# The maximal 'kinematical' invariance group for an arbitrary potential

Helvetica Physica Acta
Vol. 47, 1974.

Birkhäuser Verlag Basel

# The Maximal 'Kinematical' Invariance Group for an Arbitrary Potential

by **Charles P. Boyer**

C.I.M.A.S., Universidad Nacional Autónoma de México

(9. V. 74)

*Abstract.* We consider the problem of finding all local symmetries of the time-dependent Schrödinger equation in $n$ spatial dimensions with an arbitrary time-independent potential. This problem is reduced to the solution of a set of first-order partial differential equations for the potential. The general solution and a complete list of such potentials and their symmetry groups are then given for the cases $n = 1, 2, 3$. We give also in these cases the explicit forms for the infinitesimal generators and a discussion of the corresponding group representations.

## Introduction

In a recent work Niederer [1] has shown that the group of 'kinematical' space-time transformations (i.e. in the sense of Lie) that leave invariant the Schrödinger equation for a free particle is larger than just the Galilei group [2]; it is a group which contains, in addition to the Galilei transformations, dilations and conformal transformations. Indeed this group was used recently [3] to build a non-relativistic conformal invariant field theory, and was shown [4] for space dimension two to be an important subgroup of the relativistic conformal group.

Moreover, in further works Niederer [5, 6] has demonstrated that the invariance groups for both the harmonic oscillator potential and the linear potential are isomorphic to the free particle invariance group. In both cases the isometric mappings have been given explicitly. At first glance, this seems surprising since one would expect the harmonic oscillator to break translational symmetry and the linear potential to break rotational symmetry. The situation becomes more transparent, however, after making a unitary transformation from the Schrödinger to the Heisenberg picture.

In the present work we consider the problem of finding the maximal 'kinematical' local invariance group of space-time transformation for the non-relativistic time-dependent Schrödinger equation with an arbitrary time-independent potential. In Section I we determine the sets of first-order partial differential equations in $n$ space dimensions and one time dimension which must be satisfied in order that a non-trivial (i.e. other than time translations) invariance exists. The general solutions are then found for the cases $n = 1, 2, 3$ and a complete classification of all such potentials is given for these cases. In Section II we give a discussion of the invariance algebra for each of the potentials of the preceding section, writing down the explicit forms of the generators. While our method is infinitesimal in nature, there is no problem integrating to the group. Indeed in Section III, where we discuss some of the relevant representations, it is shown that our infinitesimal representations are integrable to the group. Furthermore, in this section the connection with certain spectrum generating algebras [7] is seen.

This work is an expanded and revised version of an earlier work with the same title [8] where only the case $n = 1$ was treated in detail. The classification of potentials in the one-dimensional case was given previously by Anderson, Kumei and Wulfman [9]. Also the invariance group for the closely related heat equation was given earlier by Blumen and Cole [10].

After this work was essentially completed, the author received a preprint [11] from Dr. U. Niederer where a similar treatment (although global rather than infinitesimal in nature) was given and essentially all the potentials were found; a complete classification of potentials, however, was not given.

## I. Classification of Potentials Admitting Symmetries

Let $\psi(x_i,t)$ be a solution to the $n$-dimension time-dependent Schrödinger equation

$$(H - i\,\partial_t)\,\psi(x_i,t) = 0 \tag{1.1}$$

where the Hamiltonian $H$ is given by

$$H = -\tfrac{1}{2}\,\partial_{x_i x_i} + V(x_i) \tag{1.2}$$

and

$$\partial_z = \frac{\partial}{\partial z} \quad \text{and} \quad \partial_{x_i x_i} = \partial^2_{x_1} + \cdots + \partial^2_{x_n}.$$

We now want to consider the set of all local space-time transformations of the form

$$T(g)\,\psi(x_i,t) = u(g, g^{-1}\cdot(x_i,t))\,\psi(g^{-1}\cdot(x_i,t)) \tag{1.3}$$

where $g^{-1}\cdot(x_i,t)$ denotes the local action of the transformation $g$ on the space-time coordinates, $u$ is for now an arbitrary function, and $\psi$ is a suitably defined function of $x_i$ and $t$. We are only interested in that subset of transformations of the form (1.3) which leave invariant the space $\mathscr{S}$ of solutions of the Schrödinger equation (1.1), i.e.

$$(H' - i\,\partial_t)\,T(g)\,\psi(x_i,t) = 0. \tag{1.4}$$

It is not difficult to show that such a subset forms a group $G$.

To find the group $G$ corresponding to the potential $V$, we consider the infinitesimal generators $L$ of $T(g)$. It is easy to see that they take the form

$$L(x_i,t) = i[a(x_i,t)\,\partial_t + b_i(x_i,t)\,\partial_{x_i} + c(x_i,t)]. \tag{1.5}$$

By defining the operator $\Delta(x_i,t) \equiv H - i\,\partial_t$, we see condition (1.4) says that $T(g)\,\psi$ must lie in the null space of $\Delta'$. Upon expanding $T(g)$ infinitesimally we can see that $\psi$ must lie in the null space of $[\Delta, L]$ or as an operator equation

$$[\Delta(x,t), L(\mathbf{x},t)] = i\lambda(\mathbf{x},t)\,\Delta(\mathbf{x},t) \tag{1.6}$$

where $\lambda(\mathbf{x},t)$ is for now an arbitrary function. It follows from the fact that $L$ is a first-order operator that $\lambda$ can contain no non-trivial operator terms. Upon inserting the explicit expressions for $\Delta$ and $L$ into (1.6) and equating the coefficients of the respective derivatives we obtain a set of coupled partial differential equations whose $\mathbf{x}$ dependence can be determined immediately, yielding the forms

$$a(\mathbf{x},t) = a(t) \tag{1.7a}$$

$$b_i(\mathbf{x},t) = \tfrac{1}{2}\dot{a}x_i + b_{ij}x_j + b_i^0(t), \quad \dot{b}_{ij} = 0 \tag{1.7b}$$

$$c(\mathbf{x},t) = -\tfrac{1}{4}\ddot{a}\mathbf{x}^2 - i\dot{\mathbf{b}}^0\cdot\mathbf{x} + c_0(t) \tag{1.7c}$$

$$\lambda(\mathbf{x},t) = \dot{a}(t) \tag{1.7d}$$

where the dot means derivative with respect to $t$. Then equating the non-derivative terms in (1.6) we arrive at a first-order partial differential equation for $V(\mathbf{x})$, viz.

$$(\tfrac{1}{2}\dot{a}x_i + b_{ij}x_j + b_i^0)\, V_{x_i} + \dot{a}V = -\tfrac{1}{4}\dddot{a}\mathbf{x}^2 - \ddot{\mathbf{b}}^0\cdot\mathbf{x} + \alpha(t) \tag{1.8}$$

where $V_{x_i} = \partial V/\partial x_i$ and

$$\alpha(t) = -i\dot{c}_0 + \frac{in\ddot{a}}{4}. \tag{1.9}$$

In order to find all solutions to (1.8) we notice that the general solution is the sum of the general solution of the homogeneous part plus a particular solution of the full inhomogeneous equation. We will see that the inhomogeneous part of the solutions specifies the time-dependence of the coefficients $a(t)$, $b_i^0(t)$, and $c_0(t)$. We can then insert these explicit expressions into (1.8) and equate the coefficients of the various powers of $t$ and obtain a set of homogeneous partial differential equations for $V$ which can then be solved by the method of characteristics. However, in order to simplify our procedure as much as possible, we will consider any two potentials to be equivalent if they can be related by rotations or translations in the underlying $n$-dimensional space, i.e. two potentials are equivalent if they lie on the same orbit under transformations of the $n$-dimensional Euclidean group $E(n)$.

