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On the description of classical Einstein relativistic two-particle systems

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Abstract. We start by considering the system of one free particle, and give a sufficiently general description of this system to include the center of mass of systems of several particles. We then pass to the system of two particles. We define the coordinates separating the center of mass and the internal system, and we discuss the dynamics. Finally we outline the construction of a more restrictive two-particle theory, and study some consequences of the definition of a particle in an external field as a two-particle system in the limit where the mass of one of the particles becomes infinite.

1. Introduction

An attempt to construct a general Hamiltonian formalism on the theory of Special Relativity encounters two obstacles. The first of these is related to the fact that the Hamiltonian formalism for the description of several particles presupposes the existence of a universal dynamical time serving as a parameter measuring the *evolution* of and *correlation* between the states of the individual particles. The second problem concerns the separation of coordinates into center of mass coordinates and internal coordinates for systems of two or more particles, and entails a reconciliation of the linear structure imposed by the conservation of 'four-momentum' law with the hyperbolic structure of the Einstein addition of velocity law.

The first difficulty is disposed of by the introduction of invariant time. Such a time concept is, in fact, available in the theory of Special Relativity if one accepts to take as invariant time the time of what Einstein calls the stationary systems [1, 2]. To avoid the second problem we have chosen a state-space which is sufficiently big to accept a linear (affine) action of the inhomogeneous Lorentz group. Certainly, in this way, 'states' are introduced which are not physical, and to get rid of them we postulate constraints. Throughout, definitions are chosen such that we, in special cases, obtain known results. Moreover, the theory is formulated in such a way that the Galilean theory is obtained in the limit where $c \to \infty$.

The coordinates used throughout, are not the usual coordinates of Minkowski space-time and momentum-space. In fact, it turns out that there exist other sets of coordinates, which are more appropriate for the discussion of the particle dynamics, in the sense that the basic definitions are more easily established and interpreted in these coordinates.

Before passing to the description of the one-particle system, we thus consider the relation between the usual coordinates (a^μ, b^μ) = usual (four-momentum, four-position)[1]), and a set of coordinates which will be basic in the further description (p^μ, q^μ) and which will be denoted (four-'momentum', four-'position').

In terms of the coordinates (a^μ, b^μ) the state-space chosen is characterized by,

$$\Omega = \{(a^\mu, b^\mu, \tau) \in \mathbb{R}^9 \mid a^{\circ 2} - \mathbf{a}^2 > 0 \text{ and } a^\circ > 0\}$$

where τ denotes the invariant time. The action of the proper Lorentz group $SO(3, 1)$ on Ω is defined by

$$\begin{aligned} a^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu a^\nu \\ b^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu b^\nu \\ \tau &\mapsto \tau \end{aligned}$$

where Λ is the usual representation of $SO(3, 1)$ on $\mathbb{R}^4$, isometric with respect to the Minkowski-metric

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The coordinates (p^μ, q^μ, t) are defined by the diffeomorphism

$$(a^\mu, b^\mu, \tau) \mapsto (p^\mu, q^\mu, t)$$

with

$$\begin{aligned} (p^\circ, \mathbf{p}) &= (a^\circ - mc, \mathbf{a}) \\ (q^\circ, \mathbf{q}) &= (b^\circ - \tau c, \mathbf{b}) \\ t &= \tau \end{aligned}$$

m is a constant, the 'kinematical' mass of the particle.

The above action of $SO(3, 1)$ is easily transformed to the coordinates (p^μ, q^μ, t); we obtain

$$\begin{aligned} p^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu p^\mu + mv^\mu(\mathbf{u}) \\ q^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu q^\mu + tv^\mu(\mathbf{u}) \\ t &\mapsto t \\ v^\mu(\mathbf{u}) &= (c(\gamma - 1), \gamma\mathbf{u}), \quad \gamma = (1 - \mathbf{u}^2/c^2)^{-1/2} \end{aligned}$$

is what appears in this theory as the four-velocity of a free 'elementary' ($\hbar = 0$) particle moving with velocity $\mathbf{u}$ (equation 9).

