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On Nonrelativistic Positive-α Landau Surfaces

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Abstract. Results previously proved for relativistic positive-α Landau surfaces are extended to the nonrelativistic case. It is proved that in the physical region, at points where no two initial and no two final particle momenta are parallel, leading surfaces are real analytic submanifolds of codimension 1. The normal to a leading surface at a manifold point is proved to determine, in an essentially unique way, the Coleman-Norton space-time picture of the corresponding multiple scattering process.

1.

The formulation of the Faddeev-Yakubovsky equations [1] has made possible a mathematically rigorous study of the nonrelativistic multichannel S-matrices generated by a large class of many-body Hamiltonian operators. Hepp and Riahi [2] have established, at least for certain repulsive interactions, that in the physical region of momentum space the nonrelativistic scattering functions are holomorphic except on positive-α Landau surfaces. Properties of these surfaces are therefore of interest.

In the relativistic case two properties are of particular interest [3]. The first is that, at points where no two initial and no two final particle momenta are parallel, the leading surfaces are real analytic submanifolds of the physical region of codimension 1. Associated with these leading surfaces are, according to Coleman and Norton [4], space-time pictures of multiple scattering processes. The second result is that at a manifold point the normal to a leading surface determines the corresponding space-time picture in an essentially unique way.

The purpose of this note is to modify the proofs of [3] so that they apply to the nonrelativistic problem. A summary of notation is given in Section II. Precise statements of the results, together with proofs, are in Section III.

2.

The formulation developed in this section is adapted from the relativistic terminology [3, 5]. As a result, particle creation and annihilation are allowed. The particles are, however, required to be stable.

Considered abstractly, a connected Feynman graph G is a collection of vertices connected by directed line segments. The vertices are labelled by an index set $\mathcal{V} =$

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$\{0, 1, \ldots, V\}$, the lines by an index set $\mathfrak{L} = \{1, \ldots, L\}$. The directions of the lines are represented by an incidence matrix $[e]$:

$$e(r, l) = +1 \text{ if line } l \in \mathfrak{L} \text{ is directed into vertex } r \in \mathfrak{V},$$
$$= -1 \text{ if line } l \text{ is directed out of vertex } r,$$
$$= 0 \text{ otherwise.}$$

The connectedness of G implies rank $[e] = V$.

Each line l of G is associated with a particle of momentum $\boldsymbol{p}_l$, nonzero mass μ_l, and energy p_l^0.

The initial and final particles of the scattering process corresponding to G are associated with the vertices. They are indexed by a set $\mathfrak{L}_0 = \{1, \ldots, N\}$, and their distribution among the vertices is described by an incidence matrix $[f]$:

$$f(r, j) = +1 \text{ if particle } j \in \mathfrak{L}_0 \text{ is incident on vertex } r \in \mathfrak{V} \text{ and is a final particle,}$$
$$= -1 \text{ is particle } j \text{ is incident on vertex } r \text{ and is an initial particle,}$$
$$= 0 \text{ otherwise.}$$

The momentum of particle j is denoted by $\boldsymbol{k}_j$, its nonzero mass by m_j, and its energy by k_j^0.

Particle stability is introduced by requiring that at each vertex there are at least two incoming and two outgoing particles. This is expressed mathematically by requiring each row of the total incidence matrix $[e, -f]$ to have at least two positive and two negative entries.

Energy and momentum are conserved at each vertex, leading to the requirement that $K = (k_1^0, \boldsymbol{k}_1; \ldots; k_N^0, \boldsymbol{k}_N)$ belong to the manifold

$$\mathfrak{E} = \{ K \mid \sum_{r,j} f(r, j)\, k_j^0 = 0 , \quad \sum_{r,j} f(r, j)\, \boldsymbol{k}_j = 0\} . \tag{2.1}$$

The physical region is defined as

$$\mathfrak{P} = \{K \in \mathfrak{E} \mid k_j^0 = (2\, m_j)^{-1}\, \boldsymbol{k}_j^2 , \ j \in \mathfrak{L}_0\} . \tag{2.2}$$

The subset of points of $\mathfrak{P}$ at which two (or more) initial particle momenta $\boldsymbol{k}_j$, or two (or more) final particle momenta, are parallel is denoted by $\mathfrak{P}_0$.