Now, since the inhomogeneous part of (1.8) is at most quadratic in $\mathbf{x}$, the general solution must have the form

$$V(x) = v_{ij}x_ix_j + g_ix_i + v_0 + \tilde{V}(x) \tag{1.10}$$

where $\tilde{V}(x)$ is a general solution of the homogeneous equation and $v_{ij}$ is a symmetric $n\times n$ matric, i.e. $v_{ij} = v_{ji}$. Thus there exists a transformation in the $n$-dimensional orthogonal group $O(n)$ which diagonalizes $v_{ij}$. Accordingly we perform this transformation and rewrite (1.10) as

$$V(x) = \tfrac{1}{2}\omega_i^2x_i^2 + g_ix_i + v_0. \tag{1.10'}$$

Inserting (1.10′) into (1.8) and equating coefficients of the powers of the components $x_i$, we obtain the set of equations

$$\dddot{a} + (2\omega_i)^2\,\dot{a} = 0 \quad \text{(no sum on } i) \tag{1.11a}$$

$$\ddot{b}_i^0 + \omega_i^2 b_i^0 = b_{ij}g_j + \frac{3\dot{a}}{2} \quad \text{(no sum on } i) \tag{1.11b}$$

$$b_{ij}(\omega_i^2 - \omega_j^2) = 0 \quad \text{(no sum on } i, j) \tag{1.11c}$$

$$\dot{c}_0 = \frac{n}{4}\ddot{a} + i g_i b_i + i v_0 \dot{a}. \tag{1.11d}$$

Equation (1.11a) tells us that our problem subdivides into four different cases:

I) $\omega_1^2 = \cdots = \omega_n^2 = 0$

II) $\omega_1^2 = \cdots = \omega_n^2 = \omega^2 > 0$

III) $\omega_1^2 = \cdots = \omega_n^2 = -\omega^2 < 0$

IV) $\omega_i^2 \neq \omega_j^2$ for some $i$ and $j$.

*Case I.* $\omega_i^2 = 0$ for all $i$

This would only be possible if $v_{ij}$ were identically zero. Then integrating (1.11a, b, d) we find

$$a(t) = a_2 t^2 + a_1 t + a_0 \tag{1.12a}$$

$$b_i^0(t) = \frac{a_2 g_i}{2} t^3 + \frac{1}{2}\left(b_{ij} g_j + \frac{3a_1}{2} g_i\right) t^2 + b_i^{(1)} t + b_i^{(2)} \tag{1.12b}$$

$$c_0(t) = i\frac{a_2 \mathbf{g}^2}{8} t^4 + i\frac{a_1 \mathbf{g}^2}{4} t^3 + i\left(\mathbf{g}\cdot\frac{b^{(1)}}{2} + a_2 v_0\right) t^2 + \left(i v_0 a_1 + \frac{n}{2} a_2\right) t + c_0 \tag{1.12c}$$

where $\mathbf{g}^2 = g_i g_i$. We notice that the parameters $a_0$, $a_1$, $a_2$, $b_i^{(1)}$, $b_i^{(2)}$, $c_0$, and $b_{ij}$ define a Lie algebra of dimension $[n(n+3)/2] + 4$. We will examine the invariance algebra more closely in the next section.

We now wish to determine all the solutions of the homogeneous part of equation (1.8). Such solutions will naturally lead to constraints among the parameters of the Lie algebra in such a way that for a given solution only a subalgebra will remain as the invariance algebra. Some of these symmetries are of the usual geometric type and are just what one expects intuitively; others are not so apparent. Since the potential has no explicit time-dependence in our problem, $a_0$ does not occur in (1.8) and all potentials admit time translation symmetry. Then inserting (1.12) into the homogeneous part of (1.8) and equating the coefficients of the various powers of $t$, we obtain the set of differential equations.

$$a_2 g_i \tilde{V}_{x_i} = 0 \tag{1.13a}$$

$$\left(b_{ij} g_j + \frac{3a_1}{2} g_i\right) \tilde{V}_{x_i} = 0 \tag{1.13b}$$

$$(a_2 x_i + b_i^{(1)}) \tilde{V}_{x_i} + 2a_2 \tilde{V} = 0 \tag{1.13c}$$

$$(\tfrac{1}{2} a_1 x_i + b_{ij} x_j + b_i^{(2)}) \tilde{V}_{x_i} + a_1 \tilde{V} = 0 \tag{1.13d}$$

where the tilde indicates solutions of the homogeneous part of (1.8). We further subdivide this case

*I.A.* $a_2 \neq 0$

Immediately we see from (1.13a) that the gradient of $\tilde{V}$ must be normal to the direction of the vector $g_i$ specifying the linear potential $g_i x_i$. In addition, combining (1.13a and b) it is also seen that the gradient of $V$ must be normal to the vector $b_{ij} g_j$ which itself is normal to $g_i$. Without loss of generality we choose $g_i = 0, i = 1, \ldots, n-1$, $g_n \neq 0$ and study the cases of $g_n \equiv 0$ and $g_n \neq 0$ separately.

