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A Mechanical Quantum Measuring Process[1])

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Abstract. A mechanistic analysis of the quantum measuring process is proposed. This takes into account both the reduction of the wave packet and the recording of the measurement on a macroscopic apparatus. An exactly solvable model is given to sustain the theory.

Outline

In this paper a mechanistic model is proposed which shows that the usual framework of quantum mechanics (when adapted to the study of infinite systems) is sufficient to deal consistently with the following two problems of the quantum measuring process: (i) the 'reduction of the wave packet;' (ii) the transfer of information from a microscopic to a macroscopic level. In Section I we present our description of the quantum measuring process. In Section II we propose an exactly solvable model to support this description. In Section III we analyze the model of Section II and show that it does indeed behave according to the general framework described in Section I. Concluding remarks are presented in Section IV.

I. The Framework

The literature on the quantum measuring process is quite extensive. The reader will find reviews in Reece [1] and Whitten-Wolfe [2]. Our description draws mainly on the recent work of Hepp [3] and Emch [4], and consolidates some aspects of both.

A measuring process involves three elements: a system $\mathscr{S}$ to be measured, a measuring apparatus $\mathscr{A}$, and an interaction which couples $\mathscr{S}$ and $\mathscr{A}$. Each of these elements will be described in some detail below.

The *system to be measured* is a quantum system described by its C^*-algebra of observables $\mathfrak{B}$ and its set of states $\mathfrak{T}$; we will denote its initial state by φ. We restrict our attention to the mathematically simple case in which the observables $R \in \mathfrak{B}$ to be measured are compatible and have jointly discrete spectrum. We therefore write: $R = \sum_{j \in J} r_j P_j$, where the index set J is countable, $r_j \in \mathrm{R}$, and the P_j's are mutually orthogonal projectors in $\mathfrak{B}$, adding up to 1.

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The *measuring apparatus* is an infinite quantum system. It is described by the C^*-algebra $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}(\Lambda)}^n$ of its quasi-local observables, and by the set of states $\mathfrak{S}$ on $\mathfrak{A}$. (For details of the C^*-algebraic description see Emch [5].) We denote the initial state of $\mathscr{A}$ by ψ.

In order for $\mathscr{A}$ to function as an apparatus which measures $R \in \mathfrak{B}$, its algebra of observables $\mathfrak{A}$ must provide a set of operators $\{M_j\}$ which are in one-to-one correspondence with the set of projectors $\{P_j\}$.

An *interaction* is switched on at $t = 0$ (the beginning of the measuring process) between the system $\mathscr{S}$ and the apparatus $\mathscr{A}$. For every observable X of interest, we denote by $\alpha_t[X]$ the time-evolute of X under this interaction. We place five requirements on this interaction.

First, we follow von Neumann [6] in requiring that the measurement be repeatable on $\mathscr{S}$. This is taken to mean that the effect of the measurement on $\mathscr{S}$ is to collapse the state φ into a mixture of the eigenstates of R. Specifically:

$$\lim_{t\to\infty} \langle \varphi \otimes \psi; \alpha_t[B \otimes I] \rangle = \sum_j \lambda_j \langle \varphi_j; B \rangle$$

for all $B \in \mathfrak{B}$, where

$$\langle \varphi_j; B \rangle = \frac{\langle \varphi; P_j B P_j \rangle}{\langle \varphi; P_j \rangle}$$

$$\lambda_j = \langle \varphi; P_j \rangle.$$

Second, the measurement must transfer information about $\mathscr{S}$ to $\mathscr{A}$, where we can 'read' it. We thus require that the effect of the measurement on the apparatus be

$$\lim_{t\to\infty} \langle \varphi \otimes \psi; \alpha_t[I \otimes M_i] \rangle = \sum_j \lambda_j \langle \psi_j; M_i \rangle$$

for all $M_i \in \{M_j\}$, where

$$\lambda_i = \langle \varphi; P_i \rangle$$

$$\langle \psi_j; M_i \rangle = \delta_{ij} \qquad \text{with 0 dispersion.}$$

Note that the λ's contain all information about the state φ available from R, so the pertinent information has indeed been transferred from $\mathscr{S}$ to $\mathscr{A}$.

Note also that we do not require the state of $\mathscr{A}$ after the measurement to take the form $\sum_j \lambda_j \psi_j$ on all of $\mathfrak{A}$. This is weaker than the usual requirement, but is physically reasonable, since the M_j's are the only observables which give us information about $\mathscr{S}$.

Third, it is physically desirable that the measurement produce an effect which the experimenter can perceive with this own senses. The infinite size of the measuring apparatus can indeed be used to give a precise meaning to the requirement that the effect of the measurement be macroscopic on $\mathscr{A}$. The M_j's will be space-averaged observables [5], and as such will belong to the center $\mathfrak{Z}_\psi = \pi_\psi(\mathfrak{A})'' \cap \pi_\psi(\mathfrak{A})'$ of the GNS representation for ψ. Other consequences of the infinite size of $\mathscr{A}$ will be touched upon in Section IV.