2. The one-particle system

A classical Einstein relativistic particle of kinematical mass $m > 0$, is a physical system associated [5].

[1]) For an alternative formulation of a relativistic dynamics employing the coordinates (a^μ, b^μ, τ), see [3, 4].

(i) the state space

$$\Omega = \{(p^\mu, q^\mu, t) \in \mathbb{R}^9 \mid (p^\circ + mc)^2 - \mathbf{p}^2 > 0 \text{ and } p^\circ > -mc\} = \Gamma \times \mathbb{R}$$

and phase space:

$$(\Gamma, \omega = dp_\mu \wedge dq^\mu).$$

(ii) the observables 'momentum', 'position' and time defined by the following functions on Ω

$$\begin{aligned} p^\mu(p^\mu, q^\mu, t) &= p^\mu \\ q^\mu(p^\mu, q^\mu, t) &= q^\mu \\ t(p^\mu, q^\mu, t) &= t; \end{aligned} \tag{1}$$

(iii) and the following action of the kinematical symmetry group

$G = SO(3, 1) \times_s \mathbb{R}^5$

the Lorentz transformations $\{(\boldsymbol{\theta}, \mathbf{u})\}$:

$$\begin{aligned} p^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu p^\nu + mv^\mu(\mathbf{u}) \\ q^\mu &\mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu q^\nu + \tau v^\mu(\mathbf{u}) \\ t &\mapsto t \end{aligned} \tag{2}$$

the 'spatial' translations $\{a^\mu\}$:

$$\begin{aligned} p^\mu &\mapsto p^\mu \\ q^\mu &\mapsto q^\mu + a^\mu \\ t &\mapsto t \end{aligned} \tag{3}$$

the time translations $\{\tau\}$:

$$\begin{aligned} p^\mu &\mapsto p^\mu \\ q^\mu &\mapsto q^\mu \\ t &\mapsto t + \tau. \end{aligned} \tag{4}$$

There also exists another set of 'natural' symplectic coordinates on Γ. These coordinates will be denoted (y^μ, x^μ), and they appear in the following construction.

Consider the function

$$\Delta m(p^\mu, q^\mu, t) = \frac{1}{c}\sqrt{(p^\circ + mc)^2 - \mathbf{p}^2} - m$$

on Γ. It is Lorentz-invariant, because the form

$$(p^\circ + mc)^2 - \mathbf{p}^2$$

is invariant under the given action (2) of G, and its interpretation follows from the observation that Δmc is the p° of the center of mass frame of reference of the particle, i.e.

$$(\Delta mc, \mathbf{0}) = L^{-1}(p^\mu)^\mu_\nu p^\nu + mw^\mu(p^\mu)$$

for

$$L(p^\mu) = \Lambda\left(\frac{\mathbf{p}c}{p^\circ + mc}\right) \quad \text{and} \quad w^\mu(p^\mu) = v^\mu\left(\frac{\mathbf{p}c}{p^\circ + mc}\right).$$

The coordinates (y, x) are defined by the symplectomorphism[2])

$$
\begin{aligned}
p^\circ &\mapsto y^\circ = \Delta m \\
\mathbf{p} &\mapsto \mathbf{y} = \mathbf{p} \\
q^\circ &\mapsto x^\circ = \frac{\sqrt{(p^\circ + mc)^2 - \mathbf{p}^2}}{p^\circ + mc} q^\circ c \\
\mathbf{q} &\mapsto \mathbf{x} = \mathbf{q} - \frac{\mathbf{p}}{p^\circ + mc} q^\circ
\end{aligned}
\tag{5}
$$

whose inverse is

$$
\begin{aligned}
y^\circ &\mapsto p^\circ = \sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2} - mc \\
\mathbf{y} &\mapsto \mathbf{q} = \mathbf{p} \\
x^\circ &\mapsto q^\circ = \frac{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2}}{(\Delta m + m)c} \frac{x^\circ}{c} \\
\mathbf{x} &\mapsto \mathbf{q} = \mathbf{x} + \frac{\mathbf{p}}{(\Delta m + m)c^2} x^\circ
\end{aligned}
$$

