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Conformal Group → Schrödinger Group → Dynamical Group – The Maximal Kinematical Group of the Massive Schrödinger Particle

by **A. O. Barut**[1])
International Centre for Theoretical Physics, Trieste, Italy

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Abstract. We determine the 15-parameter non-relativistic limit of the conformal group, equation (11), and its linear realization in the six-dimensional space, equation (22). We show that the *massive* free Schrödinger equation is invariant under a 15-parameter set of transformations enlarging the 12-parameter Schrödinger group, equations (13), (14) and (17). We show how this set arises from the conformal group and determine the quantum-mechanical representation of the generators.

I. Introduction

It has recently been pointed out that the free Schrödinger equations for a spinless particle of mass m admits a 12-parameter symmetry group (called the Schrödinger group) containing the Galilei transformations [1]–[4]. This result is, at first sight, surprising for two reasons: first, that a simple *massive* particle would obey a kind of conformal and dilatation symmetry *at all* and, second, that the dilatation and conformal transformations in question have a rather strange structure and, as they stand, are not contractions of the usual conformal transformations of the Minkowski space.

The purpose of this paper is to show that the Schrödinger group does actually arise from the conformal group by a combined process of contraction and a 'transfer' of the transformation of mass to the co-ordinates, because the mass occurs as a factor of $\partial/\partial t$ in the Schrödinger operator $(2m\partial_t - \partial_i \partial_i)$. We further show that actually a 15-parameter set of transformations, which acts as a symmetry on the space of solutions of the Schrödinger operator, can be defined. The new 3-parameter set relates to the special conformal transformations. Furthermore, we show how the dynamical group $O(2,1)$ occurs in this process, which plays the role of the spectrum-generating algebra for some classes of interactions. However, the Schrödinger group and the new quasi-conformal group still refer to a single particle; no new internal degrees of freedom are described by the representations of the larger group on the space of the Schrödinger wave functions. We also discuss the linear realization of the group in the six-dimensional space. It is thus appropriate that the largest kinematical group of the Schrödinger equation is

[1]) On leave of absence from the University of Colorado, Boulder, Colo., USA.

related in a natural way to the largest kinematical group of Maxwell's equations, namely the conformal group.

II. The Relativistic Conformal Group

We start from the Lie algebra of the 15-parameter conformal group of the Minkowski space M, which can be represented, on the space of the scalar functions over M, by the following differential operators:

$$M_{\mu\nu}=x_\mu\,\partial_\nu-x_\nu\,\partial_\mu,\quad P_\mu=\partial_\mu,\quad K_\mu=2x_\mu x^\nu\,\partial_\nu-x^2\,\partial_\mu,\quad D=x^\nu\,\partial_\nu, \tag{1}$$

where $M_{\mu\nu}$ and P_μ are the basis elements of the Poincaré sub-algebra, K_μ generates the so-called special conformal transformations, and D generates the dilatations. We also need the commutation relations, which are easily found from equation (1):

$$[M_{\mu\nu},M_{\sigma\rho}]=g_{\mu\rho}M_{\nu\sigma}+g_{\nu\sigma}M_{\mu\rho}-g_{\mu\sigma}M_{\nu\rho}-g_{\nu\rho}M_{\mu\sigma}$$

$$[M_{\mu\nu},P_\lambda]=g_{\nu\lambda}P_\mu-g_{\mu\lambda}P_\nu$$

$$[M_{\mu\nu},K_\lambda]=g_{\nu\lambda}K_\mu-g_{\mu\lambda}K_\nu;\quad [M_{\mu\nu},D]=0$$

$$[P_\mu,P_\nu]=0;\quad [P_\mu,K_\nu]=2(g_{\mu\nu}D-M_{\mu\nu})$$

$$[P_\mu,D]=P_\mu;\quad [K_\mu,K_\nu]=0;\quad [K_\mu,D]=-K_\mu. \tag{2}$$

The finite conformal transformations can be parametrized by the following non-linear realization in the Minkowski space:

$$x'_\mu=\Lambda^\nu_\mu x_\nu+a_\mu,\quad x'_\mu=\rho x_\mu\quad\text{and}\quad x'_\mu=\frac{x_\mu+c_\mu x^2}{1+2c^\mu x_\mu+c^2x^2}\,. \tag{3}$$