The nonrelativistic Landau equations for the graph G are [6]:

$$\sum_{\mathfrak{L}} e(r, l)\, p_l^0 = q_r^0(K) \equiv \sum_{\mathfrak{L}_0} f(r, j)\, k_j^0 , \quad (r \in \mathfrak{V}) , \tag{2.3a}$$

$$\sum_{\mathfrak{L}} e(r, l)\, \boldsymbol{p}_l = \boldsymbol{q}_r(K) \equiv \sum_{\mathfrak{L}_0} f(r, j)\, \boldsymbol{k}_j , \quad (r \in \mathfrak{V}) , \tag{2.3b}$$

$$p_l^0 = (2\, \mu_l)^{-1}\, \boldsymbol{p}_l^2 , \quad (l \in \mathfrak{L}) , \tag{2.3c}$$

$$\sum_{\mathfrak{L}} \eta(l, i)\, \alpha_l\, \boldsymbol{p}_l = 0 , \tag{2.3d}$$

$$\sum_{\mathfrak{L}} \eta(l, i)\, \alpha_l\, \mu_l = 0 . \tag{2.3e}$$

The matrix $[\eta]$ in (2.3d, e) is the loop matrix; it is any $L \times (L - V)$ matrix, the columns of which span the null space of the matrix $[e]$. The positive-α requirement is

$$\alpha_l \geqslant 0 , \quad (l \in \mathfrak{L}) , \tag{2.3f}$$

$$\sum \alpha_l = 1 . \tag{2.3g}$$

Solutions of (2.3) for a given K are denoted by $\{\alpha_l, \boldsymbol{p}_l\}$. Condition (2.3g) is introduced to exclude the trivial solution $\{0, \boldsymbol{p}_l\}$.

The positive-α Landau surface $\mathfrak{L}^+[G]$ is the set of all points K of $\mathfrak{E}$ at which equations (2.3) have a solution. The leading surface $\mathfrak{L}_0^+[G]$ is the subset of points of $\mathfrak{L}^+[G]$ that do not belong to any of the positive-α surfaces of contractions of G. It is trivial that $\mathfrak{L}_0^+[G]$ is empty if G is reducible in the sense that removal of one line results in a disconnected graph. Only the leading surfaces of irreducible graphs G are of interest therefore, and they can be alternatively defined as the subset of all points of $\mathfrak{L}^+[G]$ at which all solutions of (2.3) satisfy

$$\alpha_l > 0\,, \quad (l \in \mathfrak{L})\,. \tag{2.4}$$

3.

The first problem in the analysis of the nonrelativistic Landau equations is to describe how the different solutions at the same point K are related.

Proposition 1. Suppose that at $K \in \mathfrak{L}^+[G]$ there are two different solutions, $\{\alpha_l, \boldsymbol{p}_l\}$ and $\{\beta_l, \boldsymbol{r}_l\}$, of the Landau equations (2.3). Then the condition

$$\alpha_l\,(\boldsymbol{p}_l - \boldsymbol{r}_l) = \beta_l\,(\boldsymbol{p}_l - \boldsymbol{r}_l) = 0\,, \quad (l \in \mathfrak{L})\,, \tag{3.1}$$

must hold. If $K \in \mathfrak{L}_0^+[G]$, the solution $\{\alpha_l, \boldsymbol{p}_l\}$ is unique.

Proof: If (2.3a, b, c) hold for both $\boldsymbol{p}_l$ and $\boldsymbol{r}_l$, there must exist real quantities (θ_i^0, θ_i) such that

$$\boldsymbol{r}_l = \boldsymbol{p}_l + \sum_i \eta(l, i)\, \theta_i\,, \quad (l \in \mathfrak{L})\,, \tag{3.2}$$

$$\boldsymbol{r}_l^2 = \boldsymbol{p}_l^2 + \mu_l \sum_i \eta(l, i)\, \theta_i^0\,. \tag{3.3}$$

Substitution of the expression (3.2) for $\boldsymbol{r}_l$ into (3.3), multiplication by α_l, summation over l, and appeal to the loop equations (2.3d, e) yield

$$\sum_{\mathfrak{L}} \alpha_l [\sum_i \eta(l, i)\, \theta_i]^2 = 0\,. \tag{3.4}$$

The first of conditions (3.1) follows immediately from (3.2) and (3.4). The second condition follows from interchanging the two solutions $\{\alpha_l, \boldsymbol{p}_l\}$ and $\{\beta_l, \boldsymbol{r}_l\}$ in the preceding argument. To prove uniqueness at points of $\mathfrak{L}_0^+[G]$ it is sufficient to observe that, because of (3.1), other solutions of the form $\{\gamma_l = \varkappa_1\, \alpha_l + \varkappa_2\, \beta_l, \boldsymbol{p}_l\}$ exist. The only restrictions on the parameters $\varkappa_i$, $i = 1, 2$, are those imposed by (2.3f, g). Unless the solution is unique, that is unless $\alpha_l = \beta_l$ for all l, the numbers $\varkappa_i$ can clearly be chosen so that the γ_l satisfy (2.3f, g) but violate (2.4). Since this contradicts the assumption that K belongs to $\mathfrak{L}_0^+[G]$, the solution $\{\alpha_l, \boldsymbol{p}_l\}$ must be unique. QED.

