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Autor: Straumann, N.

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On Radiative Fluids

by **N. Straumann**

University of Zürich and SIN

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Abstract. The Thomas calculation of the energy-momentum tensor of radiation quanta, which are nearly in thermal equilibrium with a relativistically moving material medium, is simplified and extended to more general transport equations. It is shown that the fluid (matter plus radiation) behaves, independently of the detailed form of the Boltzmann equation, like a relativistic imperfect fluid (in the formulation of Eckart). General expressions for the coefficients of heat conduction, shear viscosity and bulk viscosity are given. Published formulae for these coefficients in special cases are corrected.

In this note we consider a fluid, consisting of some material medium in relativistic motion which is locally in thermal equilibrium, plus radiation quanta (photons, neutrinos) which are not exactly in thermal equilibrium with the material medium. The study of such fluids is relevant for various astrophysical and cosmological problems (e.g. the damping of protogalactic fluctuations; see for instance Ref. [2]).

Thomas [1] has computed the energy-momentum tensor of the radiation field by solving the relativistic transport equation to first order in an expansion around the local thermal equilibrium (i.e. to first order in the mean free path). Weinberg [2] has used the results of Thomas to show that the fluid (matter plus radiation) behaves like a relativistic imperfect fluid, as described by Eckart [3] (for a rederivation of Eckart's theory see also [2] or [4]), and has given explicit expressions for the radiative heat conduction, shear viscosity and bulk viscosity.

The derivations of Thomas [1] are rather lengthy and cumbersome. Furthermore, his transport equation takes only emission and absorption processes into account and does not include scattering processes (at least not properly). The purpose of this note is to both generalize and simplify the work of Thomas. The simplification arises in keeping the formalism manifestly Lorentz-invariant. The conclusion that the fluid behaves like a relativistic imperfect fluid (in the formulation of Eckart) does not rely on any particular assumptions about the right hand side of the Boltzmann equation. Only the coefficients for heat conduction and viscosity depend on the specific model. We show that a proper treatment of scattering changes the results for the viscosity coefficients.

1. Solution of the Transport Equation to First Order in the Mean Free Path

Let $F(k, x)$ be the distribution function of the radiation quanta. F is normalized such that $F(k, x)d^3xd^3k$ gives the number of quanta in the volume element d^3x at the space-time point x, and whose 3-momenta $\mathbf{k}$ lie within d^3k. With this definition $F(k, x)$

is Lorentz-invariant. This follows for instance from the following form of the particle current four-vector ($c = \hbar = 1$):

$$N^\mu(x) = \int k^\mu F(k, x) \frac{d^3k}{k^0},$$

and the fact that d^3k/k^0 is a Lorentz-invariant measure.

Let U^μ be the four-velocity, $(U \cdot U = 1)$, of the material medium and $T(x)$ its local temperature. We assume that $F(k, x)$ is close to the local thermal equilibrium and write

$$F = F^{(0)} + F^{(1)}, \tag{1}$$

where (we consider for definiteness photons)

$$F^{(0)}(k, x) = \frac{2}{(2\pi)^3} \frac{1}{\exp(k \cdot U/k_B T) - 1}, \tag{2}$$

and $|F^{(1)}| \ll F^{(0)}$. To first order, the transport equation has the following general form

$$k^\mu \frac{\partial}{\partial x \mu} F^{(0)} = L[F^{(1)}], \tag{3}$$

where L is a linear operator. Specific cases for L will be discussed later. At the moment, we keep the arguments very general.

Let G_x be the little group belonging to $U(x)$ (i.e. the group of all Lorentz-transformations leaving $U(x)$ invariant). G_x is isomorphic to SO(3). It is convenient to consider $F(k, x)$ as a function of $\omega = (k \cdot U)$, n^μ and x, where

$$n^\mu = k^\mu/(k \cdot U) - U^\mu. \tag{4}$$

$\omega = k \cdot U$ is obviously invariant under G_x for every point x, and n^μ transforms according to the representation D^1 of G_x. Note that $n \cdot U = 0$ and $n \cdot n = -1$.