*I.A.i.* $g_i \equiv 0$, $V_{x_i} \neq 0$ for all $i$

We can perform a translation of coordinates in (1.13c) such that $b_i = 0$ and integrating the characteristics we obtain

$$\tilde{V}(x_i) = \frac{1}{x_1^2} F\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) = \frac{1}{r^2} f(\Omega) \tag{1.14}$$

where the functions $F$ and $f$ are arbitrary functions of their arguments and $\Omega$ denotes the $(n-1)$-dimensional unit sphere parametrized as

$$\begin{aligned} x_1 &= r \sin\theta_{n-1} \ldots \sin\theta_1 \\ x_2 &= r \sin\theta_{n-1} \ldots \cos\theta_1 \\ &\vdots \\ x_n &= r \cos\theta_{n-1}. \end{aligned} \tag{1.15}$$

Thus our general solutions (1.14) are homogeneous functions of degree $-2$ and (1.13d) reduces to

$$(b_{ij} x_j + b_i^{(2)}) \tilde{V}_{x_i} = 0. \tag{1.13d$'$}$$

With no further constraints on the potential $V$ we must have $b_{ij} = b_i = 0$. Further solutions of (1.13d′) yield only geometrical symmetries and to simplify the calculations involving many angles we consider only the physically interesting cases $n = 2, 3$. We can now perform a rotation such that $b_2 = b_3 = 0$ and then consistency with (1.14) yields $b_1 = 0$. Upon integrating the characteristics for (1.13d′) we find

$$\tilde{V}(x_i) = \frac{1}{r^2} f\left(\epsilon_{ijk} b_{ij} \frac{x_k}{r}\right) \tag{1.16}$$

which reduces, for example, to

$$\tilde{V}(x_i) = \frac{1}{r^2} g(\theta_2) \quad b_{23} = b_{13} = 0, \quad b_{12} \neq 0$$

$$\tilde{V}(x_i) = \frac{C}{r^2} \quad b_{ij} \neq 0, \quad i \neq j$$

where $g(\theta_2) = f(\cos\theta_2)$, and $C$ is a constant.

*I.A.ii.* $g_n \neq 0$

Thus (1.13a) implies $\tilde{V}_{x_n} = 0$ and, performing a rotation in the 'little' group of $g$, we can write, for $n = 3$ (1.13b), as

$$b_{23}\tilde{V}_{x_2} = 0 \quad \tilde{V}_{x_3} = 0. \tag{1.13b'}$$

Then (1.13b'), together with (1.13d) and (1.14), yield

$$\tilde{V}(x_i) = \frac{1}{x_1^2} F(x_2/x_1) \qquad b_3, b_3 \neq 0 \tag{1.17a}$$

$$\tilde{V}(x_i) = \frac{C}{x_1^2 + x_2^2} \qquad b_{12}, b_3, b_3 \neq 0 \tag{1.17b}$$

$$\tilde{V}(x_i) = \frac{C}{x_1^2} \qquad b_{23}, b_2, b_2, b_3 \neq 0 \tag{1.17c}$$

where all $b$'s other than the ones mentioned vanish respectively for each case. For $n = 2$ only (1.17c) occurs with $g_2, b_2, b_1 \neq 0$.

*I.B.* $a_2 = 0$, $a_1 \neq 0$

First note that if all $b_{ij}$'s vanish we have a special case of the previous result; hence we assume not all $b_{ij}$'s vanish. Again we can make translations in (1.13d) such that $b_i^{(2)} = 0$ and rewrite (1.13d) as

$$x_i \tilde{V}_{x_i} + \beta_{ij} x_j \tilde{V}_{x_i} + 2\tilde{V} = 0 \tag{1.13d''}$$

with $\beta_{ij} = 2b_{ij}/a_1$.

*I.B.i.* $g_i \equiv 0$

Now for $n = 3$ we can perform rotations such that $\beta_{23} = \beta_{31} = 0$ and we find the solutions with $\beta_{12} = \beta$

$$\tilde{V}(x_i) = \frac{1}{r^2} F(\theta_1 - \beta \ln r, \theta_2). \tag{1.18}$$

Furthermore, for consistency with (1.18) we must have $b_i = 0$. For $n = 2$ we find (1.18) without the $\theta_2$ dependence.

*I.B.ii.* $g_n \neq 0$

We can make a rotation in the 'little' group of $g$ and choose $b_{23} = 0$, then (1.13b) becomes

$$\beta_{13}\tilde{V}_{x_1} + 3\tilde{V}_{x_3} = 0. \tag{1.13b'}$$

However, (1.13b') is consistent with (1.13d'') only if $\beta_{13} = 0$ and $V_{x_3} = 0$, in which case we find

$$\tilde{V}(x_i) = e^{-2\theta_1/\beta} f(\theta_1 - \beta \ln [r \sin \theta_2]) \qquad b_3^{(1)}, b_3^{(2)} \neq 0. \tag{1.19}$$

This case does not appear for $n = 2$.

*I.C.* $a_1 = a_2 = 0$

These are all the usual geometrical symmetries.

*I.C.i.* $g_i \equiv 0$, $V_{x_i} \neq 0$ for all $i$

Proceeding in the same way as previously we find for $n = 3$

$$\tilde{V}(x_i) = \tilde{V}(r^2) \qquad b_{ij} \neq 0, \; i \neq j \tag{1.20a}$$

$$\tilde{V}(x_i) = \tilde{V}(x_1^2 + x_2^2, x_3) \qquad b_{12} \neq 0. \tag{1.20b}$$

For $n = 2$ only (1.20a) occurs for $i, j = 1, 2$.

*I.C.ii.* $g_n \neq 0$

We find as expected for $n = 3$

$$\tilde{V}(x_i) = \tilde{V}(x_1^2 + x_2^2) \qquad b_{12}, b_3^{(1)}, b_3^{(2)} \neq 0 \tag{1.21a}$$

$$\tilde{V}(x_i) = \tilde{V}(x_1, x_2) \qquad b_3^{(1)}, b_3^{(2)} \neq 0 \tag{1.21b}$$

$$\tilde{V}(x_i) = \tilde{V}(x_1) \qquad b_{23}, b_2^{(1)}, b_2^{(2)}, b_3^{(1)}, b_3^{(2)} \neq 0 \tag{1.21c}$$

while for $n = 2$ only (1.21c) remains with $b_2^{(1)}, b_2^{(2)} \neq 0$. These results, as well as the other cases with the symmetry algebras included, have been compiled in Table I.

Table I

Potentials admitting symmetries for $n = 3$. The parameters $g$, $\omega^2$, $\omega_i^2$ can be positive, negative or zero, and $V$ and $f$ are arbitrary functions of their arguments. The cases $n = 1$, 2 can be obtained by straightforward restriction.