One should notice that the form of x^μ is partly determined by the choice of y^μ and the condition that the application should be a symplectomorphism, i.e.

$$\omega(p^\mu, q^\mu) = \omega(y^\mu, x^\mu).$$

The action (iii) of $SO(3, 1)$ can easily be transferred to the coordinates (y, x). On y it is given by

$$
\begin{aligned}
&\Delta m \mapsto \Delta m \\
&p^i \mapsto p'^i = \Lambda(\boldsymbol{\theta}, \mathbf{u})^i_\nu \left(\frac{1}{c}\sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2} - m, \mathbf{p}\right)^\nu + m\gamma u^i;
\end{aligned}
$$

and x^μ transforms according to

$$x^\mu \mapsto \Sigma(\Delta m, \mathbf{p}, \Lambda(\boldsymbol{\theta}, \mathbf{u}))^\mu_\nu x^\nu + \P^\mu(\Delta m, \mathbf{p}, \mathbf{u})t \tag{6}$$

for

$$
\begin{aligned}
&\Sigma(\Delta m, \mathbf{p}, \Lambda(\boldsymbol{\theta}))^\mu_\nu = \Lambda(\theta)^\mu_\nu \\
&\Sigma(\Delta m, \mathbf{p}, \Lambda(\mathbf{u}))^\circ_\circ = 1 \\
&\Sigma(\Delta m, \mathbf{p}, \Lambda(\mathbf{u}))^\circ_i = \frac{\Delta m + m}{\sqrt{\mathbf{p}'^2 + (\Delta m + m)^2 c^2}} \frac{u^i}{c} \\
&\Sigma(\Delta m, \mathbf{p}, \Lambda(\mathbf{u}))^i_\circ = 0 \\
&\Sigma(\Delta m, \mathbf{p}, \Lambda(\mathbf{u}))^i_j = \delta^i_j + \frac{1}{\gamma + 1} \frac{u^i u^j}{c^2} - \frac{p'^i u^j}{c\sqrt{\mathbf{p}'^2 + (\Delta m + m)^2 c^2}}
\end{aligned}
$$

---

[2]) In the quantum case $\mathbf{x}$ turns out to be the Newton–Wigner position operator.

and

$$\P^{\mu}(\Delta m, \mathbf{p}, \mathbf{u}) = \left(\frac{(\Delta m + m)(\gamma - 1)c}{\sqrt{\mathbf{p}'^2 + (\Delta m + m)^2c^2}}, \frac{-\mathbf{p}'(\gamma - 1)}{\sqrt{\mathbf{p}'^2 + (\Delta m + m)^2c^2}} + \gamma\mathbf{u}\right).$$

Moreover, the translations $a^{\mu} \in \mathbb{R}^4$ read

$$x^{\circ} \mapsto x^{\circ} + \frac{(\Delta m + m)c^2}{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2c^2}} a^{\circ}$$

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{a} - \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2c^2}} a^{\circ}$$

The dynamics of a one-particle system is by assumption described by the Hamilton equations

$$\dot{p}^{\mu} = -\partial q_{\mu}\mathscr{H}(p^{\mu}, q^{\mu}, t)$$
$$\dot{q}^{\mu} = \partial p_{\mu}\mathscr{H}(p^{\mu}, q^{\mu}, t)$$

The Hamiltonian $\mathscr{H}$ moreover, is assumed to be of such a form that the covariance condition

$$\dot{q}^{\mu}(\Lambda(\boldsymbol{\theta}, \mathbf{u})^{\mu}_{\nu}p^{\nu} + mv^{\mu}(\mathbf{u}), \Lambda(\boldsymbol{\theta}, \mathbf{u})^{\mu}_{\nu}q^{\nu}, t) = \Lambda(\theta, \mathbf{u})^{\mu}_{\nu}\dot{q}^{\nu}(p^{\mu}, q^{\mu}, t) + v^{\mu}(\mathbf{u})$$

