\, s\, J_4$ | $L_j^{(I)}(R, R, R, R)$ |
| 12 | $2\,(1-\zeta^2)^3\,\mu\, s\, J_4$ | $M_j^{(I)}(R)$ |
| 13 | $2\,(1-\zeta^2)^3\,(-J_4)$ | $M_j^{(I)}(P)$ |

$$J_1 = (1/2)\,\mu\,\varepsilon\,(I_1 + I_2) \qquad J_4 = (1/2)\,\varepsilon\, I_2$$
$$J_2 = (1/4)\,\mu^3\,\varepsilon^2\,(5\, I_1 + 6\, I_2 + 5\, I_3) \qquad J_5 = (5/4)\,\mu\,\varepsilon^2\, I_3$$
$$J_3 = (1/4)\,\mu^2\,\varepsilon^2\,(3\, I_2 + 5\, I_3) \qquad J_6 = I_1 .$$

$I_1(\varepsilon)$, $I_2(\varepsilon)$, $I_3(\varepsilon)$ are the integrals (13.2),

$$\tau = 3\,(1 + 2\,\mu)/n^2, \qquad \varrho = \frac{1}{n^2}\left(\frac{2\,\mu+1}{\mu\,\varepsilon} - 1\right),$$

$P$, $R$, $S$, are the 4-vectors defined in (11.1) and (11.3), i.e.

$$P^\mu = (1, 0, 0, 0)\, m$$
$$R^\mu = (2\,\mu + 1, 0, 0, -\alpha)\,\frac{m}{2\,n^2}$$
$$S^\mu = \left(\frac{(2\,\mu+1)^2}{2\,n^2} - \frac{1}{s}, 0, 0, -\frac{\alpha\,(2\,\mu+1)}{2\,n^2}\right)\frac{m}{n^2} .$$

Table III

| $G_j^{(I)}$ | $i = 1, \ldots, 7$ $q_j^{(I)} = K_j^{(I)}$ | $i = 8, \ldots, 11$ $q_j^{(I)} = L_j^{(I)}$ | $i = 12, 13$ $q_j^{(I)} = M_j^{(I)}$ |
|---|---|---|---|
| $G_M^{(D)}$ | $-\frac{1}{2} K_M^+$ | $-2 L_M^+$ | 0 |
| $G_E^{(D)}$ | $-\frac{1}{80} K_E^+$ | $-\frac{7}{20} L_E^+$ | $+\frac{9}{10} M_E^+$ |
| $G_E^{(F)}$ | $+\frac{5}{48} K_E^+$ | $-\frac{1}{12} L_E^+$ | $+\frac{3}{2} M_E^+$ |
| $W_4^{(D)}$ | $+\frac{1}{2} K_M^-$ | $+2 L_M^-$ | 0 |
| $s\, W_3^{(D)}$ | $-\frac{1}{20} K_E^-$ | $-\frac{7}{5} L_E^-$ | $+\frac{18}{5} M_E^-$ |
| $s\, W_3^{(F)}$ | $+\frac{5}{12} K_E^-$ | $-\frac{1}{3} L_E^-$ | $+6 M_E^-$ |

$$K_M^+ = \frac{1}{m^5} \{(A\,A)\,[(C\,D)\,B^0 + (D^*\,C)\,B^3] + 2(A\,B)\,[(C\,D)\,A^0 + (D^*\,C)\,A^3]\}$$

$$L_M^+ = \frac{1}{m^4} (B\,C)\,[(A\,D) + \zeta(D^*\,A)] \tag{15.2}$$

$$K_M^- = \frac{1}{m^5\,\alpha} \{(A\,A)\,[(C\,D)\,(p^*\,B) + (p\,B)\,(D^*\,C)] + 2(A\,B)\,[(C\,D)\,(p^*\,A) + (p\,A)\,(D^*\,C)]\}$$

$$L_M^- = \frac{1}{m^4\,\alpha} (B\,C)\,[(\zeta - \alpha)\,(A\,D) + (1 - \alpha\,\zeta)\,(D^*\,A)] \tag{15.3}$$

