Zeitschrift: Helvetica Physica Acta
Band: 42 (1969)
Heft: 6

Artikel: Unitary sum rule and the time evolution of neutral K mesons
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DOI: https://doi.org/10.5169/seals-114095

Unitary Sum Rule and the Time Evolution of Neutral K Mesons

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Abstract. The consequences of the unitary sum rule for the decay of neutral K mesons are investigated without assuming the usual semigroup property (Wigner-Weisskopf equation with constant complex Hamiltonian). A much wider class of motions in the K meson subspace then becomes possible. In particular, there may exist evolutions which do not admit any states with pure exponential decay laws. In a CP-invariant theory, however, the unitary sum rule alone implies exponential decay for the CP-eigenstates.

I. Introduction

The phenomenological description of K^0 meson decay, widely used in current practice [1], consists of the application of a generalized Wigner-Weisskopf equation in the K^0, $\overline{K^0}$ subspace to describe the evolution and decay. This equation is supplemented by the unitary sum rule of Bell and Steinberger [2], connecting the two-dimensional subspace with the space of all final states for the decaying system.

It is well-known, however, that the projection of the time evolution into a subspace cannot, in general, have the exact semigroup property if the evolution of the full space is unitary; the Wigner-Weisskopf equation with constant (non-Hermitian) Hamiltonian, on the other hand, assures the semigroup property for the solution. The assumption of constant decay rates as they arise in the unitary sum rule is also not justifiable in an exact sense in general. In this paper, however, we shall *assume* the general validity of the unitary sum rule in a generalized form (which is applicable to arbitrary states ϱ [3]) but relax the restriction of the Wigner-Weisskopf equations.

This more general viewpoint, in which we do not insist on the semigroup property, allows for a great variety of motions in the K^0 meson space. In particular we find that there are evolutions possible for which there does not exist any state ϱ with exponential decay law.

If the unitary sum rule is supplemented by the requirement of CP-invariance, we find that even without the semigroup property the CP eigenstates decay exponentially.

II. Unitary Sum Rule

Let $\mathcal{H}$ be the Hilbert space of the neutral K mesons together with all their decay products. We consider the K mesons as particles, and write

$$\mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_f \tag{1}$$

where $\mathcal{H}_K$ is the 2-dimensional K meson space and $\mathcal{H}_f$ the space of the 'final states'. Let P_K, P_f $(P_K + P_f = I)$ be the projections of $\mathcal{H}$ on $\mathcal{H}_K$, $\mathcal{H}_f$. The evolution of the entire system is defined by a semigroup of unitary operators in $\mathcal{H}$:

$$V_t = e^{-iHt} \tag{2}$$

where H is the total self-adjoint Hamiltonian of the system. Let $\{f\}$ be a set of final states forming an orthonormal basis in $\mathcal{H}_f$, and choose, for instance, the CP-eigenstates K_1, K_2 as an orthonormal basis in $\mathcal{H}_K$. Then we have the completeness relation

$$\sum_i |(K_i, V_t \psi)|^2 + \sum_f |(f, V_t \psi)|^2 = \|V_t \psi\|^2 = \|\psi\|^2 . \tag{3}$$

Suppose now the initial state to be in $\mathcal{H}_K$: Let $\psi = P_K \psi$ and introduce the reduced evolution $U_t = P_K V_t P_K$ in $\mathcal{H}_K$. If we take the time derivative of (3) we obtain

$$\frac{d}{dt} \sum_i |(K_i, P_K V_t P_K \psi)|^2 = \frac{d}{dt} \|U_t \psi\|^2 = - \frac{d}{dt} |(f, V_t P_K \psi)|^2 . \tag{4}$$

It is at this point where Bell and Steinberger [2] make the farreaching assumption that the transition probability per unit time between K- and f-states is given by the T-matrix

$$\frac{d}{dt} \sum_f |(f, V_t P_K \psi)|^2 = \sum_f |(f, T U_t \psi)|^2 . \tag{5}$$

We do not attempt here to investigate the precise validity of this assumption (some remarks will be made later), but instead consider the exact mathematical consequences of (5) for the reduced evolution U_t of the K meson system.

From (4) and (5) we get *Bell and Steinberger's unitary sum rule for pure states:*

$$- \frac{d}{dt} \|U_t \psi\|^2 = \sum_f |(f, T U_t \psi)|^2 , \tag{6}$$

i.e. [2], the transition probability from $U_t \psi$ into the final states (per unit time) equals the 'depletion' of the state $U_t \psi$ in $\mathcal{H}_K$.