Fourth, the measuring process should be mechanistic. By this we mean that the evolution, with $\mathscr{S}$ coupled to $\mathscr{A}$, should be obtained as the infinite volume limit $(\Lambda \to \infty)$ of an Hamiltonian evolution.

Our last requirement is one of stability. If a measurement were to be successful only when the apparatus begins exactly in the initial state ψ, the experimenter would have to specify completely the state of his apparatus before he could begin. This clearly would make measurement too difficult a task. We therefore place one final requirement on the measuring process. The measurement must give the same result if $\mathscr{A}$ begins in any state ψ' in some class of states $\mathfrak{S}_0 \subset \mathfrak{S}$, which are experimentally 'close' to ψ:

(i) $\lim_{t\to\infty}\langle\varphi \otimes \psi'; \alpha_t[B \otimes I]\rangle = \sum_j \lambda_j\langle\varphi_j; B\rangle,$ for all $\varphi \in \mathfrak{T}$, all $\psi' \in \mathfrak{S}_0$, and all $B \in \mathfrak{B}$;

(ii) $\lim_{t\to\infty}\langle\varphi \otimes \psi'; \alpha_t[I \otimes M_i]\rangle = \sum_j \lambda_j\langle\psi_j; M_i\rangle,$ for all $\varphi \in \mathfrak{T}$, all $\psi' \in \mathfrak{S}_0$, and all $M_i \in \{M_j\}$.

A physically appealing candidate for $\mathfrak{S}_0$ is the subset $\mathfrak{S}_\psi$ of all normal states on the von Neumann algebra $\pi_\psi(\mathfrak{A})''$ canonically associated (by the GNS construction) to $\mathfrak{A}$ and ψ. In particular, $\mathfrak{S}_\psi$ contains the set of all states ψ' obtained from ψ by (quasi-)local disturbances (see, for instance, Winnink [7]): $\langle\psi'; A\rangle = \langle\psi; B^*AB\rangle$ with $B \in \mathfrak{A}$. Even more interestingly, $\mathfrak{S}_\psi$ contains all states [8] which satisfy the equilibrium KMS condition (for an introductory review, see for instance [5]) with respect to (quasi-) local perturbations of the original evolution defining ψ. We will show that the model described in Section II is stable enough to allow $\mathfrak{S}_0 \supseteq \mathfrak{S}_\psi$.

This completes our description of the quantum measuring process.

II. Description of the Model

In order to demonstrate that the class of measuring processes described previously is not empty, we now present an exactly solvable model which fulfills all the conditions listed in Section I.

In this model the system $\mathscr{S}$ is a single $\frac{1}{2}$-spin. The algebra of observables is $\mathfrak{B} = \mathscr{B}(\mathbf{C}^2)$. We will measure the z-component σ_0^z of the spin, with eigen-projectors $P_0^{\pm}$. The system begins in an arbitrary initial state $\varphi \in \mathfrak{T}$.

The apparatus consists of two infinite non-interacting chains of $\frac{1}{2}$-spins, labeled by σ_n and τ_n. These two chains have been prepared, at finite temperature β, by placing them in the magnetic fields B_σ and B_τ, respectively. Therefore, the initial state ψ is the corresponding Gibbs state.

For times $t \geqslant 0$, the magnetic fields are switched off, and the interaction between $\mathscr{S}$ and $\mathscr{A}$ is obtained, in the limit $M \to \infty$, from the Hamiltonian $H^M = P_0^+ \otimes H_A^M$, where

$$H_A^M = J \sum_{m=1}^{M} (\sigma_m^x \tau_m^x + \sigma_m^x \tau_{m+1}^x + \sigma_m^y \tau_m^y + \sigma_m^y \tau_{m+1}^y).$$

Note that the relabeling $\sigma_m \to S_{2m}$, $\tau_m \to S_{2m-1}$, converts H_A into the usual $x - y$ Hamiltonian:

$$H_A^N = J \sum_{n=1}^{N} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y), \qquad \text{where } N = 2M.$$

We can thus describe this apparatus as a one-dimensional quantum lattice system. To each site $n \in \mathbf{Z}^+$, we attach a C^*-algebra $\mathfrak{A}_n = \mathscr{B}(\mathbf{C}^2)$. For every finite subset $\Lambda \subset \mathbf{Z}^+$, we define $\mathfrak{A}(\Lambda) = \bigotimes_{n\in\Lambda} \mathfrak{A}_n$. The algebra of quasi-local observables for the infinite system is the C^*-inductive limit $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}(\Lambda)^n}$.