$$K_E^+ = \frac{1}{m^3} \{(p\,C)\,[(A\,A)\,(B\,D) + 2(A\,B)\,(A\,D)] + (p^*\,C)\,[(A\,A)\,(B^*\,D) + 2(A\,B)\,(A^*\,D)]\}$$

$$L_E^+ = \frac{1}{s} \{(1 - \alpha\,\zeta)\,[(A\,C)\,(B\,D) + (C^*\,A)\,(B^*\,D)] - (\alpha - \zeta)\,[(C^*\,A)\,(B\,D) + (C\,A)\,(B^*\,D)]$$

$$M_E^+ = \frac{1}{s} (R\,R)\,[(1 - \alpha\,\zeta)\,(R\,A) - (\alpha - \zeta)\,(R^*\,A)] \tag{15.4}$$

$$K_E^- = \frac{1}{\alpha\,m^3} \{(p\,C)\,[(A\,A)\,(B^*\,D) + 2(A\,B)\,(A^*\,D)] + (p^*\,C)\,[(A\,A)\,(B\,D) + 2(A\,B)\,(A\,D)]\}$$

$$L_E^- = \frac{1}{\alpha\,s} \{(1 - \alpha\,\zeta)\,[(A\,B)\,(C^*\,D) + (B^*\,A)\,(C\,D)] - (\alpha - \zeta)\,[(A\,B)\,(C\,D) + (B^*\,A)\,(C^*\,D)]\}$$

$$M_E^- = \frac{1}{\alpha\,s} (R\,R)\,[1 - \alpha\,\zeta)\,(R^*\,A) + (\zeta - \alpha)\,(R\,A)] \tag{15.5}$$

$$p^\mu = p_1^\mu / m = (1, 0, 0, \alpha)$$

## 16. Numerical Results

The form factors given in the preceding section have been integrated numerically in the production channel above threshold $s \geqslant 4\,M^2$ [13]). The result is shown in Figure 1. As mentioned above, the odd parity form factors vanish identically, i.e. $W_3 = W_4 - W_1 = W_6 = 0$ or $W_4 = G_M$ for all SU(3) labels. The symmetry distinguishes

[13]) The integrals have been done by a two-step Romberg integration programme on the BULL $\Gamma$ 30-S computer of the University of Berne.

therefore a unique type of coupling which has the correct (physical) parity in the production channel but the wrong parity in the scattering channel. The non-vanishing form factors are rapidly decreasing functions. However, one has to keep in mind, that the symmetry does not fix the form factors completely, because of the unknown 'dynamical' form factor $F(s)$ multiplying each term. It is this function that would be responsible for the usual singularities one expects the form factors to have in the production channel. Concerning this point it is also important to be aware of the fact that in our exact symmetry limit all baryons and all mesons (vector- and pseudo-scalar-mesons) have the same mass.

The essential features of the numerical results are the following:

a) *Behaviour at threshold:* the form factors are integer multiples of each other, a fact that has already been derived by RÜHL [6]. Putting $G_M^{(S)} = 1$ these numbers are

| | $G_M$ | $G_E$ | $G_{PS}$ |
|---|---|---|---|
| $S$ | 1 | 1 | – |
| $D$ | 3 | 3 | 0 |
| $F$ | 2 | 2 | –3 |

b) *Electromagnetic form factors of the baryons* (see Figures 1 and 2): taking for the vector meson the quantum numbers of the photon, i.e. $G^p = 1/3\, G^{(D)} + G^{(F)}$, $G^n = -2/3\, G^{(D)}$, we get for proton and neutron the old SU(6) result $G_M^p(s) = -3/2\, G_M^n(s)$ and almost the same relation for the electric form factors. This latter result is connected, by virtue of the relation (14.2), to the very small values of $G_{PS}^{(D)}$ compared