is satisfied. In the case of a free particle ($\partial q_{\mu}\mathscr{H} = 0$) thus, the most general Hamiltonian satisfying the covariance condition is of the form

$$\mathscr{H}(p^{\mu}, q^{\mu}, t) = \frac{p^{\mu}p_{\mu}}{2m} + f(\Delta m(p^{\mu}, q^{\mu}, t))$$

where f is an arbitrary differentiable function of Δm, i.e. $f \circ \Delta m$ is a Lorentz invariant function of p^{μ}. We will *assume* that $f = \hbar$ is a constant ($> -\frac{1}{2}mc^2$) denoting the internal energy of the system. Furthermore, we postulate a relation expressed by the constraint

$$\mathscr{H} = p^{\circ} \tag{7}$$

i.e.

$$p^{\circ} = \sqrt{\mathbf{p}^2 + 2m\hbar + m^2c^2} - mc$$

which couple the center of mass of the particle with its internal 'structure' in such a way as to satisfy the Einstein law

$$\mathscr{E}_{\mathrm{CM}} = \Delta mc^2$$

where Δm is interpreted as the mass defect of the center of mass, and $\mathscr{E}_{\mathrm{CM}}$ denotes the energy of the center of mass frame,

$$\mathscr{E}_{\mathrm{CM}} = \mathscr{H}(p^{\circ}, \mathbf{0}, t).$$

In fact, in the coordinates (x, y) the Hamiltonian reads

$$\mathscr{H}(\Delta m, \mathbf{p}, x^{\mu}) = c\sqrt{\mathbf{p}^2 + (\Delta m + m)^2c^2} - (\Delta m + m)c^2 - \frac{\Delta m^2c^2}{2m} + \hbar$$

while the constraint (7) take the form

$$\Delta m + \frac{\Delta m^2}{2m} = \frac{\hbar}{c^2}. \tag{8}$$

With the given choice for a Hamiltonian, the equations of motion for a free particle are

$$\begin{aligned}
\Delta\dot{m} &= \frac{\partial \mathscr{H}}{\partial x^\circ} = 0 \\
\dot{\mathbf{p}} &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}} = \mathbf{0} \\
x^\circ &= -\frac{\partial \mathscr{H}}{\partial \Delta m} = \frac{\Delta m + m}{m} c^2 - \frac{\Delta m + m}{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2}} c^3 \\
\dot{\mathbf{x}} &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}} = \frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2}}.
\end{aligned}$$

Thus, parametrizing the solutions by the velocity $\mathbf{u}$, we obtain

$$\begin{aligned}
\Delta m &= m(\sqrt{1 + 2\hbar/mc^2} - 1) \\
\mathbf{p} &= (\Delta m + m)\gamma\mathbf{u}, \quad \gamma = (1 - \mathbf{u}^2/c^2)^{-1/2} \\
x^\circ &= \left(\frac{\gamma - 1}{\gamma} + \frac{\Delta m}{m}\right) c^2 (t - t_0) \\
\mathbf{x} &= \mathbf{u}(t - t_0) + \mathbf{a};
\end{aligned}$$

or in the coordinates (p, q),

$$\begin{aligned}
p^\mu &= (\Delta m + m)(c(\gamma - 1), \gamma\mathbf{u}) + (\Delta m c, \mathbf{0}) \\
q^\mu &= \left(\left(\left(1 + \frac{\Delta m}{m}\right)\gamma - 1\right) c, \left(1 + \frac{\Delta m}{m}\right)\gamma\mathbf{u}\right) + (0, \mathbf{a})
\end{aligned} \tag{9}$$

Moreover, the total energy is

$$\mathscr{E} = \frac{p^\mu p_\mu}{2m} + \hbar = \frac{(\Delta m + m)c^2}{\sqrt{1 - u^2/c^2}} - mc^2.$$

Notice that $\mathbf{x}$ in the above solutions does not depend on m and Δm, but that $\mathbf{q}$ does.