Let us reformulate (6) for arbitrary states in $\mathcal{H}_K$ which can be described by a density matrix ϱ. We denote by P_ψ the projector on the 1-dimensional subspace of $\mathcal{H}$ spanned by ψ; then

$$P_{U_t \psi} = U_t P_\psi U_t^+ \tag{7}$$

and therefore

$$\|U_t \psi\|^2 = Tr\, U_t P_\psi U_t^+ = Tr\, U_t^+ U_t P_\psi .$$

Furthermore, for the right hand side of (6),

$$\sum_f (f, T U_t \psi)(\psi, U_t^+ T^+ f) = Tr(P_f T U_t P_\psi U_t^+ T^+) = Tr(U_t^+ T^+ P_f T U_t P_\psi) . \tag{8}$$

Combining these results and replacing P_ψ by an arbitrary state (which can always be constructed as a convex combination of pure states of the form P_ψ), we obtain

$$- \frac{d}{dt} Tr(U_t^+ U_t \varrho) = Tr(U_t^+ T^+ P_f T U_t \varrho) \tag{9}$$

as the *extension of the unitary sum rule to arbitrary states.*

The result (9) can be written conveniently, without reference to the final states $\mathcal{H}_f$ if we introduce in $\mathcal{H}_K$ the operator $\gamma = P_K\, T^+\, P_f\, T\, P_K$ with 2×2 self-adjoint matrix elements:

$$\gamma_{ik} = \sum_f (K_i, T^+ f)\,(f, T\, K_K)\,. \qquad (10)$$

Equation (9) then appears as

$$Tr\left\{\left(U_t^+ \gamma\, U_t + \frac{d}{dt}\,(U_t\, U_t^+)\right) \varrho\right\} = 0\,, \qquad (11)$$

and since this is to hold *for arbitrary states* $\varrho = \varrho^+$, we obtain the operator equation

$$U_t^+ \gamma\, U_t + \frac{d}{dt}\,(U_t^+\, U_t) = 0\,. \qquad (12)$$

III. Time Evolution of the K^0 Meson Systems Under the Unitary Sum Rule

Every evolution U_t in $\mathcal{H}_K$ can be considered as the solution of a Schroedinger equation with time-dependent Hamiltonian $H_K(t)$

$$\dot{U}_t = -i\, H_K(t)\, U_t\,; \qquad H_K(t) = i\, \dot{U}_t\, U_t^{-1}\,. \qquad (13)$$

$H_K(t)$ is in general neither self-adjoint nor normal.

Inserting this into the unitary sum rule (12) we obtain

$$\gamma + i\,(H_K^+ - H_K) = 0 \qquad (14)$$

and conversely, (14) together with definition (13) implies the unitary sum rule (12). So the anti-Hermitian part of H_K is time-independent and H_K has the general form

$$H_K(t) = M(t) - \frac{i\,\gamma}{2}\,, \qquad (15)$$

where $M(t)$ is an arbitrary Hermitian operator.

There are several cases to consider:

A. In case $H_K(t) = M(t) - i\,\gamma/2$ is a normal operator, or, equivalently, if $M(t)$ *and* γ *commute*, then the orthogonal basis $\{\varphi_i\}$ diagonalizing γ also reduces $M(t)$ and we obtain an evolution

$$U_t = \begin{pmatrix} \exp\left(-i\int_0^t M_1(t')\,dt' - \frac{\gamma_1\, t}{2}\right) & 0 \\ 0 & \exp\left(-i\int_0^t M_2(t')\,dt' - \frac{\gamma_2\, t}{2}\right) \end{pmatrix} \qquad (16)$$

where $M_i(t)$, γ_i, are the eigenvalues of $M(t)$, γ. Although U_t does not satisfy the semi-group property (unless $M(t)$ is constant), the decay laws for the eigenstates remain of the form

$$p_i(t) = Tr\,(U_t\, P_{\varphi_i}\, U_t^+) = e^{-\gamma_i t}\,. \qquad (17)$$

B. If H_K is *not normal but time-independent*, then U_t has the semigroup property and there is a non-orthogonal basis diagonalizing U_t. The eigenvalues of U_t are $e^{-ih_i t}$ where h_i are the eigenvalues of H_K (the real and imaginary parts of h_i do not, however, correspond to the eigenvalues of M and γ). This is the case usually considered [1] for the K-decay, where *two non-orthogonal states (denoted K_L, K_S) have a pure exponential decay*