| Potential | Symmetry |
|---|---|
| $gx_3 + \omega^2 r^2/2$ | $[o(3) \oplus sl(2, R)] \supset\!\!\!\!\!+ w_3$ |
| $c/x_1^2 + gx_3 + \omega^2 r^2/2$ | $[o(2) \oplus sl(2, R)] \supset\!\!\!\!\!+ w_2$ |
| $c/(x_1^2 + x_1^2) + gx_3 + \omega^2 r^2/2$ | $[o(2) \oplus sl(2, R)] \supset\!\!\!\!\!+ w_1$ |
| $f(x_2/x_1)/x_1^2 + gx_3 + \omega^2 r^2/2$ | $sl(2, R) \supset\!\!\!\!\!+ w_1$ |
| $c/r^2 + (\omega^2/2) r^2$ | $o(3) \oplus sl(2, R)$ |
| $f(\cos\theta_2)/r^2 + \omega^2 r^2/2$ | $o(2) \oplus sl(2, R)$ |
| $f(\theta_1, \theta_2)/r^2 + \omega^2 r^2/2$ | $sl(2, R)$ |
| $f(\theta_1 - \beta \ln r, \theta_2)/r^2$ | $s_2$ (2-dimensional solvable algebra) |
| $e^{-2\theta_1/\beta} f(\theta_1 - \beta \ln[r \sin\theta_2]) + gx_3$ | $s_2 \supset\!\!\!\!\!+ w_1$ |
| $\omega^2(x_2^2 + x_3^2)/2 + \omega_3^2 x_3^2/2 + gx_3$ | $[o(2) \oplus t_1] \supset\!\!\!\!\!+ w_3$ |
| $\omega^2(x_2^2 + x_3^2)/2 + gx_3 + V(x_1)$ | $[o(2) \oplus t_1] \supset\!\!\!\!\!+ w_2$ |
| $\omega_3^2 x_3^2/2 + gx_3 + V(x_1^2 + x_2^2)$ | $[o(2) \oplus t_1] \supset\!\!\!\!\!+ w_1$ |
| $(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2)/2 + gx_3$ | $t_1 \supset\!\!\!\!\!+ w_3$ |
| $(\omega_2^2 x_2^2 + \omega_3^2 x_3^2)/2 + gx_3 + V(x_1)$ | $t_1 \supset\!\!\!\!\!+ w_2$ |
| $\omega_3^2 x_3^2/2 + gx_3 + V(x_1, x_2)$ | $t_1 \supset\!\!\!\!\!+ w_1$ |
| $V(r)$ | $o(3) \oplus t_1$ |
| $V(x_1^2 + x_2^2, x_3)$ | $o(2) \oplus t_1$ |
| $V(x_1, x_2, x_3)$ | $t_1$ |

*Case II.* $\omega_1^2 = \cdots = \omega_n^2 = \tilde{\omega}^2 > 0$

Since $x_1^2 + \cdots + x_n^2$ is an $O(n)$ invariant this case only occurs if $v_{ij} = \omega^2 \delta_{ij}$. Then solving the equations (1.11a, b, d) we find

$$a(t) = a_1 \sin 2\omega t + a_2 \cos 2\omega t + a_0 \tag{1.22a}$$

$$b_i^0(t) = b_i^{(1)} \sin \omega t + b_i^{(2)} \cos \omega t - \frac{g_i}{\omega}(a_1 \sin 2\omega t - a_2 \cos 2\omega t) + \frac{b_{ij} g_j}{\omega^2} \tag{1.22b}$$

$$\begin{aligned} c_0(t) = &\left(+\frac{n\omega}{2}\cos 2\omega t + \frac{ig^2}{2\omega^2}\cos 2\omega t + iv_0 \sin 2\omega t\right) a_1 \\ &+ \left(-\frac{n\omega}{2}\sin 2\omega t - \frac{ig^2}{2\omega^2}\sin 2\omega t + iv_0 \cos 2\omega t\right) a_2 \\ &- \frac{i}{\omega}\cos \omega t g_i b_i^{(1)} + \frac{i}{\omega}\sin \omega t g_i b_i^{(2)} + c_0. \end{aligned} \tag{1.22c}$$

Again, as in the previous case, we insert (1.22) back into (1.8) and look for all possible solutions of the homogeneous part of the equation. By equating coefficients of the independent trigonometric functions we find

$$b_{ij}\left(x_j + \frac{g_j}{\omega^2}\right)\tilde{V}_{x_j} = 0 \tag{1.23a}$$

$$b_i^{(1)} \tilde{V}_{x_i} = b_i^{(2)} \tilde{V}_{x_i} = 0 \tag{1.23b}$$

$$\left(x_i + \frac{g_i}{\omega^2}\right)\tilde{V}_{x_i} + 2\tilde{V} = 0 \quad \text{or} \quad a_1 = a_2 = 0. \tag{1.23c}$$

In this case we can always perform a translation of coordinates to choose $g_i = 0$.

*II.A.* $a_1$, $a_2$ not both zero

The solution to (1.23c) has already been given by equation (1.14). The additional geometric constraints (1.23a and b) then yield for $n = 2, 3$ the potentials listed in (1.16) and (1.17).

*II.B.* $a_1 = a_2 = 0$

The potentials are purely geometric and are given by (1.20) and (1.21) for $n = 2, 3$ (see Table I).

*Case III.* $\omega^2 = \cdots = \omega_n^2 = -\tilde{\omega}^2 < 0$

This is described precisely by Case II after substituting $\tilde{\omega}$ for $\omega$ and $\cosh \omega t$ and $\sinh \omega t$ for $\cos \omega t$ and $\sin \omega t$, respectively.

*Case IV.* $\omega_i^2 \neq \omega_j^2$ for some $i \neq j$

We immediately see from (1.11a) that $a = 0$; furthermore from (1.11c) we see that $b_{ij} = 0$ whenever $\omega_i \neq \omega_j$. Hence the maximal number of parameters occurs when only one $\omega_i$ differs from the others in which case the symmetry algebra has dimension $n[(n+1)/2] + 3$ and when the frequencies are all different the dimension is $2n + 2$.

For $n = 3$ we have the possibilities

$$V(x_i) = \tfrac{1}{2}\omega_1^2 x_1^2 + \tfrac{1}{2}\omega_2^2 x_2^2 + \tfrac{1}{2}\omega_3^2 x_3^2 + g_i x_i + v_0 \tag{1.24a}$$

with all $\omega_i^2$'s different and $b_{ij} = 0$ or

$$V(x_i) = \tfrac{1}{2}\omega^2(x_1^2 + x_2^2) + \tfrac{1}{2}\omega_3^2 x^2 + g_i x_i + v_0 \tag{1.24b}$$

with $b_{12} \neq 0$, and the symmetry algebra has dimension 8 and 9 respectively. In either case any of the corresponding $\omega_i^2$'s may be zero, positive, or negative and the corresponding solutions to (1.11b) and (1.11d) are given by (1.12b or c) when $\omega_i^2 = 0$,

$$b_i^0(t) = b_i^{(1)} \sin \omega_i t + b_i^{(2)} \cos \omega_i t \tag{1.25}$$

for $i = 1, 2, 3$ in (1.24a), $i = 3$ in (1.24b) and by (1.22*b*) with $a_1 = a_2 = 0$ for $i = 1, 2$ in (1.24b) when $\omega_i^2 \neq 0$ and by the replacement of the circular trigonometric functions by hyperbolic ones when $\omega_i^2 < 0$. One can then substitute the corresponding $b_i(t)$ and $c_0(t)$ into (1.9) and look for solutions to the homogeneous part. Now only when the frequencies are commensurable can one justify treating the trigonometric functions as independent; nevertheless, one can reduce the constraint equations to exactly the type of geometric symmetries discussed previously (see Table I).