As in Ref. [5] we expand $F^{(1)}(\omega, n^\mu, x)$ with respect to n^μ into irreducible (under G_x) polynomials

$$F^{(1)}(\omega, n^\mu, x) = A(\omega, x) + B_\mu(\omega, x)n^\mu + C_{\mu\nu}(\omega, x)(n^\mu n^\nu + \tfrac{1}{3}h^{\mu\nu}) + \cdots, \tag{5}$$

where

$$h^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu \tag{6}$$

is the projection operator on the hyperplane normal to U^μ. Since $h^\mu_\nu n^\nu = n^\mu$ we can require that $h^\mu_\nu \cdot B^\nu = B^\mu$. Similarly, we can, without loss of generality, assume that $C_{\mu\nu}$ is symmetric and satisfies the conditions

$$h^\lambda_\mu C_{\lambda\nu} = C_{\mu\nu}, \quad C^\lambda_\lambda = 0$$

(the latter since $n^\mu n_\nu + \frac{1}{3}h^\mu_\nu$ is traceless). In a comoving reference system at x (i.e. a system where $U(x) = (1, 0, 0, 0)$), the expansion (5) reduces to an expansion in terms of spherical harmonics of the unit vector $\hat{\mathbf{k}}$. The terms not written in (5) are not needed for the calculation of the energy momentum tensor.

The linear operator L is clearly a scalar with respect to the group G_x. Hence it operates in the irreducible subspaces spanned by the irreducible polynomials in (5) as a multiple of the unit operator. We can thus write

$$L[F^{(1)}] = -\omega[\kappa_0 A + \kappa_1 B_\mu n^\mu + \kappa_2 C_{\mu\nu}(n^\mu n^\nu + \tfrac{1}{3}h^{\mu\nu}) + \cdots], \tag{7}$$

where the κ_i are functions of ω and x only. For a given model they can (in principle) be calculated. Examples will be given later.

Using the transport equation (3), we next express A, B_μ and $C_{\mu\nu}$ in terms of U^μ, T and κ_i. In order to do this, we introduce a measure $d\Omega_U$ on the two-dimensional surface $S = \{k|k^2 = 0, k^0 > 0, k\cdot U = \text{const.}\}$. $d\Omega_U$ is defined to be the unique G-invariant measure on S, normalized such that

$$\int_S d\Omega_U = 4\pi.$$

In a comoving system $d\Omega_U$ reduces to the angular integration of the vector $\hat{\mathbf{k}}$. The irreducible polynomials in (7) are orthogonal with respect to $d\Omega_U$. They are normalized as follows:

$$\frac{1}{4\pi}\int_S n^\mu n^\nu d\Omega_U = -\tfrac{1}{3}h^{\mu\nu},$$

$$\frac{1}{4\pi}\int_S (n^\mu n^\nu + \tfrac{1}{3}h^{\mu\nu})(n^\sigma n^\rho + \tfrac{1}{3}h^{\sigma\rho})d\Omega_U = \tfrac{1}{15}(h^{\mu\nu}h^{\sigma\rho} + h^{\mu\sigma}h^{\nu\rho} + h^{\mu\rho}h^{\nu\sigma}) - \tfrac{1}{9}h^{\mu\nu}h^{\sigma\rho}.$$

Taking moments of (3) and using (7) we obtain

$$\frac{1}{4\pi}\int_S k^\lambda\partial_\lambda F^{(0)}d\Omega_U = -\omega\kappa_0 A, \tag{8}$$

$$\frac{1}{4\pi}\int_S n_\mu k^\lambda\partial_\lambda F^{(0)}d\Omega_U = \tfrac{1}{3}\omega\kappa_1 B_\mu, \tag{9}$$

$$\frac{1}{4\pi}\int_S (n_\mu n_\nu + \tfrac{1}{3}h_{\mu\nu})k^\lambda\partial_\lambda F^{(0)}d\Omega_U = -\tfrac{2}{15}\omega\kappa_2 C_{\mu\nu}. \tag{10}$$

With equation (2) we now evaluate the left hand sides of (8), (9) and (10). We have, putting $F^{(0)}(k, x) = \phi(\omega/T)$:

$$\frac{1}{4\pi}\int_S k^\mu\partial_\mu F^{(0)}d\Omega_U = \frac{\omega}{T}\phi'\cdot\frac{1}{4\pi}\int_S\left[\frac{1}{\omega}k^\mu k_\nu\partial_\mu U^\nu - \frac{1}{T}k^\mu\partial_\mu T\right]d\Omega_U.$$