$$p_i(t) = e^{2 Im h_i t} . \tag{18}$$

C. An interesting possibility arises if $H_K(t)$ is *not normal and is time-dependent.* Here the time evolution U_t may not satisfy the semigroup property and no pure state may exist for which the decay is exponential. As an illustration consider the particular case where $H_K(t)$ is diagonalizable for all times by a non-orthogonal system $\{\varphi_i\}$. If we denote by $h_i(t)$ the time-dependent eigenvalues of $H_K(t)$, the evolution and the decay laws of $\{\varphi_i\}$ read

$$U_t = \begin{pmatrix} \exp\left(-i\int_0^t h_1(t')\,dt'\right) & 0 \\ 0 & \exp\left(-i\int_0^t h_2(t')\,dt'\right) \end{pmatrix},$$

$$p_i(t) = \exp\left(\int_0^t 2\, Im\, h_i(t')\,dt'\right). \tag{19}$$

Clearly $\{U_t\}$ is not a semigroup and $p_i(t)$ is not exponential except for the case where $Im\, h_i(t)$ is time-independent.

As a more general illustration, we consider the evolution generated in $\mathcal{H}_K$ by the Hamiltonian

$$H_K(t) = \begin{pmatrix} m_1 - \frac{i\gamma_1}{2} & \varepsilon e^{i\omega t} \\ \varepsilon e^{-i\omega t} & m_2 - \frac{i\gamma_2}{2} \end{pmatrix}, \tag{20}$$

written in the basis for which γ is diagonal. This operator clearly satisfies the unitary sum rule (14), and has the property that it cannot be diagonalized in any time-independent basis for any $\varepsilon, \omega \neq 0$. The differential equation (13) for U_t can be solved exactly in this case, and yields

$$U_t = \frac{\exp(-[i(m_1+m_2)+(\gamma_1+\gamma_2/2)]\,t/2}{\sinh\chi} \begin{pmatrix} e^{(i/2)\omega t}\sinh\left(\chi+\frac{Dt}{2}\right) & -i\,e^{(i/2)\omega t}\sinh\frac{Dt}{2} \\ -i\,e^{-(i/2)\omega t}\sinh\frac{Dt}{2} & e^{-(i/2)\omega t}\sinh\left(\chi-\frac{Dt}{2}\right) \end{pmatrix} \tag{21}$$

with

$$D = \sqrt{\zeta^2 - 4\varepsilon^2}\,; \quad \zeta = \frac{\gamma_2-\gamma_1}{2} - i(m_1 - m_2 + \omega)\,; \quad \cosh\chi = \frac{\zeta}{2\varepsilon}. \tag{22}$$

The quantity D is defined uniquely by the requirement that $D \to \zeta$ for $\varepsilon \to 0$. We remark that (21) does not have the semigroup property. To show that in this case there does not exist any state ϱ for which the decay law is a pure exponential, we consider the decay law to second order in (small) ε:

$$
\begin{aligned}
p_\varrho(t) &= \varrho_{11}\, e^{-\gamma_1 t} + \varrho_{22}\, e^{-\gamma_2 t} \\
&+ \frac{4\,\varepsilon^2}{|\zeta|^2} \left\{ \frac{e^{-\gamma_1 t} + e^{-\gamma_2 t}}{4} - \frac{1}{2}\, e^{-(\gamma_1+\gamma_2/2)\,t} \cos(m_2 - m_1 - \omega)\, t \right\} \\
&+ \frac{2\,\varepsilon}{|\zeta|^2} \Big\{ Re\,\varrho_{12} \left[(m_2 - m_1 - \omega)\,(e^{-\gamma_2 t} - e^{-\gamma_1 t}) + e^{-(\gamma_1+\gamma_2/2)\,t} (\gamma_2 - \gamma_1) \sin(m_2 - m_1 - \omega)\, t \right] \\
&+ Im\,\varrho_{12} \left[\frac{\gamma_1 - \gamma_2}{2}\,(e^{-\gamma_1 t} + e^{-\gamma_2 t}) + e^{-(\gamma_1+\gamma_2/2)\,t} (\gamma_2 - \gamma_1) \cos(m_2 - m_1 - \omega)\, t \right] \Big\}. \qquad (23)
\end{aligned}
$$

It is easy to see that the cancellation of all terms proportional to either of the two possible pure decay laws (for, e.g., $\gamma_1 \gg \gamma_2$) is inconsistent with the condition $Tr\,\varrho^2 \leqslant 1$, i.e., $|\varrho_{12}|^2 \leqslant \varrho_{11}\,\varrho_{22}$. In this model the decay of an arbitrary state takes place in a way which involves both lifetimes as well as oscillatory contributions.