According to this description, it seems reasonable to consider $\mathbf{x}$ as the observable describing the position of the particle in real space. This follows from the form of the solutions as well as from the covariance of $\mathbf{x}$, which transforms by a 'Lorentz–Fitzgerald' contraction under a special Lorentz transformation. In fact, let $\mathbf{x} = \mathbf{a}$ denote the position of a particle at rest; then by (6),

$$\mathbf{x} \to \mathbf{x}' = \mathbf{a} - \frac{\gamma}{\gamma - 1} \frac{\mathbf{a} \cdot \mathbf{u}}{c^2} \mathbf{u} + \mathbf{u}t$$

or

$$\mathbf{x}' = \mathbf{a}\sqrt{1 - u^2/c^2} + \mathbf{u}t$$

$$\mathbf{x}' = \mathbf{a}$$

is the position in a frame moving with velocity $\mathbf{u}$. Notice that since $\Sigma(\Delta m, \mathbf{p}, \Lambda(\mathbf{u}))^i_o = 0$, $\mathbf{x}$ transforms independently of $\mathbf{x}^\circ$. The coordinates q^μ moreover, are by assumption the coordinates in which the interactions are local.

3. The two-particle system

A system of two particles of restmasses $m_1 > 0$ and $m_2 > 0$ is by hypothesis described by the state-space

$$\Omega = \{(p_1^\mu, q_1^\mu, p_2^\mu, q_2^\mu, t) \in \Gamma_1 \times \Gamma_2 \times \mathbb{R}\} = \Gamma_{12} \times \mathbb{R}.$$

The observables momentum and position for the individual particles are defined as for the one-particle system (1). Moreover, the action of G on Ω is given for the individual particles as in (2) to (5).

A system of coordinates which will be useful in the description of such a composite system are the 'barycentric coordinates' of the center of mass frame of reference. These coordinates are defined by the application

$$(p_1^\mu, q_1^\mu, p_2^\mu, q_2^\mu) \mapsto (P^\mu, Q^\mu, p^\mu, q^\mu)$$

where P, Q are the coordinates describing the center of mass $(m_1 > m_2)$

$$P^\mu = p_1^\mu + p_2^\mu$$

$$Q^\mu = \frac{1}{m_1 + m_2}(m_1 q_1^\mu + m_2 q_2^\mu,$$
$$+ (m_1 p_{2\alpha} - m_2 p_{1\alpha}) L(p_1^\mu + p_2^\mu)^\alpha_\beta L^{-1}(p_1^\mu + p_1^\mu)^{\beta,\mu}{}_\nu (q_2^\nu - q_1^\nu)$$

and (p, q) are the coordinates which describe the internal system

$$p^\mu = L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu \frac{m_1 p_2^\nu - m_2 p_1^\nu}{m_1 + m_2}$$

$$q^\mu = L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu (q_2^\nu - q_1^\nu)$$

for

$$L(p_1^\mu + p_2^\mu) = L(P^\mu) = \Lambda\left(\frac{\mathbf{P}c}{P^\circ + Mc}\right)$$

$$L'^\mu = \partial_{P_\mu} L.$$

The application thus defined is a symplectomorphism with respect to the canonical symplectic form

$$\omega = dp_{1\mu} \wedge dq_1^\mu + dp_{2\mu} \wedge dq_2^\mu$$

on Γ_{12}. The inverse is given by

$$
\begin{aligned}
p_1^\mu &= \frac{m_1}{m_1 + m_2} P^\mu - L(P^\mu)^\mu_\nu p^\nu \\
q_1^\mu &= Q^\mu + p_\alpha L^{-1}(P^\mu)^\alpha_\beta L(P^\mu)^{\beta\prime\mu}_{\ \ \nu} q^\nu - \frac{m_2}{m_1 + m_2} L(P^\mu)^\mu_\nu q^\nu \\
p_2^\mu &= \frac{m_2}{m_1 + m_2} P^\mu + L(P^\mu)^\mu_\nu p^\nu \\
q_2^\mu &= Q^\mu + p_\alpha L^{-1}(P^\mu)^\alpha_\beta L(P^\mu)^{\beta\prime\mu}_{\ \ \nu} q^\nu - \frac{m_1}{m_1 + m_2} L(P^\mu)^\mu_\nu q^\nu.
\end{aligned}
\tag{10}
$$