Proposition 1 is stronger than the corresponding relativistic result, where (3.1) has been proved only under the assumption that $K \in \mathfrak{L}_0^+[G]$. This difference can be traced to the more complicated form of the relativistic counterpart of (2.3c). Efforts to circumvent this complication, thereby strengthening the relativistic result, have so far failed.

The next step of the analysis of the Landau equations is to restrict attention to irreducible graphs and to write the equations in a slightly different form. Let K be any point of $\mathfrak{L}_0^+[G]$, and let $\{\alpha_l, \boldsymbol{p}_l\}$ be the corresponding solution of (2.3) and (2.4). The loop equations (2.3d, e) imply that $\{\alpha_l, \boldsymbol{p}_l\}$ has a form compatible with

$$\alpha_l \, \mu_l = \Delta_l(X^0) \equiv \sum_{\mathfrak{V}} e(r, l) \, x_r^0 \, ,$$

$$\alpha_l \, \boldsymbol{p}_l = \Delta_l(\boldsymbol{X}) \equiv \sum_{\mathfrak{V}} e(r, l) \, \boldsymbol{x}_r \, . \tag{3.5}$$

The real quantity $X = (X^0, \boldsymbol{X}) = (x_0^0, \ldots, x_V^0, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_V)$ is unconstrained by the loop equations but is, of course, constrained by (2.3f, g) and (2.4). Substitution of (3.5) and (2.3c) into (2.3a, b) yields the following new form for the Landau equations (2.3) and (2.4):

$$q_r^0(K) = f_r^0(X) \equiv 1/2 \sum_{\mathfrak{L}} e(r, l) \, \mu_l [\Delta_l(X^0)]^{-2} \, [\Delta_l(\boldsymbol{X})]^2 \, , \quad (r \in \mathfrak{V}) \, , \tag{3.6a}$$

$$\boldsymbol{q}_r(K) = \boldsymbol{f}_r(X) \equiv \sum_{\mathfrak{L}} e(r, l) \, \mu_l [\Delta_l(X^0)]^{-1} \, \Delta_l(\boldsymbol{X}) \, , \quad (r \in \mathfrak{V}) \, , \tag{3.6b}$$

$$\Delta_l(X^0) > 0 \, , \quad (l \in \mathfrak{L}) \, , \tag{3.7}$$

$$\sum_{\mathfrak{L}} \mu_l^{-1} \, \Delta_l(X^0) = 1 \, . \tag{3.8}$$

Equations (3.6) are, in the sequel, frequently denoted by the expression $Q(K) = F(X)$.

Consider X now as a variable. The mapping $F(X)$ defined by (3.6) clearly is well defined and real analytic so long as (3.7) is satisfied. It is homogeneous of degree zero and translationally invariant. That is, if X and Y are related by

$$y_r^0 = \lambda \, x_r^0 + a^0 \, ,$$

$$\boldsymbol{y}_r = \lambda \, \boldsymbol{x}_r + \boldsymbol{a} \, , \tag{3.9}$$

where $(\lambda \neq 0, a^0, \boldsymbol{a})$ are arbitrary, then $F(X) = F(Y)$. This invariance implies that the null space of the matrix $[H(X)]$, the Jacobian matrix of the mapping F evaluated at the point X, contains all vectors Y of the form (3.9). For no point X, therefore, does this null space have dimension less than five.

The important point is that if $K \in \mathfrak{L}_0^+[G]$ and $Q(K) = F(X)$, then the dimension of the null space of $[H(X)]$ is exactly five. The equations that $Y = (Y^0, \boldsymbol{Y})$ must satisfy to belong to the null space of $[H(X)]$ are, explicitly,

$$\sum_{\mathfrak{L}} e(r, l) \, \mu_l [\Delta_l(X^0)]^{-2} \, \Delta_l(\boldsymbol{X}) \cdot \boldsymbol{V}_l = 0 \, , \tag{3.10a}$$

$$\sum_{\mathfrak{L}} e(r, l) \, \mu_l [\Delta_l(X^0)]^{-1} \, \boldsymbol{V}_l = 0 \, , \tag{3.10b}$$

where

$$\boldsymbol{V}_l = \Delta_l(\boldsymbol{Y}) - [\Delta_l(X^0)]^{-1} \, \Delta_l(Y^0) \, \Delta_l(\boldsymbol{X}) \, . \tag{3.10c}$$