## II. The Invariance Algebra

It is the purpose of this section to analyse in more detail the nature of the symmetry algebras for the potentials listed in the previous section. The structure of the generators differs from case to case so we will study each separately. It is also mentioned that the generators are constructed to be hermitian with respect to the usual scalar product of non-relativistic quantum mechanics

$$(\psi_1, \psi_2) = \int_{-\infty}^{\infty} d^n x \psi_1^*(\mathbf{x}, t)\, \psi_2(\mathbf{x}, t). \tag{2.1}$$

*Case I.* $V(x_i) = g_i x_i + v_0 + \tilde{V}(x_i)$

The full symmetry group is attained when the homogeneous solution $\tilde{V}(x_i)$ vanishes identically. In this case the functions (1.12) together with (1.8) and (1.7) yield the generators

$$A_0 = i\,\partial_t \tag{2.2a}$$

$$A_1 = i\left[2t\,\partial_t + \left(x_i - \frac{3g_i}{2}t^2\right)\partial_{x_i} + 3itg_i x_i - \frac{ig^2}{2}t^3 + 2iv_0 t + \frac{n}{2}\right] \tag{2.2b}$$

$$A_2 = i\left[t^2\,\partial_t + \left(tx_i - \frac{t^3 g_i}{2}\right)\partial x_i - \frac{i}{2}x^2 + \frac{3it^2}{2}g_i\,x_i - \frac{ig^2}{8}t^4 + iv_0\,t^2 + \frac{n}{2}t\right] \tag{2.2c}$$

$$B_i^{(1)} = i\left(t\,\partial_{x_i} - ix_i + \frac{i}{2}g_i\,t^2\right) \tag{2.2d}$$

$$B_i^{(2)} = i(\partial_{x_i} + ig_i\,t) \tag{2.2e}$$

$$E = \mathbf{1} \tag{2.2f}$$

$$L_{ij} = i\left[x_j\,\partial_{x_i} - x_i\,\partial_{x_j} + \frac{t^2}{2}(g_j\,\partial_{x_i} - g_i\,\partial_{x_j}) - it(g_j\,x_i - g_i\,x_j)\right] \tag{2.2g}$$

corresponding to the parameters $a_0$, $a_1$, $a_2$, $b_i^{(1)}$, $b_i^{(2)}$, $\bar{c}_0$ and $b_{ij}/2$ respectively. The generators (2.2) span a Lie algebra $\tilde{s}_n$ of dimension $[n(n+3)/2]+4$ as mentioned previously. We can call this the centrally extended conformal Galilei algebra or, following Niederer, the centrally extended Schrödinger algebra. The structure of $\tilde{s}_n$ is best exhibited by constructing the combinations

$$I_3 = \tfrac{1}{2}(A_0 + A_2 - g_i\,B_i^{(1)} - v_0\,E) \tag{2.3a}$$

$$I_2 = \tfrac{1}{2}(A_0 - A_2 - g_i\,B_i^{(1)} - v_0\,E) \tag{2.3b}$$

$$I_1 = \tfrac{1}{2}A_1 \tag{2.3c}$$

which is seen to be a representation of the algebra of the special linear group $SL(2,R) = S_p(2,R) \simeq SU(1,1)^{2-1} \simeq SO_0(2,1)$. The generators (2.2d, e and f) yield a representation of the $(2n+1)$-dimensional Heisenberg–Weyl algebra $w_n$ with commutation relations

$$[B_i, B_j] = i\,\delta_{ij}\,E \tag{2.4}$$

where $E$ being a central generator commutes with everything. Now it can be seen that $w_n$ is an invariant subalgebra of $\tilde{s}_n$ and that the $L_{ij}$ commute with the generators (2.3) of the $sl(2,R)$ subalgebra. Furthermore, the generators $L_{ij}$ span the Lie algebra $o(n)$, although, as mentioned by Niederer, they do not generate the usual rotation group action unless $g_i \equiv 0$. Thus we will have the algebraic structure

$$\tilde{s}_n \simeq [o(n) \oplus sl(2,R)] \;\dot{+}\; w_n \tag{2.5}$$

where $\oplus$ and $\dot{+}$ denotes direct and semidirect sum respectively and the invariant subalgebra appears to the right. For completeness we write down the additional non-vanishing commutation relations giving rise to the above structure

$$[L_{ij}, L_{hl}] = i(\delta_{il}\,L_{jh} - \delta_{jl}\,L_{ih} - \delta_{ih}\,L_{jl} + \delta_{jh}\,L_{il}) \tag{2.6a}$$

$$[L_{ij}, B_h^{(\alpha)}] = i(\delta_{jh}\,B_i^{(\alpha)} - \delta_{ih}\,B_j^{(\alpha)}) \tag{2.6b}$$

$$[I_3, B_i^{(1)}] = [I_2, B_i^{(1)}] = \frac{i}{2}B_i^{(2)} \qquad [I_1, B_i^{(1)}] = \frac{i}{2}B_i^{(1)} \tag{2.6c}$$

$$[I_3, B_i^{(2)}] = [I_2, B_i^{(2)}] = -\frac{i}{2}B_i^{(1)} \qquad [I_1, B_i^{(2)}] = \frac{i}{2}B_i^{(2)}. \tag{2.6d}$$

An important subalgebra of $\tilde{s}_n$ is the extended Galilei algebra $\tilde{g}_n$ obtained by replacing the $sl(2,R)$ algebra by the algebra of a one-parameter subgroup generated by $(A_0 - g_i B_i)$, with the structure $\tilde{g}_n \simeq [o(n) \oplus t_1] \oplus w_n$ where $t_1$ denotes the aforementioned one-dimensional subalgebra which is *not* the generator of time translations unless $g_i \equiv 0$. Notice the generator of time translations $A_0$ does not in general commute with the rotations $L_{ij}$. This is to be expected since the $g_i$ term breaks rotational invariance. What is, however, somewhat surprising is the appearance of the full $o(n)$ as a symmetry even when $g_i \neq 0$.

As seen from the calculations in the previous section, any additional potential $\tilde{V}(x_i)$ appearing as a solution of the homogeneous part of (1.9) does so only at the expense of losing some of the symmetry. For example, a potential of the type (1.14) can be a solution only when $g_i \equiv 0$ and then it only admits the $sl(2,R)$ symmetry given explicitly by (2.2a–c) with $g_i = 0$. Further restrictions on the potential (1.14) allows the addition of more symmetry of the usual geometric type, and these are indicated by the non-vanishing parameters listed under each case of Section I.A. One has only to associate the corresponding generator of (2.1) with these parameters. Likewise for the purely geometric symmetries of Case I.C, the non-vanishing parameters have been listed previously and of course, the generators $A_1$ and $A_2$ cannot occur (see table).