The relativistic massive scalar particle wave equation,

$$(\Box^2-m^2c^2)\,\varphi=0,\quad(\hbar=1),$$

is *formally* invariant under the conformal group, provided the mass m is transformed as $\Box^2$, i.e.

$$m^2\to e^{2\alpha}m^2,\quad\text{under dilatations,}$$

$$m^2\to\sigma(x)^2m^2,\quad\text{under global special conformal transformations,} \tag{4}$$

where $\sigma(x)=1+2c^\mu x_\mu+c^2x^2$ (see equation (3)). This is *not* a symmetry transformation for the particle of definite mass m, because it connects the states of the particle of mass m with those of other particles of mass m'. However, after the contraction to the non-relativistic case, we shall see that the transformation (4) of m, can be transferred to the co-ordinates, so that we get back a symmetry transformation for a particle of fixed mass m. This is because, after the contraction, the mass occurs as a factor of $\partial/\partial t$ or of $\partial^2/\partial x^2$.

III. The Contraction

The group contraction is carried out by simply setting in equation (1):

$$\partial_0 \rightarrow mc + \frac{1}{c}\partial_t. \tag{5}$$

Then the generators of the conformal group become

$$P_0 = mc + \frac{1}{c}\partial_t, \quad P_i = \partial_i, \quad M_{ij} = (x_i \partial_j - x_j \partial_i)$$

$$M_{0i} = c(t\partial_i - mx_i) - \frac{1}{c} x_i \partial_t$$

$$K_i = c^2(2x_i mt - t^2 \partial_i) + (2x_i t\partial_t - 2x_i x_\kappa \partial_\kappa + x^2 \partial_i)$$

$$K_0 = c^3 mt^2 + c(t^2 \partial_t - 2tx_\kappa \partial_\kappa + m\mathbf{x}^2) + \frac{1}{c}\mathbf{x}^2 \partial_t$$

$$D = c^2 mt + (t\partial_t - x_\kappa \partial_\kappa). \tag{6}$$

At the same time, the wave operator becomes

$$(\Box^2 - m^2 c^2) \rightarrow (2m\partial_t - \partial_i \partial_i) + \frac{1}{c^2}\partial_t \partial_t, \tag{7}$$

i.e. the Schrödinger operator plus an additional term of order $1/c^2$. From equations (5) and (6), various groups can be identified in different orders in c.

Corresponding to the expansion (6) of the generators P_μ, $M_{\mu\nu}$, K_μ, D, we introduce the following redefinition of the associated group parameters a_μ, $\Lambda_{\mu\nu}$, c_μ and ρ:

$$a_0 = c\tau$$

$$\Lambda_{0i} = \frac{1}{c} v_i$$

$$\rho = \rho$$

$$c_0 = \frac{1}{c}\alpha$$

$$c_i = \frac{1}{c^2}\beta_i. \tag{8}$$

With these substitutions the non-linear realization of the conformal transformations in the Minkowski space, given in equation (3), become

$$\left.\begin{aligned} x'_i &= v_i t + \Lambda_i^j x_j + a_i \\ t' &= \Lambda_0^0 t + \tau + \frac{1}{c^2}\mathbf{v}\cdot\mathbf{x} \end{aligned}\right\} \tag{9a}$$

$$x'_\mu = \rho x_\mu \tag{9b}$$

$$\left.\begin{aligned} t' &= \frac{t + \alpha t^2 - \frac{1}{c^2}\mathbf{x}^2}{(1+\alpha t)^2 + \frac{1}{c^2}[2\boldsymbol{\beta}\cdot\mathbf{x} - \mathbf{x}^2 - \beta^2\,\mathbf{x}^2]} \\ x'_i &= \frac{x_i + \beta_i t^2 - \frac{1}{c^2}\beta_i\,\mathbf{x}^2}{(1+\alpha t)^2 + \frac{1}{c^2}[2\boldsymbol{\beta}\cdot\mathbf{x} - \mathbf{x}^2 - \beta^2\,\mathbf{x}^2]}\,. \end{aligned}\right\} \tag{9c}$$