To properly define the initial state ψ and the time evolution, we first consider the finite system $\mathfrak{B} \otimes \mathfrak{A}([1, N])$. For $t \geqslant 0$, we define the time evolution α_t^N by

$$\alpha_t^N[C] = \exp[iH^N t]C\exp[-iH^N t] \qquad \text{for all} \quad C \in \mathfrak{B} \otimes \mathfrak{A}([1, N]).$$

On $\mathfrak{A}_n$ the state ψ_n is given by the density matrix

$$\rho_n = \exp(-\beta h_n)/Tr[\exp(-\beta h_n)] = \tfrac{1}{2}(1 + S_n^z \tanh \beta B_n), \qquad \text{where } h_n = -B_n S_n^z.$$

We then construct the state ψ^N on $\mathfrak{A}([1, N])$ as $\psi^N = \bigotimes_{n=1}^N \psi_n$. Clearly this state extends uniquely to a state $\psi = \bigotimes_{n \in Z^+} \psi_n$ on $\mathfrak{A}$ such that $\psi|_{\mathfrak{A}([1,N])} = \psi^N$ for all N. The fact that the ψ^N (and ψ) are product states will significantly simplify the analysis.

Since we wish to use this apparatus to measure σ_0^z, we need to find a set of operators $\{M_\pm\} \in \mathfrak{Z}_\psi$ which corresponds to the spectral projectors $P_0^\pm$ of σ_0^z. We first define the magnetization of the finite even (odd) chain as

$$M_\sigma^N = \frac{2}{N}\sum_{n=1}^{N/2} S_{2n}^z; \quad M_\tau^N = \frac{2}{N}\sum_{n=1}^{N/2} S_{2n-1}^z.$$

Now let $M_-^N = (\tanh \beta B_\sigma - \tanh \beta B_\tau)^{-1}(M_\sigma^N - M_\tau^N)$. We then let

$$M_- = w - \underset{N\to\infty}{\text{op}\lim}\, \pi_\psi(M_-^N),$$

and

$$M_+ = I - M_-$$

be the operators in $\mathfrak{Z}_\psi$ corresponding to $P_0^\pm$.

III. Analysis of the Model

In this section we will prove that the model described in Section II is indeed a measuring process, as defined in Section I. Thus we must show that the interaction between $\mathscr{S}$ and $\mathscr{A}$ reduces the wave function of $\mathscr{S}$; that it transfers information from $\mathscr{S}$ to $\mathscr{A}$; that the effect upon $\mathscr{A}$ is macroscopic; that the model is mechanistic; and that the measurement gives the same result for any state $\psi' \in \mathfrak{S}_\psi$.

By construction, the model is clearly mechanistic. The observables $M_\pm$ we have chosen to correspond with the eigenprojectors of σ_0^z are space averages, so the macroscopicity condition is satisfied automatically. In Theorem 1, we show the reduction of the wave function and the transfer of information for the simple case in which the apparatus begins in the initial state ψ. Our answer to the stability question is to be found in Theorem 2.

Theorem 1. Let $\lambda_\pm \equiv \langle \varphi; P_0^\pm \rangle$, $\langle \varphi_\pm; B\rangle \equiv (\lambda_\pm)^{-1}\langle \varphi; P_0^\pm B P_0^\pm\rangle$ for all $\varphi \in \mathfrak{T}$, all $B \in \mathfrak{B}$, and ψ, α_t^N, $M_\pm^N$, H_A^N be defined as in Section II. Then

(i) $\lim_{t\to\infty}\lim_{N\to\infty} \langle \varphi \otimes \psi; \alpha_t^N[B \otimes I]\rangle = \sum_j \lambda_j \langle \varphi_j; B\rangle \qquad$ for all $\varphi \in \mathfrak{T}$, $B \in \mathfrak{B}$.

(ii) $\lim_{t\to\infty}\lim_{N\to\infty} \langle \varphi \otimes \psi; \alpha_t^N[I \otimes M_\pm^N]\rangle = \sum_j \lambda_j \langle \psi_j; M_\pm\rangle \qquad$ where $\langle \psi_-; M_\pm\rangle \equiv$
$\langle \psi; M_\pm\rangle$,
$\langle \psi_+; M_\pm\rangle \equiv \lim_{t\to\infty}\lim_{N\to\infty} \langle \psi; \exp[iH_A^N t]M_\pm^N \exp[-iH_A^N t]\rangle$,
$\langle \psi_j; M_i\rangle = \delta_{ij} \qquad$ with zero dispersion.

Proof. We first notice that since

$$\exp[\pm iH^N t] = \exp[\pm itP_0^+ \otimes H_A^N] = P_0^+ \otimes \exp[\pm itH_A^N] + P_0^- \otimes I,$$

we have

$$\begin{aligned}\langle \varphi \otimes \psi; \alpha_t^N[B \otimes I]\rangle &= \langle \varphi \otimes \psi; P_0^+ BP_0^+ \otimes I + P_0^- BP_0^- \otimes I \\ &\quad + P_0^- BP_0^+ \otimes \exp[-iH_A^N t] + P_0^+ BP_0^- \otimes \exp[iH_A^N t]\rangle \\ &= \lambda_+\langle \varphi_+ ; B\rangle + \lambda_-\langle \varphi_- ; B\rangle + \langle \varphi; P_0^- BP_0^+\rangle \\ &\quad \times \langle \psi; \exp[-iH_A^N t]\rangle + \langle \varphi; P_0^+ BP_0^-\rangle\langle \psi; \exp[iH_A^N t]\rangle\end{aligned}$$

Furthermore

$$\begin{aligned}\langle \varphi \otimes \psi; \alpha_t^N[I \otimes M_\pm^N]\rangle &= \langle \varphi \otimes \psi; P_0^- \otimes M_\pm^N + P_0^+ \otimes \exp[iH_A^N t]M_\pm^N \exp[-iH_A^N t]\rangle \\ &= \lambda_-\langle \psi; M_\pm^N\rangle + \lambda_+\langle \psi; \exp[iH_A^N t]M_\pm^N \exp[-iH_A^N t]\rangle.\end{aligned}$$

Hence the Theorem follows directly from Lemmata 1 and 2 below.