Under the integral sign we can replace $k^\mu\partial_\mu T$ by (see (4)) $\omega U^\mu\partial_\mu T$ and $k^\mu k_\nu\partial_\mu U^\nu$ by $\omega^2 n^\mu n_\nu\partial_\mu U^\nu$. Hence we obtain

$$\frac{1}{4\pi}\int_S k^\mu\partial_\mu F^{(0)}d\Omega_U = -\frac{\omega^2}{T}\phi'(\omega/T)\cdot\left[\frac{1}{T}U^\mu\partial_\mu T + \frac{1}{3}U^\mu_{,\mu}\right]. \tag{11}$$

Consequently (8) gives

$$A = \frac{1}{\kappa_0}\frac{\omega}{T}\phi'\left[\frac{1}{T}U^\mu\partial_\mu T + \frac{1}{3}U^\mu_{,\mu}\right]. \tag{12}$$

Similarly, we find from (9) and (10)

$$B_\mu = \frac{1}{\kappa_1}\frac{\omega}{T}\phi'\left[\frac{1}{T}h^\nu_\mu\partial_\nu T - U^\nu\partial_\nu U_\mu\right], \tag{13}$$

$$C_{\mu\nu} = -\frac{1}{2\kappa_2}\frac{\omega}{T}\phi'[h^\lambda_\mu\partial_\lambda U_\nu + h^\lambda_\nu\partial_\lambda U_\mu - \tfrac{2}{3}h_{\mu\nu}U^\lambda_{,\lambda}]. \tag{14}$$

2. Energy-momentum Tensor of the Radiation Field

The energy-momentum tensor $T_R^{\mu\nu}$ of the photons is given by

$$T_R^{\mu\nu}(x) = \int k^\mu k^\nu F(k, x) \frac{d^3k}{k^0}. \tag{15}$$

From $F^{(0)}$ we obtain the expected contribution:

$$\int k^\mu k^\nu F^{(0)} \frac{d^3k}{k^0} = \int_0^\infty d\omega\omega^3\phi(\omega/kT)\cdot\int (n^\mu + U^\mu)(n^\nu + U^\nu)d\Omega_U$$
$$= aT^4(U^\mu U^\nu - \tfrac{1}{3}h^{\mu\nu}).$$

We add this to the matter tensor $T_M^{\mu\nu}$, which is taken to be that of an ideal fluid:

$$T_M^{\mu\nu} = \rho_M U^\mu U^\nu - p_M(g^{\mu\nu} - U^\mu U^\nu),$$

where p_M and ρ_M are the material pressure and energy density. The total energy-momentum tensor $T^{\mu\nu}$ of the fluid (matter plus radiation) is thus

$$T^{\mu\nu} = \rho(T, n)U^\mu U^\nu - p(T, n)(g^{\mu\nu} - U^\mu U^\nu) + \int k^\mu k^\nu F^{(1)} \frac{d^3k}{k^0}. \tag{16}$$

Here T and n are the temperature and number density of the material medium, and $p(T, n)$, $\rho(T, n)$ are the total pressure and energy density, that matter and radiation would have if they were in thermal equilibrium at temperature T.

From (5), (12), (13) and (14) a straightforward calculation gives

$$\int k^\mu k^\nu F^{(1)}(k, x) \frac{d^3k}{k^0} = -\frac{4aT^4}{\bar{\kappa}_0}\left(U^\mu U^\nu - \frac{1}{3}h^{\mu\nu}\right)\left(\frac{1}{T}U^\lambda T_{,\lambda} + \frac{1}{3}U^\lambda_{,\lambda}\right)$$
$$+ \frac{4aT^3}{3\bar{\kappa}_1}(h^{\mu\lambda}U^\nu + h^{\nu\lambda}U^\mu)(T_{,\lambda} - TU^\sigma U_{\lambda,\sigma})$$
$$+ \frac{4aT^4}{15\bar{\kappa}_2}h^{\mu\sigma}h^{\nu\rho}\left(U_{\sigma,\rho} + U_{\rho,\sigma} - \frac{2}{3}g_{\sigma\rho}U^\lambda_{,\lambda}\right). \tag{17}$$

Here $\bar{\kappa}_i$ denotes the Rosseland mean of κ_i:

$$\frac{1}{\bar{\kappa}_i} = \frac{\int_0^\infty (1/\kappa_i)\omega^4\phi'(\omega/T)\,d\omega}{\int_0^\infty \omega^4\phi'(\omega/T)\,d\omega}. \tag{18}$$

For $\bar{\kappa}_0 = \bar{\kappa}_1 = \bar{\kappa}_2$ equation (17) reduces to the Thomas expression.