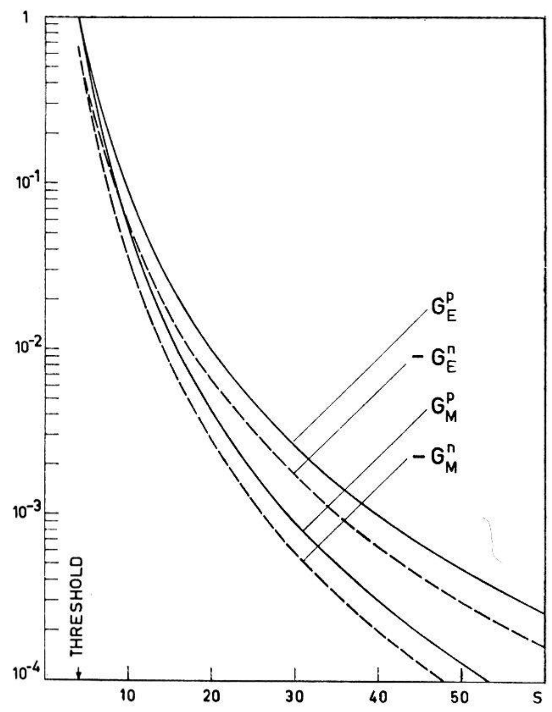


Figure 1

The electromagnetic form factors of proton and neutron, as given by the symmetry with the arbitrary «function $F(s)$ = const. Normalization is arbitrary: $G_E^p\,(s = 4\,M^2) = G_M^p\,(s = 4\,M^2) = 1$ $M^2 = 1$.

with $G_{PS}^{(F)}$. The magnetic form factors decrease by a factor $s^{1/2}$ faster than the electric form factors.

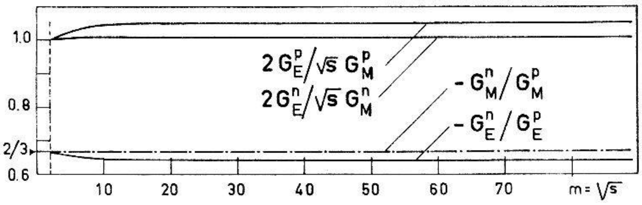


Figure 2

Ratios of form factors, predicted uniquely by the symmetry. $G_M^n/G_M^p = -2/3$; $2\, G_E^n/\sqrt{s}\, G_M^n \approx 1$ within 1% which is the estimated error of the numerical integrals.

c) *Asymptotic behaviour for* $s \to \infty$ (see Figure 3): in the high energy limit an asymptotic expansion of the integrals is possible with the leading terms

$$G_M^{(D)} = F(s)\, s^{-9/2}\, C_M^{(D)}; \qquad C_M^{(D)} \approx 36.1$$
$$G_E^{(D)} = F(s)\, s^{-4}\, C_E^{(D)}; \qquad C_E^{(D)} \approx 18.2$$
$$G_E^{(F)} = F(s)\, s^{-4}\, C_E^{(F)}; \qquad C_E^{(F)} \approx 12.9$$

Again the expansion coefficients have to be evaluated numerically. The fact that $G_M$ decreases by a power $s^{1/2}$ faster than $G_E$ and $G_{PS}$ dominates therefore the whole range $s > 4\, M^2$. The numerical data even suggest that the following relation may be exact