Lemma 1. With ψ and H_A^N defined as in Section II

$$\lim_{N\to\infty} \langle \psi; \exp[\pm iH_A^N t]\rangle = 0.$$

Lemma 2. With ψ, H_A^N, M_-^N defined as in Section II

(i) $\langle \psi_- ; M_-\rangle = \langle \psi_- ; (M_-)^2\rangle = 1,$ where $\langle \psi_- ; \circ\rangle \equiv \langle \psi; \circ\rangle$;

(ii) $\langle \psi_+ ; M_-\rangle \equiv \lim_{t\to\infty} \lim_{N\to\infty} \langle \psi; \exp[iH_A^N t]M_-^N \exp[-iH_A^N t]\rangle = 0$;

(iii) $\langle \psi_+ ; (M_-)^2\rangle \equiv \lim_{t\to\infty} \lim_{N\to\infty} \langle \psi; \exp[iH_A^N t](M_-^N)^2 \exp[-iH_A^N t]\rangle = 0.$

Proof of Lemma 1. First, we must diagonalize the Hamiltonian H_A^N. We follow the procedure of Lieb, Schultz, and Mattis [9]. We redefine H_A^N in terms of the Fermion operator

$$C_n = \begin{cases} S_n^+ \exp\left[-\tfrac{1}{2}\pi i \sum_{k=1}^{n-1} (1 + S_k^z)\right] & \text{for } n = 1, 2, \ldots, N \\ C_1 & \text{for } n = N + 1. \end{cases}$$

(This choice of boundary conditions makes the analysis simpler.) We then have

$$H_A^N = 2J \sum_{n=1}^{N} (C_n^* C_{n+1} + C_{n+1}^* C_n).$$

We next introduce the normal mode Fermion operators

$$\eta_k = N^{-1/2} \sum_k e^{ikn} C_n \qquad \text{where } k = \frac{2\pi p}{N}, p = 1, 2, \ldots, N.$$

These operators diagonalize H_A^N:

$$H_A^N = 4J \sum_k \cos k\eta_k^* \eta_k.$$

We have thus

$$\exp[\pm iH_A^N t] = \prod_k (1 + h_k \eta_k^* \eta_k), \qquad \text{where } h_k = \exp(\pm 4iJt \cos k) - 1.$$

In order to evaluate the expectation value of this operator in the state ψ, we first notice that ψ is the Gibbs state for the evolution obtained from the free Hamiltonian: $h^N = -\sum_{n=1}^N B_n S_n^z$. Hence ψ is quasi-free and gauge-invariant, so that

$$\langle\psi;\, \eta_{k_1}^*\eta_{k_1}\ldots\eta_{k_T}^*\eta_{k_T}\rangle = \sum_{P\in\mathscr{P}_T} (-)^P \prod_{i=1}^{T} \langle\psi;\, \eta_{k_i}^*\eta_{k_{p(i)}}\rangle.$$

We now evaluate $\langle\psi;\, \eta_k^*\eta_l\rangle$:

$$\begin{aligned}\langle\psi;\, \eta_k^*\eta_l\rangle &= \frac{1}{N}\sum_{mn} e^{-ikm}e^{iln}\langle\psi;\, C_m^* C_n\rangle = \frac{1}{2N}(1+\tanh\beta B_\sigma)\sum_{n=1}^{N/2} e^{-2i(k-l)n}\\ &\quad + \frac{1}{2N}(1+\tanh\beta B_\tau)\sum_{n=1}^{N/2} e^{-i(2n-1)(k-l)}\\ &= \tfrac{1}{4}(2+\tanh\beta B_\sigma+\tanh\beta B_\tau)\delta_{kl} + \tfrac{1}{4}(\tanh\beta B_\sigma - \tanh\beta B_\tau)\delta_{k\bar{l}}\\ &\equiv a\delta_{kl} + b\delta_{k\bar{l}} \qquad \text{where } \bar{l} = l \pm \pi.\end{aligned}$$