Case I.B, however, presents somewhat of an exception. We find that the potential (1.18) admits the two-dimensional solvable algebra, $\mathscr{s}_2$ given by (2.2a) and for $n = 2, 3$

$$A_1 = i\left[2t\,\partial_t + r\,\partial_r + \frac{\beta}{2}\partial_{\theta_1} + 2iv_0 t + \frac{n}{2}\right] \tag{2.7}$$

where we have used the spherical coordinates (1.15).

The relevant commutator is simply

$$[A_0, A_1] = 2iA_0. \tag{2.8}$$

Furthermore, the potentials of the form (1.19) admit the additional geometric symmetries whose generators are given by (2.2d), (2.2e) and (2.2f) with $i = 3$. Together with (2.2a) and (2.7) they span a five-dimensional solvable algebra with structure $\mathscr{s}_2 \oplus w_1$.

*Case II*

In a similar way the full symmetry algebra for this case is obtained form (1.22) where, from our equivalence, we can set $g_i = 0$ as before. We obtain the generators

$$A_0 = i\,\partial_t \tag{2.9a}$$

$$A_1 = i\left[\sin 2\omega t\,\partial_t + \omega\cos 2\omega\, t\mathbf{x}\cdot\partial_{\mathbf{x}} + i\omega^2 x_2 \sin 2\omega t + \frac{n\omega}{2}\cos 2\omega t + iv_0 \sin 2\omega t\right] \tag{2.9b}$$

$$A_2 = i\left[\cos 2\omega t\, \partial_t - \omega \sin 2\omega t \mathbf{x}\cdot\partial_{\mathbf{x}} + i\omega^2 \mathbf{x}^2 \cos 2\omega t - \frac{n\omega}{2} \sin 2\omega t + iv_0 \cos 2\omega t\right] \tag{2.9c}$$

$$B_i^{(1)} = i[\sin \omega t x_i - i\omega x_i \cos \omega t] \tag{2.9d}$$

$$B_i^{(2)} = i[\cos \omega t\, \partial_{x_i} + i\omega x_i \sin \omega t] \tag{2.9e}$$

$$E = \mathbf{1} \tag{2.9f}$$

$$L_{ij} = i(x_j\, \partial_{x_i} - x_i\, \partial_{x_j}). \tag{2.9g}$$

It can be shown by a straightforward calculation that

$$I_3 = \frac{1}{2\omega}(A_0 - v_0 E) \quad I_1 = \frac{1}{2\omega} A_1 \quad I_2 = \frac{1}{2\omega} A_2 \tag{2.10}$$

form an $sl(2, R)$ subalgebra and with the substitution $B_i^{(\alpha)} \to B_i^{(\alpha)}/\sqrt{\omega}$ we obtain the same Lie algebra as discussed previously; i.e. the symmetry algebra of the harmonic oscillator, free particle, and linear potential are isomorphic. This isomorphism has been exhibited by Niederer [5, 6].

Furthermore, the addition of further potentials which are solutions of the homogeneous part of (1.8) has been discussed and the non-vanishing parameters listed. For each case we have the corresponding symmetries and the isomorphism to the corresponding subcase of I. Notice that in Case II there are no symmetries corresponding to Case I.B.

*Case III*

Again this case can be handled by the mere substitution $\omega^2 \to -\omega^2$; sin, cos $\to$ sinh, cosh in Case II. Hence, the symmetry algebra for the repulsive harmonic oscillator is isomorphic with the three listed previously. Again the addition of further potentials follows exactly as Case II.

*Case IV*

As discussed in the previous section, the maximal symmetry algebra is attained in this case when the absolute value of all frequencies but one are equal and the dimension is $[n(n+1)/2] + 3$. On the other hand when all the frequencies are different, the dimension of the algebra is only $2n + 2$. Clearly from (1.11c) we see there is a direct correspondence between rotational symmetries which form proper subalgebras of $o(n)$ and equal frequencies. It is also mentioned that frequencies which are commensurable will also admit certain discrete symmetries; however, since our discussion is limited to local symmetries, these are not treated here.

The symmetry algebra is given when $\omega_i \neq 0$ by (2.9d–f) with $\omega$ replaced by $\omega_i$ in each $B_i$ and (2.9g) whenever $\omega_i^2 = \omega_j^2$, and when $\omega_i = 0$ by (2.2d–f) and (2.2g) for all $i, j$ such that $\omega_i^2 = \omega_j^2 = 0$. For $n = 3$ the only two cases are given by (1.24a and b) with symmetry algebras $t_1 \oplus\!\!\!\!\supset w_3$ and $[o(2) \oplus t_1] \oplus\!\!\!\!\supset w_3$ respectively.

The addition of any further potentials (see table) yield geometric constraints as discussed previously. Again it should also be clear that if $\omega_i^2 < 0$ the circular trigonometric functions are replaced by hyperbolic ones.

## III. Representations

Induced representations of a global group isomorphic to $\tilde{S}_n$ have been discussed previously [12] for the case $n = 3$, and in this case three invariants were found corresponding to non-relativistic mass, spin, and the Casimir invariant for an internal $sl(2, R)$ (different than that given by (2.3)). In our case the mass has been scaled out and the spin does not appear, so we are dealing with degenerate representations. In order to discuss these representations it is convenient to consider yet other isomorphic realizations of the algebras of the previous section. These are obtained by everywhere setting $t = 0$ and making the replacement

$$i\,\partial_t \to H \tag{3.1}$$

which holds on $\mathscr{S}$.

That is, we are passing from the Schrödinger picture to the Heisenberg picture [13]. For example, from (2.9) we get the generators familiar from the radial harmonic oscillator [7]

$$\mathscr{A}_0 = -\tfrac{1}{2}\partial_{x_i x_i} + \frac{\omega^2}{2}\mathbf{x}^2 \tag{3.2a}$$

$$\mathscr{A}_1 = i\left(\mathbf{x}\cdot\partial_x + \frac{n}{2}\right) \tag{3.2b}$$

$$\mathscr{A}_2 = -\tfrac{1}{2}\partial_{x_i x_i} - \frac{\omega^2}{2}\mathbf{x}^2 \tag{3.2c}$$

$$\mathscr{B}_i^{(1)} = \omega x_i \tag{3.2d}$$

$$\mathscr{B}_i^{(2)} = i\,\partial_{x_i} \tag{3.2e}$$

$$E = \mathbf{1} \tag{3.2f}$$

$$\mathscr{L}_{ij} = i(x_j\,\partial_{x_j} - x_i\,\partial_{x_j}). \tag{3.2g}$$

Notice we can obtain (2.9) from (3.2) by applying[1] $e^{it\mathrm{Ad}\mathscr{A}_0}$ and replacing $\mathscr{A}_0 = H$ by $i\,\partial_t$. We can use this procedure for each of the algebras discussed in the previous section to obtain generators which differ from (3.2) only by a change of basis.