The proof of this proposition follows from the way this application is constructed. In fact, Γ_{12} is naturally identified with the cotangent bundle T^*M of the momentum space

$$M = \{(p_1^\mu, p_2^\mu) \in \mathbb{R}^8 \mid (p_i^\circ + m_i c) - \mathbf{p}_i^2 > 0 \quad \text{and} \quad p_i^\circ > -m_i c, \quad i = 1, 2\}$$

and the symplectic form $\omega = -d\sigma$ where

$$\sigma = q_{1\mu}\, dp_1^\mu + q_{2\mu}\, dp_2^\mu.$$

Moreover, the application

$$\phi : (p_1^\mu, p_2^\mu) \mapsto (P^\mu, p^\mu)$$

is a diffeomorphism. In fact, ϕ is the composite map

$$
\begin{aligned}
(p_1^\mu, p_2^\mu) &\mapsto \left(p_1^\mu + p_2^\mu, \frac{m_1 p_2^\mu - m_2 p_1^\mu}{m_1 + m_2}\right) \\
&\mapsto \left(p_1^\mu + p_2^\mu, L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu \frac{m_1 p_2^\nu - m_2 p_1^\nu}{m_1 + m_2}\right)
\end{aligned}
$$

Thus, introducing the notation

$$
\begin{aligned}
u^\mu &= p_1^\mu, & u^{4+\mu} &= p_2^\mu \\
x^\mu &= q_1^\mu, & x^{4+\mu} &= q_2^\mu \\
v^\mu &= P^\mu, & v^{4+\mu} &= p^\mu \\
y^\mu &= Q^\mu, & y^{4+\mu} &= q^\mu \\
u_\mu &= p_{1\mu} = g_{\mu\nu} p_1^\nu, & u_{4+\mu} &= g_{\mu\nu} p_2^\nu, \text{ etc.}
\end{aligned}
$$

for $\mu = 0, 1, 2, 3$, we can write

$$
\begin{aligned}
v^\alpha &= \phi^\alpha(u^\alpha) \\
u^\alpha &= \phi^{-1\alpha}(v^\alpha) \quad \alpha = 0, 1, 2, \ldots, 7 \\
\sigma &= x_\alpha\, du^\alpha
\end{aligned}
$$

Moreover, with this notation,

$$y^\alpha(x^\alpha, u^\alpha) = \left(\frac{\partial \phi_\beta^{-1}}{\partial v_\alpha}\right)(\phi^\alpha(u^\alpha))x^\beta$$

$$x^\alpha(y^\alpha, v^\alpha) = \left(\frac{\partial \phi_\beta}{\partial u_\alpha}\right)(\phi^{-1\alpha}(v^\alpha))y^\beta,$$

and we may compute

$$\begin{aligned} y_\alpha(u^\alpha, x^\alpha)\, dv^\alpha(u^\alpha) &= x^\beta \left(\frac{\partial \phi_\beta^{-1}}{\partial v_\alpha}\right)(\phi^\alpha(u^\alpha))\, d_\phi^\alpha(u^\alpha) \\ &= x^\beta \left(\frac{\partial \phi_\beta^{-1}}{\partial v_\alpha}\right)(\phi^\alpha(u^\alpha)) \left(\frac{\partial \phi^\alpha}{\partial u^\gamma}\right) du^\gamma \\ &= x^\beta \left(\frac{\partial}{\partial u^\gamma}(\phi^{-1} \circ \phi)_\beta\right)(u^\alpha)\, du^\gamma \\ &= x_\alpha\, du^\alpha, \end{aligned}$$

which proves that our application is a symplectomorphism.