This expectation value is therefore 0 unless k is equal to l, or to its 'conjugate' $l \pm \pi$. Hence, ψ acts as a product state, except on 'conjugate pairs'. Therefore we can write

$$\begin{aligned}\langle\psi;\, \exp[\pm iH_A^N t]\rangle &= \langle\psi;\, \prod_{k=2\pi/N}^{2\pi} (1 + h_k\eta_k^*\eta_k)\rangle\\ &= \prod_{k=2\pi/N}^{\pi} \langle\psi;\, (1+h_k\eta_k^*\eta_k)(1+h_{\bar{k}}\eta_{\bar{k}}^*\eta_{\bar{k}})\rangle\\ &= \prod_{k=2\pi/N}^{\pi} [1 + a(h_k + h_{\bar{k}}) + (a^2 - b^2)h_k h_{\bar{k}}]\\ &= \prod_{k=2\pi/N}^{\pi} \tfrac{1}{2}\{1 + \tanh\beta B_\sigma \tanh\beta B_\tau\\ &\quad + (1 - \tanh\beta B_\sigma\tanh\beta B_\tau)\cos(4Jt\cos k)\}.\end{aligned}$$

Clearly

$$|\tfrac{1}{2}\{1 + \tanh\beta B_\sigma\tanh\beta B_\tau + (1 - \tanh\beta B_\sigma\tanh\beta B_\tau)\cos(4Jt\cos k)\}| \leqslant 1$$

for all k. Further, unless $4Jt\cos k = 2\pi p$ (p an integer), this expression is strictly less than 1. We therefore have a product of $\frac{1}{2}N$ terms, all bounded by one, most of which are strictly less than one. Consequently, for all $t > 0$

$$\begin{aligned}|\langle\psi;\, \exp(\pm iH_A^N t)\rangle| \leqslant \prod_{k=2\pi/N}^{\pi} |\tfrac{1}{2}\{1 + \tanh\beta B_\sigma\tanh\beta B_\tau + (1 - \tanh\beta B_\sigma\tanh\beta B_\tau)\\ \times\cos(4Jt\cos k)\}| \to 0 \qquad \text{as } N\to\infty.\end{aligned}$$

Q.E.D.

Proof of Lemma 2. Since $\langle\psi_-;\, \circ\rangle \equiv \langle\psi;\, \circ\rangle$, part (i) is obviously true. Next, as in Lemma 1, we write H_A^N and $(M_\sigma^N - M_\tau^N)$ in terms of the normal modes

$$H_A^N = 4J\sum_k \cos k\,\eta_k^*\eta_k,$$

$$M_\sigma^N - M_\tau^N = \frac{4}{N}\sum_k \eta_k^*\eta_{\bar{k}}.$$

Then

$$\exp[+iH_A^N t](M_\sigma^N - M_\tau^N)\exp[-iH_A^N t]$$
$$= \frac{4}{N}\sum_k \{\exp[4iJt\cos k\eta_k^*\eta_k]\eta_k^* \exp[-4iJt\cos k\eta_k^*\eta_k]$$
$$\times \exp[4iJt\cos \bar{k}\eta_{\bar{k}}^*\eta_{\bar{k}}]\eta_{\bar{k}} \exp[-4iJt\cos \bar{k}\eta_{\bar{k}}^*\eta_{\bar{k}}]$$
$$= \frac{4}{N}\sum_k (1 + h_k)\eta_k^*\eta_{\bar{k}}(1 + h_{\bar{k}}^*) = \frac{4}{N}\sum_k \exp[8iJt\cos k]\eta_k^*\eta_{\bar{k}}.$$

Using the expression for $\langle\psi; \eta_k^*\eta_l\rangle$ found in Lemma 1, we have

$$\langle\psi; \exp[iH_A^N t](M_\sigma^N - M_\tau^N)\exp[-iH_A^N t]\rangle$$
$$= \frac{1}{N}(\tanh \beta B_\sigma - \tanh \beta B_\tau)\sum_k \exp[8iJt\cos k].$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_k \exp[8iJt\cos k] = \frac{1}{2\pi}\int_0^{2\pi}\exp[8iJt\cos k]\,dk = J_0(8Jt).$$

Since the Bessel function $J_0(t)$ behaves like $t^{-1/2}$ for large t, we have

$$\lim_{t\to\infty}\lim_{N\to\infty}\langle\psi; \exp[iH_A^N t]M_-^N \exp[-iH_A^N t]\rangle = \lim_{t\to\infty} J_0(8Jt) = 0.$$

By a similar computation we get

$$\lim \langle\psi; \exp[itH_A^N](M_-^N)^2 \exp[-itH_A^N]\rangle = J_0(8Jt)^2$$
$$= (\lim_{N\to\infty}\langle\psi; \exp[itH_A^N]M_-^N \exp[-itH_A^N]\rangle)^2.$$

And thus

$$\langle\psi_+; (M_-)^2\rangle = (\langle\psi_+; M_-\rangle)^2 = 0. \qquad \text{Q.E.D.}$$

We now turn to the stability problem. Theorem 2 below shows that our model is indeed resilient enough to withstand the substitution of any ψ' in $\mathfrak{S}_\psi$ for the initial state ψ in Theorem 1.