Furthermore, if we consider the group elements $e^{\theta L}$, the exponents have the form

$$\begin{aligned} a_0\,P_0 &= mc^2\,\tau + \tau\partial_t \\ a_i\,P_i &= a_i\,\partial_i \\ \Lambda_{ij}\,M_{ij} &= \Lambda_{ij}\,M_{ij} \\ \Lambda_{0i}\,M_{0i} &= v_i(t\partial_i - mx_i) - \frac{1}{c^2}v_i\,x_i\,\partial_t \\ \rho D &= \rho[mc^2\,t + (t\partial_t - x_\kappa\,\partial_\kappa)] \\ c_0\,K_0 &= \alpha[mc^2\,t^2 + (t^2\,\partial_t - 2tx_\kappa\,\partial_\kappa + mx^2)] + \frac{1}{c^2}\alpha x^2\,\partial_t \\ c_i\,K_i &= \beta_i(2x_i\,mt - t^2\,\partial_i) + \frac{\beta_i}{c^2}(2x_i\,t\partial_t - 2x_i\,x_\kappa\,\partial_\kappa + x^2\,\partial_i)\,. \end{aligned} \tag{10}$$

IV. Non-Relativistic Conformal Group

Equations (6), (9) and (10) are still exact. Now we go to the limit $c \to \infty$ (contraction). First, in equations (9), we obtain the co-ordinate form of the galilean transformations, plus the dilatations, and the non-relativistic special conformal transformations

$$x'_\mu = \rho x_\mu$$

$$t' = \frac{t}{1+\alpha t}\,, \quad x'_i = \frac{x_i + \beta_i t^2}{(1+\alpha t)^2}\,. \tag{11}$$

But the 15-parameter group of the non-relativistic transformations does not commute with the Schrödinger operator $(2m\partial_t - \partial_i \partial_i)$, which is the contraction of the relativistic wave operator according to equation (7). However, because in the massive wave operator $(\Box^2 - m^2 c^2)$ we have formally also transformed mass m according to equation (4), the transformations (11) are the symmetry operation for the Schrödinger operator, if m also transforms according to (4):

$$m' = \rho m$$

$$m' = (1 + \alpha t)\, m \quad \text{(independent of } \boldsymbol{\beta}\text{)}. \tag{12}$$

Conversely, we can fix m, but change the transformation property of t and $\mathbf{x}$ in such a way that we obtain symmetry operations for the Schrödinger operator $(2m\partial_t - \partial_i \partial_i)$ with fixed mass m. Comparing (11) and (12), we see that this is easily possible for dilatations and the time component of the conformal transformation with parameter α. Thus, instead of (11) and (12), we can write

$$m' = m,$$

$$t' = \rho^2 t, \quad \mathbf{x}' = \rho \mathbf{x},$$

for dilatation, and

$$t' = \frac{t}{(1 + \alpha t)}, \quad \mathbf{x}' = \frac{\mathbf{x} + \boldsymbol{\beta} t^2}{(1 + \alpha t)}, \tag{13}$$

for special conformal transformations.

Equation (13) together with the galilean transformations are the transformations of the generalized Schrödinger group. (The β term is not present in Ref. [1].)

V. Quantum Mechanical Representation

Now we come to the representations of the generators. In equations (10) we recognize in the first four lines the generators of the Galilei group, or rather the quantum mechanical extension of the Galilei group, if we drop terms of order $1/c^2$ in the non-relativistic limit. The term $mc^2 \tau$ in the first can be treated as $\tau \partial_t$ because mc^2 has the transformation property of ∂_t. If we treat m in the generator of the pure Galilei transformations, $\Lambda_{0i} M_{0i}$, as a number, we arrive at the commutation relation $[M_{0i}, P_j] = im\,\delta_{ij}$, and consequently to the superselection rule for mass [5], which is thus a result of the basic limit in equation (5). We further observe in equations (10) the occurrence of terms $mc^2 t$ and $mc^2 t^2$ in the generators D and K_0, respectively. Because we are now interested in the symmetry transformations of the Schrödinger operator $2m\partial_t - \partial_i \partial_i$, for fixed m, we must treat mc^2 in these two expressions again as ∂_t, and we obtain, denoting the modified operators by a tilde:

$$\tilde{D} = 2t\partial_t - x_\kappa \partial_\kappa + \tfrac{3}{2} = 2t\partial_t - \tfrac{1}{2}(x_\kappa \partial_\kappa + \partial_\kappa x_\kappa)$$

$$\tilde{K}_0 = 2t^2 \partial_t - 2t x_\kappa \partial_\kappa + m\mathbf{x}^2 + 3t = 2t^2 \partial_t - t(x_\kappa \partial_\kappa + \partial_\kappa x_\kappa) + m\mathbf{x}^2. \tag{14}$$

Equations (14), together with the Lie algebra of the galilean group, indeed form the Lie algebra of the Schrödinger group [1]. We find

$$\left.\begin{aligned} &[\tilde{D}, \tilde{K}_0] = 2\tilde{K}_0 \\ \text{and} \\ &[\partial_t, \tilde{D}] = 2\partial_t = 2\frac{\partial \tilde{D}}{\partial t}, \quad [\partial_t, \tilde{K}_0] = 2\tilde{D} = 2\frac{\partial \tilde{K}_0}{\partial t}. \end{aligned}\right\} \tag{15a}$$

Whereas the galilean generators commute with the Schrödinger operator $S = 2m\partial_t - \partial_i \partial_i$, we now have

$$[S, \tilde{D}] = 2S, \quad [S, \tilde{K}_0] = 4tS, \tag{15b}$$

so that $\tilde{D}$ and $\tilde{K}_0$ are the symmetry operators on the space $\mathscr{H}$ of solutions of the Schrödinger equation, $S\psi = 0$. On $\mathscr{H}$ we can replace $i\partial_t$ by $(1/2m)p^2$.

The 3-parameter Lie algebra (15) generated by $\tilde{D}$, $\tilde{K}_0$, ∂_t, isomorphic to $su(1,1)$, is related to the *dynamical algebra* of the harmonic oscillator at $t = 0$, i.e. $\tilde{D}(0) = -\frac{1}{2}(\mathbf{p}\cdot\mathbf{q} + \mathbf{q}\cdot\mathbf{p})$, $\tilde{K}_0(0) = \mathbf{q}^2$, $H_0 = \frac{1}{2}\mathbf{p}^2$. For the oscillator we diagonalize $H_0 + \frac{1}{2}\tilde{K}_0(0)$ with discrete spectrum. For a free particle we diagonalize $\frac{1}{2}\mathbf{p}^2$; but it is interesting that even for a free particle there exists a generator of a symmetry transformation with a discrete spectrum, namely $H_0 + \frac{1}{2}\tilde{K}_0(0)$. It is also interesting that we can diagonalize the generator $\tilde{D}$; the corresponding solutions are *self-similar* (auto-model) solutions of the wave equation. Of course, $\tilde{D}(t)$, $\tilde{K}_0(t)$, ... are related to $\tilde{D}(0)$, $\tilde{K}_0(0)$, ... by the relation

$$\tilde{D}(t) = e[-itH_0]\, \tilde{D}(0)\, e[itH_0], \quad \text{etc.}$$

Finally, we observe the last line in equations (10). The extraction of operators K_i, whose commutator with $2m\partial_t - \partial_i \partial_i$ would be a multiple of K_i, is here more involved. We first notice that (up to order $1/c^2$)

$$[2m\partial_t - \partial_l \partial_l, 2x_i t\partial_t - 2x_i x_\kappa \partial_\kappa + 2x_i + x_\kappa x_\kappa \partial_i] + [\partial_t \partial_t, 2x_i mt - t^2 \partial_i] = 4x_i(2m\partial_t - \partial_\kappa \partial_\kappa).$$