We note that (20) cannot be constructed in a CP invariant theory. This follows immediately from the fact that γ would be diagonal in the CP basis, but $H_K(t)$, according to (20), is then not diagonal. In the next section we discuss the effect of symmetries and CP invariance.

IV. Symmetry and CP-Invariance

A symmetry is mathematically defined as a self-adjoint operator A in $\mathcal{H}$ which commutes with the time-evolution V_t at all times. If such a symmetry commutes also with the projection P_K of $\mathcal{H}$ into $\mathcal{H}_K$ (as does e.g. CP), then $A_K = P_K\ A\ P_K$ commutes with U_t at all times and is thus a *symmetry in* $\mathcal{H}_K$. This means that the eigenspaces of A_K reduce the operators U_t and if A_K is non-degenerate (which we may assume since in the 2-dimensional K^0-space $\mathcal{H}_K$ all degenerate operators are trivial) its eigenvectors φ_i diagonalize U_t at all times. Hence the unitary sum rule (12) reads, in the basis φ_i,

$$U_{ii}^+\,\gamma_{ik}\,U_{kk} + \frac{d}{dt}\,(U_{ii}^+\,\delta_{ik}\,U_{kk}) = 0 \qquad (24)$$

and we have, for $i \neq k$:

$$U_{ii}^+\,\gamma_{ik}\,U_{kk} = 0 \qquad \gamma_{ik} = 0\,. \qquad (25)$$

So the operator $\gamma = P_K\,T^+\,P_f\,T\,P_K$ in $\mathcal{H}_K$ commutes also with the symmetry A_K.

For $i = k$ equation (24) yields the first order differential equations

$$\gamma_{ii}\,|U_{ii}|^2 + \frac{d}{dt}\,|U_{ii}|^2 = 0 \qquad (26)$$

for the square of the modulus of the diagonal elements in the U_t-matrices. Together with the initial condition $U_0 = I$, (26) has the unique solutions $|U_{ii}|^2 = \exp(-\gamma_{ii}\,t)$ and the evolution

$$U_t = \begin{pmatrix} |U_{11}|\ e^{i\,\alpha_1(t)} & 0 \\ 0 & |U_{22}|\ e^{i\,\alpha_2(t)} \end{pmatrix} = \begin{pmatrix} e^{-(\gamma_{11}/2)\,t}\ e^{i\,\alpha_1(t)} & 0 \\ 0 & e^{-(\gamma_{22}/2)\,t}\ e^{i\,\alpha_2(t)} \end{pmatrix} \qquad (27)$$

is determined up to the real phases $\alpha_i(t)$. With this U_t the eigenstates φ_i of the symmetry operator A_K have the pure exponential decay laws

$$p_i(t) = e^{-\gamma_{ii} t} .$$

It is easy to see that if two pure states have pure exponential decay with different life-times, no other state ϱ, pure or mixed, can have pure exponential decay. Hence we have found that *in the presence of a symmetry A_K the eigenstates φ_i of A_K decay exponentially with respective life-times $1/\gamma_{ii}$ and these are the only states with pure exponential decay.*

If, in particular, the symmetry is CP, we conclude that *in a CP-invariant theory satisfying the unitary sum rule the two states with pure exponential decay are exactly the CP-eigenstates K_1 and K_2, and the decay constants are* [2]

$$\Gamma_i = \gamma_{ii} = \sum_f |(f, T K_i)|^2 . \tag{28}$$

The unitary sum rule in its strongest form (12) thus rules out any attempt of saving CP-invariance for the decay of neutral K-mesons, although it provides possibilities for evolutions which are more general than those usually considered. *In particular it does not imply that U_t is a semigroup nor that it is generated by a constant phenomenological Hamiltonian H_K as commonly postulated in the Wigner-Weisskopf scheme for K^0-decay.*

V. Some Possible Alternatives to the Unitary Sum Rule

The unitary sum rule depends on the validity of equation (5), which links the transitions $K \to f$ per unit time to the scattering amplitude T. It depends on the detailed properties of the S-matrix and should be verified whenever an explicit model is considered.