Theorem 2. Let: $\varphi \in \mathfrak{T}$, $B \in \mathfrak{B}$, ψ, α_t^N, M_j^N, M_j be defined as in Section II; λ_j, φ_j, ψ_j be as in Theorem 1; and ψ' be any state in $\mathfrak{S}_\psi$. Then

(i) $\lim_{t\to\infty}\lim_{N\to\infty}\langle\varphi\otimes\psi'; \alpha_t^N[B\otimes I]\rangle = \sum_j \lambda_j\langle\varphi_j; B\rangle,$

(ii) $\lim_{t\to\infty}\lim_{N\to\infty}\langle\varphi\otimes\psi'; \alpha_t^N[I\otimes M_\pm^N]\rangle = \sum_j \lambda_j\langle\psi_j; M_\pm\rangle.$

Because the detailed computations involved in the proof of this theorem are rather tedious, we only outline the argument. See [2] for the detailed proofs. We begin with three lemmata.

Lemma 3. Let ψ be the state defined in Section II; ψ_B for $B \in \mathfrak{A}$ be defined by $\langle\psi_B; A\rangle = \langle\psi; B^*AB\rangle$ for all $A \in \mathfrak{A}$; $\{C^N\}$ be a sequence of operators in $\mathfrak{A}$ such that

(i) $C^N \in \mathfrak{A}([1, N])$ for all N,
(ii) $\|C^N\| \leqslant C < \infty$ for all N,
(iii) $\lim_{N\to+\infty}\langle\psi - \psi_B; C^N\rangle = 0$ for all $B \in \mathfrak{A}_0 = \bigcup_\Lambda \mathfrak{A}(\Lambda).$

Then: $\lim_{N\to\infty}\langle\psi - \psi'; C^N\rangle = 0$ for all $\psi' \in \mathfrak{S}_\psi$.

Proof. We first show that $\lim_{N\to\infty}\langle\psi - \psi_B; C^N\rangle = 0$ for all $B \in \mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}(\Lambda)}^n$. If $B \in \mathfrak{A}$, for all $\epsilon > 0$, there exists a $B_0 \in \mathfrak{A}_0$ such that

$$\|B - B_0\|(2\|B\| + \|B - B_0\|) < \epsilon/2C.$$

Assumption (iii) implies that there exists an N_ϵ such that

$$|\langle\psi - \psi_{B_0}; C^N\rangle| < \epsilon/2 \qquad \text{for all } N > N_\epsilon.$$

Then

$$\begin{aligned}|\langle\psi - \psi_B; C^N\rangle| &\leqslant |\langle\psi; C^N - B_0^*C^NB_0\rangle| + |\langle\psi; B_0^*C^NB_0 - B_0^*C^NB\rangle| \\ &\quad + |\langle\psi; B_0^*C^NB - B^*C^NB\rangle| \\ &\leqslant |\langle\psi - \psi_{B_0}; C^N\rangle| + \|B - B_0\|C(2\|B\| + \|B - B_0\|) \\ &< \epsilon, \qquad \text{for all } N > N_\epsilon.\end{aligned}$$

This proves our first assertion that $\lim_{N\to\infty}\langle\psi - \psi_B; C^N\rangle = 0$ for $B \in \mathfrak{A}$.

We next show that, for every $\psi' \in \overline{\{\psi_B \mid B \in \mathfrak{A}\}}^n$, $\lim_{N\to\infty}\langle\psi - \psi'; C^N\rangle = 0$. For all $\epsilon > 0$, there exists a $B \in \mathfrak{A}$ such that $\|\psi' - \psi_B\| < \epsilon/2C$. Choose N_ϵ such that $|\langle\psi - \psi_B; C^N\rangle| < \epsilon/2$, for all $N > N_\epsilon$. Then

$$\begin{aligned}|\langle\psi - \psi'; C^N\rangle| &\leqslant |\langle\psi - \psi_B; C^N\rangle| + |\langle\psi_B - \psi'; C^N\rangle| \\ &< \epsilon/2 + \|\psi' - \psi_B\|C \\ &< \epsilon, \qquad \text{for all } N > N_\epsilon.\end{aligned}$$

The last step of the proof consists in showing that $\overline{\{\psi_B \mid B \in \mathfrak{A}\}}^n = \mathfrak{S}_\psi$. We first notice (Prop. 2.4.8.(ii) in [10]) that $\overline{\{\psi_B \mid B \in \mathfrak{A}\}}^n = \mathfrak{V}_\psi$, the set of all vector states on $\pi_\psi(\mathfrak{A})$, with $\{\psi_\psi, \mathscr{H}_\psi, \Psi\}$ denoting the GNS triple associated to ψ. Since, on the other hand, ψ is KMS for the evolution obtained from $h^N = -\sum_{n=1}^N B_nS_n^z$, ψ is faithful. Thus, Ψ is a cyclic and separating vector for $\pi_\psi(\mathfrak{A})''$ in $\mathscr{H}_\psi$. Therefore, (Thm. 4, p. 233 in [11]) $\mathfrak{V}_\psi = \mathfrak{S}_\psi$. Q.E.D.

