

Zeitschrift: Helvetica Physica Acta
Band: 42 (1969)
Heft: 1

Artikel: A note on the commutation relations of field operators
Autor: Schneider, Walter
DOI: <https://doi.org/10.5169/seals-114061>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 16.05.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A Note on the Commutation Relations of Field Operators

by **Walter Schneider**

Department of Mathematics, Imperial College, London

(10. V. 68)

Abstract. Let $\phi(\cdot)$ and $\psi(\cdot)$ be two fields transforming according to finite irreducible representations of $SL(2, \mathbb{C})$. Then the (anti-) commuting of two properly chosen components of $\phi(x)$ and $\psi(y)$, (x, y) varying in a domain $G \subset R^4 \times R^4$, implies the vanishing of all (anti-) commutators between any component of $\phi(x)$ and $\psi(y)$ respectively.

We consider a Wightman theory [1, 2] containing the fields $\phi(\cdot)$ and $\psi(\cdot)$ which are assumed to transform according to the irreducible representations $[\phi, q]$ and $[\psi, s]$ of $SL(2, \mathbb{C})$ respectively. Accordingly we have

$$\begin{aligned} U(A) \phi(x) U(A)^{-1} &= S_1(A^{-1}) \phi(A(A) x) \\ U(A) \psi(x) U(A)^{-1} &= S_2(A^{-1}) \psi(A(A) x) \end{aligned} \quad (1)$$

where $A \rightarrow U(A)$ is the unitary continuous representation of $SL(2, \mathbb{C})$ in the Hilbert space $\mathcal{H}$ on which the fields act as operator-valued distributions. (1) and the following equations hold on a dense linear set $D \subset \mathcal{H}$ in the sense of distribution theory. The field operators as well as $U(A)$ map D into D . $A \rightarrow S_1(A) = (A^{\otimes p})_{sym} \otimes (\bar{A}^{\otimes q})_{sym}$ is the irreducible representation of $SL(2, \mathbb{C})$ characterized by $[\phi, q]$, and similar for $[\psi, s]$. Finally, $A \rightarrow A(A)$ is the canonical homomorphism from $SL(2, \mathbb{C})$ onto $L_+^\uparrow$, explicitly $A_\nu^\mu(A) = 1/2 \operatorname{Tr} \sigma_\mu A \sigma_\nu A^*$.

With the above-mentioned assumptions we shall prove the following

Theorem: If the (anti-) commutator

$$[\phi_{0,q}(x), \psi_{r,0}(y)]_{(\pm)} \equiv \phi_{0,q}(x) \psi_{r,0}(y)^{(\pm)} \psi_{r,0}(y) \phi_{0,q}(x) \quad (2)$$

between the distinguished components $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$ vanishes, (x, y) varying in the domain $G \subset R^4 \times R^4$, then

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_{(\pm)} = 0, \quad (x, y) \in G \quad (3)$$

for all components $\phi_{h,k}(x)$ and $\psi_{m,n}(y)$. (Instead of the usual dotted and undotted spinor indices with values 1 or 2 we use the numbers h and k to characterize the components of $\phi(\cdot)$, k (h) being the number of (un-)dotted indices of value 1.)

Proof: We insert the following one-parametric subgroups of $SL(2, \mathbb{C})$

$$A_1(\lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \quad A_2(\lambda) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} \quad A_3(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \quad A_4(\lambda) = \begin{pmatrix} 1 & 0 \\ -i\lambda & 1 \end{pmatrix} \quad (4)$$

into (1) and get, taking the derivative at $\lambda = 0$

$$\begin{aligned} [M_1, \phi_{h,k}(x)]_- &= D_1(x) \phi_{h,k}(x) + h \phi_{h-1,k}(x) + k \phi_{h,k-1}(x) \\ [M_2, \phi_{h,k}(x)]_- &= D_2(x) \phi_{h,k}(x) + i h \phi_{h-1,k}(x) - i k \phi_{h,k-1}(x) \\ [M_3, \phi_{h,k}(x)]_- &= D_3(x) \phi_{h,k}(x) + (p - h) \phi_{h+1,k}(x) + (q - k) \phi_{h,k+1}(x) \\ [M_4, \phi_{h,k}(x)]_- &= D_4(x) \phi_{h,k}(x) + i (p - h) \phi_{h+1,k}(x) - i (q - k) \phi_{h,k+1}(x) \end{aligned} \quad (5)$$

and similar Equations (5') for $\psi_{m,n}$. $i M_k, k = 1, 2, 3, 4$, are the self-adjoint generators of the one-parametric unitary groups $U_k(\lambda) = U(A_k(\lambda))$; M_k maps D into D [1]. $D_k(x)$ are linear differential operators of the form $\sum_{\mu\nu} \alpha_k^{\mu\nu} x_\mu \partial_\nu$, $\alpha_k^{\mu\nu} = -\alpha_k^{\nu\mu}$.