In particular, the situation may arise where (11) holds only for certain states ϱ. Since we do not have available a complete theory of K-meson decay, we have no way to decide whether this situation occurs. Nevertheless, we may pose the following converse question: *Suppose we want a CP-invariant theory of K-decay compatible with the main experimental fact that the long-lived K produces 2π-states at times comparable to $1/\Gamma_L$. For what states ϱ does the unitary sum rule* (11) *still hold?*

In the CP basis K_i the CP-invariant evolution U_t is diagonal. Writing equation (11) out for this case we obtain the first order differential equation

$$\left(\gamma_{11} |U_{11}|^2 + \frac{d}{dt} |U_{11}|^2\right) \varrho_{11} + \left(\gamma_{22} |U_{22}|^2 + \frac{d}{dt} |U_{22}|^2\right) \varrho_{22} = 0 . \tag{29}$$

If this equation were to hold for arbitrary states ϱ_{11}, $\varrho_{22} = 1 - \varrho_{11}$, it would immediately decouple as in (26) with the solutions $|U_{ii}| = \exp(-\gamma_{ii}/2\, t)$, and for large times $t \sim 1/\gamma_{22}$ the K_1-component would have decayed so that no long-lived 2π-production could occur. So let us assume solutions of a more general type

$$|U_{ii}|^2 = \alpha_i\, e^{-\Gamma_1 t} + (1 - \alpha_i)\, e^{-\Gamma_2 t} ; \qquad 0 \leqslant \alpha_i \leqslant 1 \tag{30}$$

compatible with $U_0 = I$. (Note that $|U_{ii}|^2$ remains positive for all times.) Inserting this into (29) we find

$$\frac{\alpha_1}{\alpha_2} = -\frac{1-\varrho_{11}}{\varrho_{11}} \frac{\gamma_{22}-\Gamma_1}{\gamma_{11}-\Gamma_1} ; \qquad \frac{1-\alpha_1}{1-\alpha_2} = -\frac{1-\varrho_{11}}{\varrho_{11}} \frac{\gamma_{22}-\Gamma_2}{\gamma_{11}-\Gamma_2} . \tag{31}$$

Again this can hold for all states ϱ if and only if $\Gamma_i = \gamma_{ii}$ and either $\alpha_1 = 1$, $\alpha_2 = 0$ or $\alpha_1 = 0$, $\alpha_2 = 1$, in which cases (30) is trivial. If, on the other hand, (29) is postulated to hold just for one fixed state, (31) gives nontrivial possibilities for the choice of α_i and the Γ_i's in terms of γ, uniquely fixing ϱ_{11}. According to (30) the decay of the CP-eigenstates K_i would then not be purely exponential and the 2π production at large time could be adjusted to experiment.

It is amusing to note that the form (30), consistent with the existence of a long-lived 2π mode, does not admit any states ϱ_S, ϱ_L with pure exponential decay laws even if we abandon the unitary sum rule altogether. Long-lived 2π production implies that $0 < \alpha_1 < 1$ in (30). The decay law for arbitrary ϱ is

$$Tr\, U_t \varrho U_t^+ = [\varrho_{11} \alpha_1 + (1 - \varrho_{11}) \alpha_2]\, e^{-\Gamma_S t} + [\varrho_{11} (1 - \alpha_1) + (1 - \varrho_{11}) (1 - \alpha_2)]\, e^{-\Gamma_L t} .$$

For pure decay laws for ϱ_S and ϱ_L, we therefore require

$$(\varrho_S)_{11} (1 - \alpha_1) + (1 - (\varrho_S)_{11}) (1 - \alpha_2) = 0 ; \qquad (\varrho_L)_{11} \alpha_1 + (1 - (\varrho_L)_{11}) \alpha_2 = 0 .$$

Solving for ϱ_S, ϱ_L, we find

$$0 \leqslant (\varrho_S)_{11} = \frac{1-\alpha_2}{\alpha_1-\alpha_2} \leqslant 1 ; \qquad 0 \leqslant (\varrho_L)_{11} = \frac{\alpha_2}{\alpha_2-\alpha_1} \leqslant 1 .$$

But these inequalities contradict $0 < \alpha_1 < 1$.

We wish to thank G. Mohan and H. Neumann for helpful discussions.

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