Lemma 4. Let ψ and H_A^N be defined as before, and ψ' be in $\mathfrak{S}_\psi$. Then

$$\lim_{N\to\infty}\langle\psi'; \exp[\pm iH_A^Nt]\rangle = 0.$$

Proof. Since $\lim_{N\to\infty}\langle\psi; \exp[\pm iH_A^Nt]\rangle = 0$ (Lemma 1), and since $C^N = \exp[\pm iH_A^Nt]$ satisfies conditions (i) and (ii) of Lemma 3, it is sufficient to show that

$$\lim_{N\to\infty}\langle\psi_B; \exp[\pm iH_A^Nt]\rangle = 0$$

for $B \in \mathfrak{A}_0$.

We first show that, for finite T:

$$\begin{aligned}&|\langle\psi; \eta_{k_1}^* \dots \eta_{k_s}^*\eta_{l_1} \dots \eta_{l_{s'}} \exp[\pm iH_A^Nt]\eta_{k_{s+1}}^* \dots \eta_{k_T}\eta_{l_{s'+1}} \dots \eta_{l_T}\rangle| \\ &\quad\leqslant |\langle\psi; \eta_{k_1}^* \dots \eta_{k_s}^*\eta_{l_1} \dots \eta_{l_s} \prod_{i=1}^{T}{}'(H_{k_i}H_{\overline{k_i}}H_{l_i}H_{\overline{l_i}})\eta_{k_{s+1}}^* \dots \eta_{k_T}^*\eta_{l_{s'+1}} \dots \eta_{l_T}\rangle| \\ &\qquad\times |\langle\psi; \prod_{\substack{p=2\pi/N \\ p\neq k,l}}^{\pi} H_pH_{\bar{p}}\rangle|.\end{aligned}$$

where $H_p = 1 + (\exp[\pm 4iJt\cos p] - 1)\eta_p^*\eta_p$, and $\prod'$ designates a product over distinct 'conjugate pairs'. This follows easily from the fact that ψ behaves like a product state for the η's, except on 'conjugate pairs'.

It is then possible to show that, for finite T

$$|\langle\psi; C_{n_1}^* \dots C_{n_s}^* C_{m_1} \dots C_{m_{s'}} \exp[\pm iH_A^N t] C_{n_{s+1}}^* \dots C_{n_T}^* C_{m_{s'+1}} \dots C_{m_T}\rangle|$$
$$\leqslant 1/N^T \sum_{q_1,\dots,q_T=1}^{N} T!2^T |\langle\psi; \prod_{\substack{p=2\pi/N \\ p\neq q}}^{\pi} H_p H_{\bar{p}}\rangle|.$$

The expectation value

$$\langle\psi; \prod_{\substack{p=2\pi/N \\ p\neq q}}^{\pi} H_p H_{\bar{p}}\rangle = \prod_{\substack{p=2\pi/N \\ p\neq q}}^{\pi} \tfrac{1}{2}\{1 + \tanh\beta B_\sigma \tanh\beta B_\tau$$
$$+ (1 - \tanh\beta B_\sigma \tanh\beta B_\tau)\cos(4Jt\cos p)\}$$

is the same product as that which appears in the proof of Lemma 1, except that at most T factors have been removed. Since T is finite, this expectation value will go to zero as $N \to \infty$, and therefore

$$|\langle\psi; C_{n_1}^* \dots C_{n_s}^* C_{m_1} \dots C_{m_{s'}} \exp[\pm iH_A^N t] C_{n_{s+1}}^* \dots C_{n_T}^* C_{m_{s'+1}} \dots C_{m_T}\rangle| \to 0$$
$$\text{as } N \to \infty.$$

Now, if $B \in \mathfrak{A}_0$, it can be written as:

$$B = \sum_{r=1}^{R} \bigotimes_{n=1}^{N_0} \{\alpha_n^r I_n + \beta_n^r C_n^* C_n + \gamma_n^r C_n + \delta_n^r C_n^*\},$$

where R and N_0 are finite, and $\alpha, \beta, \gamma, \delta \in \mathbf{C}$. Since ψ and $\exp[\pm iH_A^N t]$ are gauge-invariant, $\langle\psi_B; \exp[\pm iH_A^N t]\rangle \equiv \langle\psi; B^* \exp[\pm iH_A^N t]B\rangle$ will be a *finite* sum of terms of the form

$$\langle\psi; C_{n_1}^* \dots C_{n_s}^* C_{m_1} \dots C_{m_{s'}} \exp[\pm iH_A^N t] C_{n_{s+1}}^* \dots C_{n_T}^* C_{m_{s'+1}} \dots C_{m_T}\rangle.$$

Therefore

$$\lim_{N\to\infty} \langle\psi_B; \exp[\pm iH_A^N t]\rangle = 0 \qquad \text{for } B \in \mathfrak{A}_0. \qquad \text{Q.E.D.}$$

Lemma 5. Let: ψ, M_-^N, H_A^N be defined as before; and ψ' be in $\mathfrak{S}_\psi$. Then

$$\lim_{N\to\infty} \langle\psi - \psi'; \exp[iH_A^N t] M_-^N \exp[-iH_A^N t]\rangle = 0.$$

Proof. Since $C^N = M_-^N$ satisfies assumptions (i) and (ii) of Lemma 3, it is sufficient to show that $\lim_{N\to\infty}\langle\psi - \psi_B; \exp[iH_A^N t] M_-^N \exp[-iH_A^N t]\rangle = 0$ for $B \in \mathfrak{A}_0$.

