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## A Note on the Commutation Relations of Field Operators

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(10. V. 68)

*Abstract.* Let  $\phi(\cdot)$  and  $\psi(\cdot)$  be two fields transforming according to finite irreducible representations of  $SL(2, C)$ . Then the (anti-) commuting of two properly chosen components of  $\phi(x)$  and  $\psi(y)$ ,  $(x, y)$  varying in a domain  $G \subset R^4 \times R^4$ , implies the vanishing of all (anti-) commutators between any component of  $\phi(x)$  and  $\psi(y)$  respectively.

We consider a Wightman theory [1, 2] containing the fields  $\phi(\cdot)$  and  $\psi(\cdot)$  which are assumed to transform according to the irreducible representations  $[p, q]$  and  $[r, s]$  of  $SL(2, C)$  respectively. Accordingly we have

$$\begin{aligned} U(A) \phi(x) U(A)^{-1} &= S_1(A^{-1}) \phi(A(A)x) \\ U(A) \psi(x) U(A)^{-1} &= S_2(A^{-1}) \psi(A(A)x) \end{aligned} \quad (1)$$

where  $A \rightarrow U(A)$  is the unitary continuous representation of  $SL(2, C)$  in the Hilbert space  $\mathcal{H}$  on which the fields act as operator-valued distributions. (1) and the following equations hold on a dense linear set  $D \subset \mathcal{H}$  in the sense of distribution theory. The field operators as well as  $U(A)$  map  $D$  into  $D$ .  $A \rightarrow S_1(A) = (A^{\otimes p})_{sym} \otimes (\bar{A}^{\otimes q})_{sym}$  is the irreducible representation of  $SL(2, C)$  characterized by  $[p, q]$ , and similar for  $[r, s]$ . Finally,  $A \rightarrow A(A)$  is the canonical homomorphism from  $SL(2, C)$  onto  $L_+^\uparrow$ , explicitly  $A_\nu^\mu(A) = 1/2 \operatorname{Tr} \sigma_\mu A \sigma_\nu A^*$ .

With the above-mentioned assumptions we shall prove the following

*Theorem:* If the (anti-) commutator

$$[\phi_{0,q}(x), \psi_{r,0}(y)]_{(+)} \equiv \phi_{0,q}(x) \psi_{r,0}(y)^{(\pm)} \psi_{r,0}(y) \phi_{0,q}(x) \quad (2)$$

between the distinguished components  $\phi_{0,q}(x)$  and  $\psi_{r,0}(y)$  vanishes,  $(x, y)$  varying in the domain  $G \subset R^4 \times R^4$ , then

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_{(+)} = 0, \quad (x, y) \in G \quad (3)$$

for all components  $\phi_{h,k}(x)$  and  $\psi_{m,n}(y)$ . (Instead of the usual dotted and undotted spinor indices with values 1 or 2 we use the numbers  $h$  and  $k$  to characterize the components of  $\phi(\cdot)$ ,  $k$  ( $h$ ) being the number of (un-)dotted indices of value 1.)

*Proof:* We insert the following one-parametric subgroups of  $SL(2, C)$

$$A_1(\lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \quad A_2(\lambda) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} \quad A_3(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \quad A_4(\lambda) = \begin{pmatrix} 1 & 0 \\ -i\lambda & 1 \end{pmatrix} \quad (4)$$

into (1) and get, taking the derivative at  $\lambda = 0$

$$\begin{aligned}[M_1, \phi_{h,k}(x)]_- &= D_1(x) \phi_{h,k}(x) + h \phi_{h-1,k}(x) + k \phi_{h,k-1}(x) \\ [M_2, \phi_{h,k}(x)]_- &= D_2(x) \phi_{h,k}(x) + i h \phi_{h-1,k}(x) - i k \phi_{h,k-1}(x) \\ [M_3, \phi_{h,k}(x)]_- &= D_3(x) \phi_{h,k}(x) + (p-h) \phi_{h+1,k}(x) + (q-k) \phi_{h,k+1}(x) \\ [M_4, \phi_{h,k}(x)]_- &= D_4(x) \phi_{h,k}(x) + i (p-h) \phi_{h+1,k}(x) - i (q-k) \phi_{h,k+1}(x)\end{aligned}\quad (5)$$