We can now define $\mathscr{I}_i$ as given by (2.10) with the $A$'s being replaced by the $\mathscr{A}$'s given above. A straightforward calculation then gives the Casimir operator for the $sl(2, R)$ subalgebra as

$$C = \mathscr{I}_3^2 - \mathscr{I}_1^2 - \mathscr{I}_2^2 = \tfrac{1}{8}\mathscr{L}_{ij}\mathscr{L}_{ij} + \frac{n(n-4)}{16} \tag{3.3}$$

---

[1]) This is a unitary transformation from $L^2(R_n)$ onto itself since, as will be seen shortly, $A_0$ is self-adjoint on a suitably defined domain which is dense in $L^2(R_n)$.

so we find that $C - \frac{1}{8}\mathscr{L}_{ij}\mathscr{L}_{ij} = n(n-4)/16$ is an invariant. Moreover, an orthonormal basis in $L^2(R_n)$ is given by

$$\psi_{NL}(x_i) = N_{NL}\, e^{-\omega r^2/2}(\sqrt{\omega}r)^{l+n-2}\, L_N^{l+(n-2/2)}(\omega r^2)\, Y_L(\Omega) \tag{3.4}$$

where we have used the spherical coordinates (1.15) and the $Y_L(\Omega)$ are the $n$-dimensional spherical harmonics [14] with labels $L = (l_{n-2}, \ldots, l_1)$, $l = l_{n-2}$ satisfying $l_1 = -l_2, \ldots, l_2, l_i = 0, \ldots, l_{i+1}, i = 2, \ldots, n-3$, and $L_N(r^2)$ are Laguerre polynomials. Furthermore, since

$$\tfrac{1}{2}\mathscr{L}_{ij}\mathscr{L}_{ij}\psi_{NL}(x_k) = l(l+n-2)\,\psi_{nL}(x_k) \tag{3.5a}$$

we find

$$C\psi_{NL}(x_i) = \frac{1}{4}\left(l+\frac{n}{2}\right)\left(l+\frac{n}{2}-2\right)\psi_{NL} = k(k-1)\,\psi_{NL} \tag{3.5b}$$

which can be formally identified with Bargmann's [15] discrete series $D_k^+$ with $k = (l+n/2)/2$ which is single valued on the $SL(2,R)$ manifold for $n$ even and double valued for $n$ odd [16].

By a straightforward calculation one can compute the Nelson [17] operator for the representation (3.2)

$$N \equiv \mathscr{I}_3^2 + \mathscr{I}_1^2 + \mathscr{I}_2^2 + \tfrac{1}{2}(\mathscr{B}_1^{(2)^2} + \mathscr{B}_1^{(1)^2} + \cdots + \mathscr{B}_n^{(2)^2}$$
$$+ \mathscr{B}_n^{(1)^2}) + \tfrac{1}{8}\mathscr{L}_{ij}\mathscr{L}_{ij} = 2\mathscr{I}_3(\mathscr{I}_3+1). \tag{3.6}$$

Now the operator $\mathscr{I}_3$ is known [18] to be self-adjoint on the domain $\mathscr{D}$ of functions $f \in L^2(R_n)$ such that i) $\partial_i f \in L^2(R_n)$ for each component, ii) $\int \mathbf{x}^2|f(x)|^2 d^n x < \infty$, iii) $\partial_{x_i x_i} f$ exists and $\mathscr{I}_3 f \in L^2(R_n)$, where here $\partial_i$ means derivative in the generalized sense, since the members of $\mathscr{D}$ are not necessarily continuous. It is easy to see that the linear span of functions (3.4), denoted by $\mathscr{D}_N$, is a dense subspace of $\mathscr{D}$ which is invariant under all of the operators (3.2). Thus it follows that $N$ is a self-adjoint operator in $L^2(R_n)$ which is essentially self-adjoint when restricted to $\mathscr{D}_N$. Hence, by a theorem of Nelson [17], there is a unique unitary representation of a connected and simply connected Lie group $G_n$ having infinitesimal generators (3.2). Furthermore, the representation of $G_n$ is irreducible on $L^2(R_n)$ since it can easily be seen that the algebra (3.2) is irreducible on $\mathscr{D}_N$.

Now each of the four different potentials $V = 0$, $x_i$, $(\pm\omega^2 x^2)/2$ correspond to choosing [19] the Hamiltonian $H$ as an appropriate linear combination of the generators (3.2). Thus we can reconstruct the algebra and group representations in the Schrödinger picture by applying the unitary map $e^{it\,\mathrm{Ad}\,H}$ in each case, and the four cases are thus unitarily equivalent. The explicit mappings have been given by Niederer [5, 6].

Next we consider the case of an additional potential of the form (1.14) which has as its invariance algebra in the Schrödinger picture only the $sl(2,R)$ subalgebra given, for example, by (2.10). In this case the Hamiltonian is given by

$$H = -\tfrac{1}{2}\partial_{x_i x_i} + \frac{\omega^2}{2}\mathbf{x}^2 + \frac{f(\Omega)}{\mathbf{x}^2} \tag{3.7}$$

Transforming formally to the Heisenberg picture we obtain equations (3.2a–c) with the $f(\Omega)/\mathbf{x}^2$ term added to (3.2a) and (3.2c).

We can now separate out the radial and angular parts, finding

$$\mathscr{A}_0^r = -\tfrac{1}{2}\partial_{rr} - \frac{n-1}{2r}\partial_r + \frac{\omega^2}{2}r^2 + \frac{\alpha}{2r^2} \tag{3.8a}$$

$$\mathscr{A}_1^r = i\omega\left(r\,\partial_r + \frac{n}{2}\right) \tag{3.8b}$$

$$\mathscr{A}_2^r = -\tfrac{1}{2}\partial_{rr} - \frac{n-1}{2r}\partial_r - \frac{\omega^2}{2}r^2 + \frac{\alpha}{2r^2} \tag{3.8c}$$

where $\alpha/2$ is an eigenvalue of the operator $-\Delta(\Omega) + f(\Omega)$, $\Delta(\Omega)$ being the Laplace Beltrami operator on the $(n-1)$-dimensional sphere. Now following the same procedure as before, the representation (3.8) can be integrated to a unique representation of the universal covering group of $SL(2, R)$ provided $\alpha > -(n-2)^2/4$. A computation of the Casimir operator for the algebra (3.8) yields the $c$-number

$$C_r = -3/16 + [\alpha + (n-1)(n-3)/4]/4. \tag{3.9}$$

This describes the multivalued analytic representations [16] of $SL(2, R)$. Notice from the lower bound on $\alpha$ we obtain $C^r > -1/4$ which corresponds precisely to the series $D_k^+$, $k > 1/2$. Again the representation of the algebra (3.8) and hence of the group can be shown to be irreducible on $L^2(0, \infty)$.