Multiply (3.10a) by y_r^0 and sum over r, multiply (3.10b) by $\boldsymbol{y}_r$ and sum over r, and take the difference of the two expressions. The result is

$$\sum_{\mathfrak{L}} \mu_l [\Delta_l(X^0)]^{-1} \, \boldsymbol{V}_l^2 = 0 \, . \tag{3.11}$$

It now follows from (3.5), (3.7), (3.10c) and (3.11) that there must exist nonzero $\varkappa_1$ and $\varkappa_2$ such that $\{\gamma_l = \mu_l^{-1} (\varkappa_1 \Delta_l(X^0) + \varkappa_2 \Delta_l(Y^0)), \mu_l \Delta_l(X^0)^{-1} \Delta_l(\mathbf{X})\}$ is a solution to (2.3) and (2.4). According to Proposition 1, the only possibility is that $\Delta_l(Y^0) = \lambda \Delta_l(X^0)$, where λ is some nonzero parameter, for all $l \in \mathfrak{L}$. This, and (3.11), implies in turn that Y has the form (3.9). Consequently the null space of $[H(X)]$ has dimension five.

The preceding argument also shows that the null space of $[H(X)]$ has dimension greater than five if $Q(K) = F(X)$ and $K \notin \mathfrak{L}_0^+[G]$. Thus, the following proposition is proved.

Proposition 2. Let the notation of the preceding paragraphs be adopted. Then, at a point X satisfying (3.7), the rank of $[H(X)]$ is $(4V - 1)$ if and only if $Q(K) = F(X)$ and $K \in \mathfrak{L}_0^+[G]$.

As shown by (3.9) the point X is not completely specified by the Landau equations. The ambiguity is resolved, consistent with (3.7) and (3.8), by supposing that X belongs to the set

$$\Omega^+ = \{Y = (Y^0, \mathbf{Y}) \mid \sum y_r^0 = \sum \mathbf{y}_r = 0;\ \Delta_l(Y^0) > 0, l \in \mathfrak{L};\ \sum \mu_l^{-1} \Delta_l(Y^0) = 1\}. \tag{3.12}$$

Suppose now that $K \in \mathfrak{L}_0^+[G]$ and that $X \in \Omega^+$ is the unique point satisfying $Q(K) = F(X)$. Then, there is a neighborhood $\Delta \subset \Omega^+$ of X on which the restriction $\tilde{F}$ of F to Ω^+ is a nonsingular real analytic mapping. This implies in turn that the set $\{K \in \mathfrak{E} \mid Q(K) \in \tilde{F}(\Delta)\}$, which is a subset of $\mathfrak{L}_0^+[G]$, is a real analytic submanifold of $\mathfrak{E}$ of codimension 1.

From this observation it is only a small step to the following result.

Proposition 3. Let K be any point of $\mathfrak{L}_0^+[G]$, and let X be the (unique) point of Ω^+ that satisfies $Q(K) = \tilde{F}(X)$. Then there are neighborhoods $\mathfrak{N} \subset \mathfrak{E}$ of K and $\Delta \subset \Omega^+$ of X such that $\mathfrak{N} \cap \mathfrak{L}_0^+[G] = \mathfrak{N} \cap \tilde{F}(\Delta)$. Consequently, the set $\mathfrak{L}_0^+[G]$, if it is not empty, is a real analytic submanifold of $\mathfrak{E}$ of codimension 1.

Proof: The Proof is taken from Appendix E of [3] and is sketched here only for completeness. Let $\Delta \subset \Omega^+$ be some fixed neighborhood of X on which $\tilde{F}$ is nonsingular. In order for the proposition to be false, there must be a sequence of points $K_i \in \mathfrak{L}_0^+[G]$ converging to K such that no subsequence of the corresponding points $X_i \in \Omega^+$ converges to X. Without loss of generality it may be assumed that the sequence $\{K_i\}$ is contained in a compact set. Equations (3.6a) then insure that the sequence $\{X_i\}$ is contained in a compact subset of the closure of Ω^+. The sequence $\{X_i\}$ therefore has a limit point $\tilde{X}$. If $\tilde{X} \in \Omega^+$, the continuity of $\tilde{F}$ implies that $Q(K) = \tilde{F}(\tilde{X})$ and hence that $\tilde{X} = X$. If $\tilde{X} \notin \Omega^+$, then $\tilde{X}$ corresponds to some diagram that is a contraction of G. By returning to the original Landau equations (2.3), it is easily proved that $\tilde{X}$ corresponds to a solution $\{\alpha_l, \mathbf{p}_l\}$ at K that violates (2.4). Since $K \in \mathfrak{L}_0^+[G]$, the only possibility is that $\tilde{X} \in \Omega^+$ and hence that the sequence $\{X_i\}$ has a such subsequence converging to X. QED.