We transform the second commutator on the left-hand side (again using $mc^2 \sim \partial_t$) into the form

$$\left[2m\,\partial_t - \partial_l \partial_l, -\frac{2}{m}t^2 \partial_t \partial_i + 2tx_i \partial_t - x_i\right] + \text{terms commuting with } S. \tag{16}$$

The final form of the generators is then

$$\tilde{K}_i = -\frac{2}{m}t^2 \partial_t \partial_i + 4x_i t\partial_t - 2x_i x_\kappa \partial_\kappa + x_\kappa x_\kappa \partial_i + x_i, \text{ satisfying}$$

$$[2m\partial_t - \partial_l \partial_l, \tilde{K}_j] = 4x_j(2m\partial_t - \partial_i \partial_i). \tag{17}$$

The reason that $\tilde{K}_i$ were missed in Ref. [1] is due to the fact that this operator contains a second-order term $\partial_t \partial_i$, whereas in [1] only first-order differential operators were considered.

The operator $\tilde{K}_j$ has, in addition to (17) – which, by the way, is the analogue of (15b) – the following properties:

$$[\tilde{K}_i, \tilde{K}_j] = 0$$

$$[\partial_t, \tilde{K}_i] = \frac{\partial \tilde{K}_i}{\partial t} = -4(M_{0i})\frac{t}{m}$$

$$[\tilde{K}_i, M_{0j}] = 0$$

$$[\tilde{K}_i, P_j] = 2\delta_{ij} D - 2M_{ij}$$

$$[\tilde{K}_i, D] = -\tilde{K}_i. \tag{18}$$

From equation (2) these are all expected relations. Only the commutator $[\tilde{K}_i, \tilde{K}_0]$ does not seem to give anything simple, without the additional terms in (16).

VI. Linear Representation in Six-Dimensional Space

We now consider the linear realization of the conformal group in the six-dimensional space. With

$$\eta^\mu = \kappa x^\mu, \quad \lambda = \kappa x^2, \tag{19}$$

the six-dimensional space has the co-ordinates $(\eta^\mu, \kappa, \lambda)$. The dilatations and the special conformal transformations on this space are implemented by

$$D: \eta'^\mu = \eta^\mu, \quad \kappa' = \rho^{-1}\kappa, \quad \lambda' = \rho\lambda$$

$$C: \eta'^\mu = \eta^\mu + c^\mu \lambda$$

$$\kappa' = 2c_\nu \eta^\nu + \kappa + c^2 \lambda$$

$$\lambda' = \lambda. \tag{20}$$

In the non-relativistic limit it is convenient to define similarly

$$\tilde{\eta}^0 \equiv \kappa t, \quad \tilde{\eta}^i = \kappa x^i, \quad \tilde{\lambda} = \kappa t^2. \tag{21}$$

Then from (8) we have immediately the linear realization of (11):

$$\tilde{\eta}_0' = \tilde{\eta}^0 + \alpha\tilde{\lambda}$$

$$\tilde{\eta}^i = \eta^i + \beta^i \tilde{\lambda}$$

$$\kappa' = \kappa + 2\alpha\tilde{\eta}^0 + \alpha^2 \tilde{\lambda}$$

$$\tilde{\lambda}' = \tilde{\lambda}. \tag{22}$$

Finally, the generators of the conformal group in the linear realization are given by, in terms of definitions (21),

$$D = -\kappa \frac{\partial}{\partial\kappa} + \tilde{\lambda} \frac{\partial}{\partial\tilde{\lambda}}$$

$$K_\mu = -\tilde{\lambda} \frac{\partial}{\partial\tilde{\eta}^\mu} - 2\tilde{\eta}_\mu \frac{\partial}{\partial\kappa}. \tag{23}$$

VII. Conclusion

We have shown that the Schrödinger operator $(2m\partial_t - \partial_i \partial_i)$ of a free non-relativistic particle admits a 15-parameter set of invariant transformations. This set results precisely from the conformal group by contraction and by transfer of the scale properties of mass m to the co-ordinates.

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