Similar treatments can be given for the other invariance algebras listed in Section II. Of particular interest is the $(2n+2)$-dimensional solvable algebra from the anisotropic harmonic oscillator (Case IV). The UIR's of this algebra and corresponding group for the case $n = 1$ have been given previously by Miller [20]. In the $n$-dimensional case we have the generators

$$\mathscr{A}_0 = -\tfrac{1}{2}\partial_{x_i x_i} + \frac{\omega_i^2}{2}x_i^2 \tag{3.10a}$$

$$\mathscr{B}_i = \omega_i x_i \quad \text{(no sum on } i\text{)} \tag{3.10b}$$

$$\mathscr{B}_i = i\,\partial_{x_i} \tag{3.10c}$$

$$E = 1. \tag{3.10d}$$

Now we have a dense set in $L^2(R_n)$ on which $\mathscr{A}_0$ is diagonal given by the Hermite polynomials [14]

$$\psi_{N_i}(x_i) = \prod_{i=1}^{n} C_{N_i}\exp\left(-\frac{\omega_i}{2}x_i^2\right)H_{N_i}(\sqrt{\omega_i}\,x_i) \tag{3.11}$$

where $C_{N_i}$ are left unspecified.

Again $\mathscr{A}_0$ is self-adjoint on $\mathscr{D}$ as before and we can integrate to a unique unitary representation of the group. Moreover, this representation is irreducible, as can be seen by applying (3.10b and c) to the basis vectors (3.11). The UIR labels are just given by the frequencies $(\omega_1, \ldots, \omega_n)$.

## Conclusion

We have given a complete classification of all time-independent potentials for the non-relativistic time-dependent Schrödinger equation with spatial dimensional $n = 1, 2, 3$ which admit non-trivial local symmetry groups. In addition, we have determined all of the symmetries and discussed the representations of the more interesting cases.

The directions for future research are many. It would be interesting to consider the same problem for time-dependent potentials, an example of which has already been given [11]. Furthermore, the addition of a vector potential by the usual minimal coupling scheme gives rise to more complicated equations which can be simplified by the use of gauge invariance. An example of a constant magnetic field exhibiting such symmetries has recently been given by Boon and Seligman [21]. A complete classification in this case would be desirable.

There is currently work in progress in collaboration with Kalnins and Miller on the problem of separation of variables for the two-space dimensional case and the connection with second-order symmetry operators. Such a connection has already been established [22] for the time-independent Schrödinger equation in two dimensions. The connection between the separation of variables and the four potentials admitting the maximal invariance algebra in the one-space dimensional case has already been established by Kalnins and Miller [19] as mentioned previously.

Finally, as mentioned in the Introduction, the case $n = 2$ has a direct bearing on the study of conformal invariance in relativistic mechanics when viewed from the infinite momentum reference frame [4].

## Acknowledgments

During the course of this work the author has profited from discussions with Professors W. Miller, Jr., and M. Moshinsky, Drs. E. Kalnins, J. Patera, T. Seligman, P. Winternitz and K. B. Wolf. A pertinent communication from Dr. U. Niederer and the receipt of material prior to publication is gratefully acknowledged. I would also like to thank the members of the Centre de Recherches Matemátique, Université de Montréal, where part of this work was done.

REFERENCES

[1] U. NIEDERER, Helv. Phys. Acta *45*, 802 (1972).
[2] J-M. LEVY-LEBLOND, J. Math. Phys. *4*, 776 (1963).
[3] C. R. HAGEN, Phys. Rev. *D5*, 377 (1972).
[4] G. DOMOKOS, Commun. Math. Phys. *26*, 15 (1972); G. BURDET, M. PERRIN and P. SORBA, Commun. Math. Phys. *34*, 85 (1973).
[5] U. NIEDERER, Helv. Phys. Acta *46*, 191 (1973).
[6] U. NIEDERER, Universitat Zürich preprint—Talk in Proceedings of the 2nd International Colloquium on Group Theoretical Methods in Physics, 1973, University of Nijmegen.
[7] H. BACRY and J. L. RICHARDS, J. Math. Phys. *8*, 2230 (1967); P. CORDERO and G. C. GHIRARDI, Nuovo Cimento *2A*, 217 (1971); P. CORDERO and G. C. GHIRARDI, Forschr. der Physik, *20* 105 (1972); Y. DOTHAN, Phys. Rev. *D2*, 2944 (1970); M. MOSHINSKY, Lecture: Notes of the Latin American School of Physics, Mexico, D.F. (1971); M. MOSHINSKY, T. H. SELIGMAN and K. B. WOLF, J. Math. Phys. *13*, 901 (1972).
[8] C. P. BOYER, preprint Comun. Tecn. 53, 1973, CIMAS, Universidad Nacional de México.
[9] R. L. ANDERSON, S. KUMEI and C. E. WULFMAN, Rev. Mex. Fis. *21*, 1 (1972).
[10] G. W. BLUMAN and J. D. COLE, J. Math. and Mech. *18*, 1025 (1969).

[11] U. NIEDERER, Universitat Zürich preprint, Dec. 1973.
[12] P. ROMAN, J. J. AGHASSI, R. M. SANTILLI and P. L. HUDDLESTON, Nuovo Cimento *12A*, 185 (1972).
[13] M. MOSHINSKY, S. KUMEI, T. SHIBUYA and C. E. WULFMAN, UNAM preprint 72–4; M. MOSHINSKY, SIAM J. Appl. Math. *25*, 193 (1973).
[14] A. ERDERLYI et al., *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York 1953).
[15] V. BARGMANN, Ann. Math. *48*, 568 (1947).
[16] Since the group $SL(2, R)$ is infinitely connected, there exists a continuum of multi-valued representations which are the analogs of Bargmann's discrete series. These have been classified and discussed in detail by L. PUKÁNSKY, Math. Ann. *156*, 96 (1964).
[17] E. NELSON, Ann. Math. *70*, 572 (1959).
[18] T. KATO, *Perturbation Theory for Linear Operators*, Chap. VI, Section 4 (Springer-Verlag, New York 1966).
[19] E. G. KALNINS and W. MILLER, JR., Centre de Recherches Mathemátique preprint 369, 1974, Université de Montréal. The authors have herein given an elegant connection between the existence of the four potentials and the separation of variables of the Schrödinger equation for one space and one time dimension.
[20] W. MILLER, JR., Commun. Pure and Appl. Math. *18*, 679 (1965).
[21] M. H. BOON and T. H. SELIGMAN, J. Math. Phys. *14*, 1224 (1973).
[22] P. WINTERNITZ, Y. SMORODINSKII, M. UHLIR and I. FRIS, Soviet J. Nucl. Phys. *4*, 444 (1967).