Now the dissipative term $\Delta T^{\mu\nu}$ in the phenomenological theory of Eckart reads (see Ref. [2])

$$\Delta T^{\mu\nu} = \chi(h^{\mu\lambda}U^\nu + h^{\nu\lambda}U^\mu)(T_{,\lambda} - TU^\sigma U_{\lambda,\sigma})$$
$$+ \eta h^{\mu\sigma}h^{\nu\rho}(U_{\sigma,\rho} + U_{\rho,\sigma} - \tfrac{2}{3}g_{\sigma\rho}U^\lambda_{,\lambda}) + \zeta h^{\mu\nu}U^\lambda_{,\lambda}. \tag{19}$$

If the first term in (17) would be absent, the expression (17) would be consistent with (19). The discrepancy between (17) and (19) arises, as has been pointed out by Weinberg [2], because the temperature in the general theory of Eckart does not coincide with the temperature of the material medium which appears in (16) and (17). If one redefines the temperature in (16) and (17) according to the Eckart definition, then the result (17) is indeed consistent with (19). This can be shown along the same lines as in [2]. The second and the third term in (17) remain unchanged, whereas the first term is replaced by

$$\frac{4aT^4}{\bar{\kappa}_0}\cdot\left[\frac{1}{3}-\frac{(\partial p/\partial T)_n}{(\partial \rho/\partial T)_n}\right]^2 h^{\mu\nu}U^{\lambda}_{,\lambda}$$

and has thus the form of the last term in (19). For the coefficients χ, η and ζ one obtains

$$\chi = \frac{4aT^3}{3\bar{\kappa}_1}, \tag{20}$$

$$\eta = \frac{4aT^4}{15\bar{\kappa}_2}, \tag{21}$$

$$\zeta = \frac{4aT^4}{\bar{\kappa}_0}\left[\frac{1}{3}-\frac{(\partial p/\partial T)_n}{(\partial \rho/\partial T)_n}\right]^2. \tag{22}$$

These equations generalize the results obtained by Weinberg [2].

3. A Specific Model

If the material medium is not degenerate and if the temperature is not too high, then the transport equation has the following form (see e.g. Ref. [6])

$$\begin{aligned} k^{\mu}\partial_{\mu}F &= -\omega n\sigma_a(\omega)(1-e^{-\omega/kT})(F-F^{(0)}) \\ &\quad - \omega n\sigma_s(\omega)\left[F(k)-\int p(n',n)F(\omega,n')\,d\Omega'_U\right]. \end{aligned} \tag{23}$$

Here $\sigma_a(\omega)$ and $\sigma_s(\omega)$ are the absorption and scattering cross sections, respectively, and

$$p(n',n) = \frac{1}{\sigma_s}\frac{d\sigma_s(k',k)}{d\Omega'_U}.$$

In (23) we have only included coherent scattering. For Thomson scattering we have

$$p(n',n) = \frac{1}{4\pi}\left[1+\tfrac{3}{4}(n^{\mu}n^{\nu}+\tfrac{1}{3}h^{\mu\nu})(n'_{\mu}n'_{\nu}+\tfrac{1}{3}h_{\mu\nu})\right]. \tag{24}$$

If we assume that the dominant scattering process is Thomson scattering, we obtain for the coefficients κ_i in (7), from (23) and (24), the following expressions:

$$\begin{aligned} \kappa_0(\omega) &= n\sigma_a(\omega)(1-e^{-\omega/kT}), \\ \kappa_1(\omega) &= n[\sigma_a(\omega)(1-e^{-\omega/kT})+\sigma_{Th}], \\ \kappa_2(\omega) &= n[\sigma_a(\omega)(1-e^{-\omega/kT})+\tfrac{9}{10}\sigma_{Th}]. \end{aligned} \tag{25}$$

In the particular case in which Thomson scattering dominates, we obtain from (20), (21), (22) and (25)

$$\begin{aligned} \chi &= \frac{4aT^3}{3n\sigma_{Th}}, \\ \eta &= \frac{10}{9}\frac{4aT^4}{15n\sigma_{Th}}, \\ \zeta &= 0. \end{aligned} \tag{26}$$

These expressions differ in two respects from Weinberg's results: (i) η is larger by a factor 10/9 (this has also been pointed out in Ref. [5]); (ii) the *bulk viscosity vanishes*. These differences come about because the transport equation used by Thomas does not properly describe scattering processes. From (22) and (25), one sees that only absorption processes contribute to the bulk viscosity.

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