For arbitrary operators X, Y, Z mapping D into D , the following identity holds on D :

$$[X, [Y, Z]_-]_\varrho + \varrho [Z, [Y, X]_-]_\varrho = [Y, [X, Z]_\varrho]_- \quad (6)$$

where

$$[A, B]_\varrho = A B + \varrho B A, \quad \varrho = \pm 1.$$

Applying (6) to $\psi_{m,n}(y), M_i, \phi_{h,k}(x)$ we get

$$[\psi_{m,n}(y), [M_i, \phi_{h,k}(x)]_-]_\varrho + \varrho [\phi_{h,k}(x), [M_i, \psi_{m,n}(y)]_-]_\varrho = 0 \quad (7)$$

if

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (8)$$

holds. Together with (8), also the equations

$$[D_i(x) \phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (9)$$

$$[\phi_{h,k}(x), D_i(y) \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (10)$$

are valid, G being a domain.

Inserting (5), (5') into (7) and taking into account (9), (10) we are left with the equations

$$\begin{aligned} h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho \pm k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho \\ + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho \pm \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \end{aligned} \quad (11) \quad (12)$$

$$\begin{aligned} (p - h) [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho \pm (q - k) [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho \\ + \varrho (r - m) [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho \pm \varrho (s - n) [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0. \end{aligned} \quad (13) \quad (14)$$

Adding and subtracting (11) and (12), (13) and (14), leads to

$$h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho = 0 \quad (15)$$

$$k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho + \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \quad (16)$$

$$(p - k) [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho + \varrho (r - m) [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho = 0 \quad (17)$$

$$(q - k) [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho + \varrho (s - n) [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0 \quad (18)$$

(15)–(18) are valid only if (8) holds. This is the case for $h = 0$, $k = q$ and $m = r$, $n = 0$ according to the assumption in the theorem. Therefore, making use of (15), (8) holds for $h = 0$, $k = q$ and $m = r - 1$, $n = 0$. Again using (15), (8) holds for $h = 0$, $k = q$, $m = r - 2$, $n = 0$ and so on. (8) being valid now for $h = 0$, $k = q$, $n = 0$ and all m , repeated use of (17) extends the validity of (8) to $k = q$, $n = 0$, all h , all m . (16) extends this result to $n = 0$, all h , all k , all m and finally, by (18) we end with the statement of the theorem.

Remark: Usually one considers the domain $G = \{(x, y)/(x - y)^2 < 0\}$. In this case the vanishing of either the commutator or the anticommutator between components of field operators at space-like separated points is called locality. Our theorem shows that locality need be assumed only between $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$.

In the cases $\psi(\cdot) = \phi^*(\cdot)$ [1–3], $\psi(\cdot) = \phi(\cdot)$ [1, 2, 4] the choice $\varrho = (-1)^{p+q+1}$ is enforced by the positivity condition. In any other case ϱ is arbitrary, but there always exist sufficiently many symmetries with the help of which new fields can be defined such that $\varrho = \min \{(-1)^{p+q+1}, (-1)^{r+s+1}\}$ [1, 2, 5].

Acknowledgements

The author is indebted to Prof. R. JOST for stimulating this work and to Prof. H. JONES, F.R.S. for his kind hospitality at the Imperial College of Science and Technology.

References

- [1] R. JOST, *The General Theory of Quantized Fields*, Am. Math. Soc., Providence (1965).
- [2] R. F. STREATER and A. S. WIGHTMAN, *PCT, Spin and Statistics* (Benjamin, New York 1964).
- [3] N. BURGOYNE, N.C. 8, 607 (1958).
- [4] G. F. DELL'ANTONIO, Ann. Phys. 16, 153 (1961).
- [5] H. ARAKI, J. Math. Phys. 2, 267 (1961).