and similar Equations (5') for  $\psi_{m,n}$ .  $i M_k$ ,  $k = 1, 2, 3, 4$ , are the self-adjoint generators of the one-parametric unitary groups  $U_k(\lambda) = U(A_k(\lambda))$ ;  $M_k$  maps  $D$  into  $D$  [1].  $D_k(x)$  are linear differential operators of the form  $\sum_{\mu\nu} \alpha_k^{\mu\nu} x_\mu \partial_\nu$ ,  $\alpha_k^{\mu\nu} = -\alpha_k^{\nu\mu}$ .

For arbitrary operators  $X, Y, Z$  mapping  $D$  into  $D$ , the following identity holds on  $D$ :

$$[X, [Y, Z]_-]_\varrho + \varrho [Z, [Y, X]_-]_\varrho = [Y, [X, Z]_\varrho]_- \quad (6)$$

where

$$[A, B]_\varrho = A B + \varrho B A, \quad \varrho = \pm.$$

Applying (6) to  $\psi_{m,n}(y)$ ,  $M_i$ ,  $\phi_{h,k}(x)$  we get

$$[\psi_{m,n}(y), [M_i, \phi_{h,k}(x)]_-]_\varrho + \varrho [\phi_{h,k}(x), [M_i, \psi_{m,n}(y)]_-]_\varrho = 0 \quad (7)$$

if

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (8)$$

holds. Together with (8), also the equations

$$[D_i(x) \phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (9)$$

$$[\phi_{h,k}(x), D_i(y) \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (10)$$

are valid,  $G$  being a domain.

Inserting (5), (5') into (7) and taking into account (9), (10) we are left with the equations

$$\begin{aligned}h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho \pm k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho \\ + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho \pm \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0\end{aligned}\quad (11) \quad (12)$$

$$\begin{aligned}(p-h)[\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho \pm (q-k)[\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho \\ + \varrho(r-m)[\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho \pm \varrho(s-n)[\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0.\end{aligned}\quad (13) \quad (14)$$

Adding and subtracting (11) and (12), (13) and (14), leads to

$$h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho = 0 \quad (15)$$

$$k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho + \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \quad (16)$$

$$(p-k)[\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho + \varrho(r-m)[\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho = 0 \quad (17)$$

$$(q-k)[\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho + \varrho(s-n)[\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0 \quad (18)$$

(15)–(18) are valid only if (8) holds. This is the case for  $h = 0, k = q$  and  $m = r, n = 0$  according to the assumption in the theorem. Therefore, making use of (15), (8) holds for  $h = 0, k = q$  and  $m = r - 1, n = 0$ . Again using (15), (8) holds for  $h = 0, k = q, m = r - 2, n = 0$  and so on. (8) being valid now for  $h = 0, k = q, n = 0$  and all  $m$ , repeated use of (17) extends the validity of (8) to  $k = q, n = 0$ , all  $h$ , all  $m$ . (16) extends this result to  $n = 0$ , all  $h$ , all  $k$ , all  $m$  and finally, by (18) we end with the statement of the theorem.

Remark: Usually one considers the domain  $G = \{(x, y) / (x - y)^2 < 0\}$ . In this case the vanishing of either the commutator or the anticommutator between components of field operators at space-like separated points is called locality. Our theorem shows that locality need be assumed only between  $\phi_{0,q}(x)$  and  $\psi_{r,0}(y)$ .

In the cases  $\psi(\cdot) = \phi^*(\cdot)$  [1–3],  $\psi(\cdot) = \phi(\cdot)$  [1, 2, 4] the choice  $\varrho = (-1)^{p+q+1}$  is enforced by the positivity condition. In any other case  $\varrho$  is arbitrary, but there always exist sufficiently many symmetries with the help of which new fields can be defined such that  $\varrho = \min \{(-1)^{p+q+1}, (-1)^{r+s+1}\}$  [1, 2, 5].

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