$$G_E^{(D)} = \frac{1}{2}\, s^{1/2}\, G_M^{(D)}\,.$$

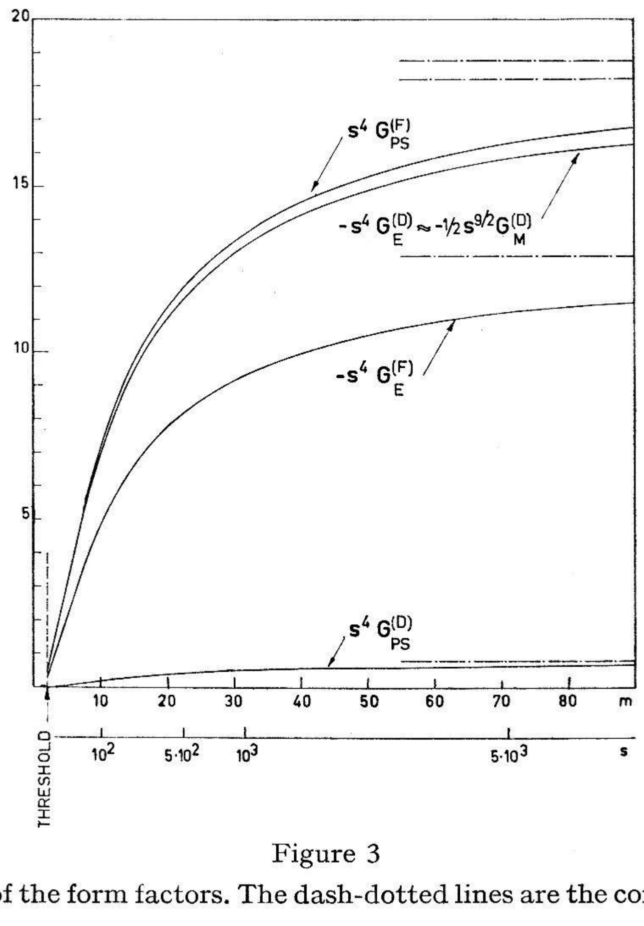


Figure 3

Asymptotic behaviour of the form factors. The dash-dotted lines are the corresponding asymptotes.

d) *The form factors below threshold:* The threshold is no singular point of the functions and the integral representation works in principle also in the region $0 < s < 4 M^2$ where we can no longer apply the symmetry directly. The integrals stay real although the parameter $\alpha$ becomes imaginary. The corresponding integrals have not been evaluated so far, although there may be some interest in these data from the point of view of dispersion theory: the important contributions to the form factors in the scattering channel are connected to the singularities in the crossed channel *below* the two-nucleon-threshold.

At $s = 0$, finally, the form factors become singular and one cannot expect e.g. to find a meaningful ratio $G_E^p/G_M^p$.

## Acknowledgment

We are indepted to Dr. H. Bebié and Dr. M. Zulauf for many helpful discussions. Concerning the computer work, we appreciated the continuous assistance lent by the members of the Institute of Applied Mathematics and by the computer operators.

## References

[1] For an excellent review of the various attempts at a relativistic generalization of SU(6) see H. Rüegg, W. Rühl and T. S. Santhanam, Helv. phys. Acta *40*, 9 (1967).
[2] P. Budini and C. Fronsdal, Phys. Rev. Lett. *14*, 968 (1965). For a more detailed account of the Budini-Fronsdal-theory see C. Fronsdal, Proc. of the Seminar on High Energy Physics and Elementary Particles, Trieste (1965). See also Ref. [1].
[3] L. O'Raifeartaigh, Phys. Rev. Lett. *14*, 332, 575 (1965), Phys. Rev. *139*, B1053 (1965).
[4] I. M. Gelfand, M. A. Neumark, *Unitäre Darstellungen der klassischen Gruppen*, Berlin 1957.
[5] H. Leutwyler and V. Gorgé, Helv. phys. Acta *41*, 171 (1968).
[6] W. Rühl, Nuovo Cim. *44*, 572, 659 (1966) and Nuovo Cim. *47A*, 355 (1967).
[7] The baryon-meson vertex for the case of the meson representation proposed by Fronsdal is currently being investigated by Bamberg. See P. G. Bamberg, Oxford Univ. preprint (1967), for preliminary results.
[8] V. Bargman and E. Wigner, Proc. Nat. Acad. Sci. U.S. *34*, 211 (1948).
[9] See e.g. B. Sakita, Phys. Rev. Lett. *13*, 643 (1964).
[10] G. Cocho, C. Fronsdal, Harun-Ar-Rashid and R. White, Phys. Rev. Lett. *17*, 275 (1966); H. Leutwyler, Phys. Rev. Lett. *17*, 156 (1966).
[11] C. Fronsdal and R. White, Phys. Rev. *151*, 1287 (1966).
[12] G. Feldmann and P. Mathews, Phys. Rev. *151*, 1176 (1966). See also C. Fronsdal, Phys. Rev. *156*, 1653 (1967).
[13] A. A. Makarov, Nguyen van Hieu, P. Winternitz, Fortschr. Phys. *14*, 771 (1966).
[14] See W. Rühl, Nuovo Cim. *44*, 659 (1966) for further details.
[15] W. Rühl, Nuovo Cim. *43*, 171 (1966).