By Proposition 3 there are at each point $\bar{K} \in \mathfrak{L}_0^+[G]$ a neighborhood $\mathfrak{N} \subset \mathfrak{E}$ of $\bar{K}$ and a real analytic function Λ, defined and with nonzero gradient ∇_K and Λ on $\mathfrak{N}$, such that

$$\mathfrak{N} \cap \mathfrak{L}_0^+[G] = \{K \in \mathfrak{N} \mid \Lambda(K) = 0\}. \tag{3.13}$$

Since K enters the Landau equations only in the combination $Q(K)$, the function Λ can depend only on the same combination. That is, there is a real analytic function $\Phi(Q)$, with nonvanishing gradient $\nabla_Q \Phi$, such that $\Lambda(K) = \Phi(Q(K))$. The gradient $\nabla_Q \Phi(Q(\bar{K}))$ must, because $\Phi(\tilde{F}(X))$ vanishes identically on a neighborhood of the point $\bar{X} \in \Omega^+$ corresponding to $\bar{K}$, belong to the null space of $[H(\bar{X})]$. This implies that $\nabla_Q \Phi(Q(\bar{K}))$ has the form (3.9) and hence that $\nabla_K \Lambda(\bar{K})$ has the form

$$(\partial\Lambda/\partial k^0)\,(\bar{K}) = \sum_{\mathfrak{V}} f(r, j)\,[\lambda\,\bar{x}_r^0 + a^0]\,,$$

$$(\partial\Lambda/\partial \boldsymbol{k}_j)\,(\bar{K}) = \sum_{\mathfrak{V}} f(r, j)\,[\lambda\,\bar{\boldsymbol{x}}_r + \boldsymbol{a}]\,, \quad (j \in \mathfrak{L}_0)\,, \tag{3.14}$$

Equations (3.14) lead immediately to the following proposition.

Proposition 4. Let $K \in \mathfrak{L}_0^+[G]$, and let X be the (unique) corresponding point in Ω^+. Let Λ be the real analytic function, the zeros of which define $\mathfrak{L}_0^+[G]$ in a neighborhood of K. Then (3.14) must hold. The quantities $(x_r^0, \boldsymbol{x}_r)$ have the natural interpretation [4] as the space-time positions of the scatterings that make up the multiple scattering process corresponding to G. Thus, apart from an over-all scaling and translation, the normal at K to $\mathfrak{L}_0^+[G]$ defines the corresponding space-time picture of the process.

Propositions 3 and 4 can now be applied to prove the final result.

Proposition 5. The set $(\mathfrak{P} - \mathfrak{P}_0) \cap \mathfrak{L}_0^+[G]$ is either empty or a submanifold of $(\mathfrak{P} - \mathfrak{P}_0)$ of codimension 1.

Proof: Let $K \in (\mathfrak{P} - \mathfrak{P}_0) \cap \mathfrak{L}^+[G]$. The condition for the proposition to be false at K is that the gradient $\nabla\Lambda$ have zero projection on the tangent space of $\mathfrak{P}$.

This implies that $\nabla\Lambda$ has form

$$(\partial\Lambda/\partial k_j^0)\,(\bar{K}) = \sigma_j\,a^0 + \gamma_j\,,$$

$$(\partial\Lambda/\partial \boldsymbol{k}_j)\,(\bar{K}) = \sigma_j\,\boldsymbol{a} - m_j^{-1}\,\gamma_j\,\boldsymbol{k}_j\,, \quad (j \in \mathfrak{L}_0)\,, \tag{3.15}$$

$$\sigma_j = \sum_{\mathfrak{V}} f(r, j)\,.$$

The quantities $(\gamma_1, \ldots, \gamma_N, a^0, \boldsymbol{a})$ are arbitrary. Particle stability conditions, together with (3.14), now imply, just as in relativistic theory, that all of the γ_j are zero. This means that $\bar{X}$ in (3.14) is zero, an impossibility since $\bar{X} \in \Omega^+$. The proposition must therefore be true. QED.

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