For each $B \in \mathfrak{A}_0$, there exists an $N_0 < \infty$ such that $B \in \mathfrak{A}([1, N_0])$. Choose $N > N_0$. Then

$$\exp[iH_A^N t] M_-^N \exp[-iH_A^N t]$$
$$= \frac{4}{N}(\tanh\beta B_\sigma - \tanh\beta B_\tau)^{-1} \sum_k \exp[8iJt\cos k]\eta_k^*\eta_{\bar{k}}$$
$$= \frac{4}{N^2}(\tanh\beta B_\sigma - \tanh\beta B_\tau)^{-1} \sum_k \exp[8iJt\cos k] \sum_{m,n=1}^{N} e^{i\pi n} e^{-ik(m-n)} C_m^* C_n.$$

The sums over m and n can each be divided into a sum over sites from 1 to N_0, and a sum over sites from $N_0 + 1$ to N. If one index is larger than N_0, and the other is smaller than or equal to N_0, the expectation value for the linear form $\psi - \psi_B$ will be zero, since $\psi - \psi_B$ acts like a gauge-invariant product form for sites beyond N_0. We therefore have

$$\begin{aligned}&\langle\psi - \psi_B; \exp[iH_A^N t]M_-^N \exp[-iH_A^N t]\rangle\\ &\quad = (\tanh\beta B_\sigma - \tanh\beta B_\tau)^{-1}\frac{4}{N^2}\sum_k \exp[8iJt\cos k]\Bigg\{\sum_{m,n=1}^{N_0} e^{i\pi n}e^{-ik(m-n)}\\ &\quad\times\langle\psi - \psi_B; C_m^* C_n\rangle + \sum_{m,n=N_0+1}^{N} e^{-i\pi n}e^{-ik(m-n)}\langle\psi - \psi_B; C_m^* C_n\rangle\Bigg\}.\end{aligned}$$

The first sum inside the brackets is a finite sum, majorized by $2N_0^2$. The second term is identically zero, since ψ and ψ_B are different only on the first N_0 sites. Therefore

$$|\langle\psi - \psi_B; \exp[iH_A^N t]M_-^N \exp[-iH_A^N t]\rangle| \leqslant (\tanh\beta B_\sigma - \tanh\beta B_\tau)^{-1}(8N_0^2)/N.$$

Since N_0 is fixed and finite, we have

$$\lim_{N\to\infty}\langle\psi - \psi_B; \exp[iH_A^N t]M_-^N \exp[-iH_A^N t]\rangle = 0 \qquad \text{for all } B \in \mathfrak{A}_0. \qquad \text{Q.E.D.}$$

Proof of Theorem 2. The proof of this theorem proceeds in a manner analogous to that of Theorem 1, using Lemmata 4 and 5 in place of Lemmata 1 and 2.

Q.E.D.

Theorem 2 completes the demonstration that this model behaves like a quantum measuring process, as defined in Section I.

IV. Concluding Remarks

The model defined in Section II and analyzed in Section III shows conclusively that the quantum measuring process can be cast, as outlined in Section I, in the general framework of quantum mechanics. Specifically, a time-independent Hamiltonian interaction between the system $\mathscr{S}$ to be measured and the apparatus $\mathscr{A}$ can be rigged up in such a manner that the following essential conditions are satisfied. As time proceeds to $+\infty$. (i) the initial state φ of $\mathscr{S}$ experiences von Neumann's 'reduction of the wave packet'; (ii) the relevant information contained in φ is transferred to $\mathscr{A}$; (iii) this effect becomes macroscopic on $\mathscr{A}$; and (iv) the process is stable under a large class of perturbations of the initial state ψ of $\mathscr{A}$.

As a particular consequence of property (iii), the reading of the result of the measurement on $\mathscr{A}$ belongs to the realm of classical mechanics, where von Neumann's [6], [12] infinite regression is innocuous.

The main point of this paper being to prove the consistency of the scheme proposed in Section I, we have constructed one specific, exactly solvable, mechanistic model. As one might have expected, a price has to be paid for simplicity and exact solvability. First, for simplicity's sake, we have restricted ourselves to the measurement of observables with discrete spectrum; this should ultimately be dispensed with. Second, our interaction is admittedly somewhat contrived. One peculiarity is that the reduction of the wave packet is instantaneous (see proofs of Lemmata 1 and 4). The effect of the interaction on the apparatus, however, is much tamer; the limit is approached as $t^{-1/2}$ (see proofs of Lemmata 2 and 5).

Finally, we have taken the apparatus to be an infinite system. As usual, the main justification for this idealization is to avoid the spurious recurrences associated with large but finite systems. It also has the advantage of bringing into focus the macroscopic aspect of a quantum measuring process in which information is transferred from the microscopic quantum level to the classical level of description.

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