The transformation properties of the coordinates (P, Q, p, q) under the inhomogeneous Lorentz-group $SO(1, 3) \times_s \mathbb{R}^4$ are easily determined,

$$P^\mu \mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu P^\nu + (m_1 + m_2)v^\mu(\mathbf{u})$$

$$p^\mu \mapsto \Lambda(\boldsymbol{\theta}_w)^\mu_\nu p^\nu$$

$$q^\mu \mapsto \Lambda(\boldsymbol{\theta}_w)^\mu_\nu q^\nu$$

where $\boldsymbol{\theta}_w = \boldsymbol{\theta}_w(\boldsymbol{\theta}, \mathbf{u}, P^\mu)$ is the angle of the rotation

$$\Lambda(\boldsymbol{\theta}_w)^\mu_\nu = L^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu P^\nu + (m_1 + m_2)v^\mu(\mathbf{u}))^\mu_\alpha \Lambda(\boldsymbol{\theta}, \mathbf{u})^\alpha_\beta L(P^\mu)^\beta_\nu.$$

The transformation law for Q^μ is somewhat more complicated and we will not write down the explicite expression for the transformed of Q^μ; we notice however, that it is linear in Q^μ and q^μ.

It follows thus that the center of mass is defined in much the same way[3]) as the particle of restmass $M = m_1 + m_2$; in fact,

$$(P^\circ + Mc)^2 - \mathbf{P}^2 = (\Delta M + M)^2 c^2 > 0 \quad \text{and} \quad P^\circ > -Mc$$

on Γ_{12} and $Q^\mu \in \mathbb{R}^4$. We can thus define (X, Y)-coordinates for the center of mass in the same way as for the one-particle system (5), and determine their transformation properties. It turns out that the transformation properties of X are more complicated in this case than in the one-particle case. Without giving the explicit expression we notice only that $\mathbf{X}$ transforms independently of X° and q°.

The dynamics of a two-particle system is by assumption described by the Hamilton equations. Moreover, we consider Hamiltonians of the form

$$\mathscr{H}(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{P^\mu P_\mu}{2M} + \hbar(p^\mu, q^\mu, t) \tag{11}$$

[3]) Except for the transformation properties of Q^μ. The transformed of Q^μ depends on the internal coordinates.

and impose the constraint

$$\alpha(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \mathscr{H}(P^\mu, Q^\mu, p^\mu, q^\mu, t) - P^\circ c = 0, \tag{12}$$

i.e.

$$\Delta M + \frac{\Delta M^2}{2M} = \frac{\hbar}{c^2}.$$

We also postulate a second constraint

$$\beta(P^\mu, Q^\mu, p^\mu, q^\mu, t) = p^\circ - \frac{m_1 - m_2}{Mc} \frac{\hbar(p^\mu, q^\mu, t)}{\sqrt{1 + 2\hbar(p^\mu, q^\mu, t)/Mc^2}} = 0.$$

Notice that while (12) couple the motion of the center of mass to the internal system, (13) is a constraint on the internal degrees of freedom only.

The form (11) of the Hamiltonian follows from the covariance conditions on the motion of the individual particles, and the assumption that the center of mass behaves as a free particle and moreover, does not influence the motion of the internal system. The only a priori conditions we have on the form of $\hbar$ however, is that it should be invariant under rotations, and that in the case of no interaction,

$$\hbar(p^\mu, q^\mu, t) = \frac{p^\mu p_\mu}{2m} + \hbar_1 + \hbar_2$$

where $m = m_1 m_2/(m_1 + m_2)$ and $\hbar_1$, $\hbar_2$ are the constants denoting the internal energies. Thus in this case,

$$\mathscr{H}(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + \hbar_1 + \hbar_2.$$

or,

$$\mathscr{H}(p_1^\mu, q_1^\mu, p_2^\mu, q_2^\mu, t) = \frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} + \hbar_1 + \hbar_2.$$

Moreover, the constraints (12) and (13) imply the constraints

$$\Delta m_1 + \frac{\Delta m_1^2}{2m_1} = \frac{\hbar_1}{c^2}$$

$$\Delta m_2 + \frac{\Delta m_2^2}{2m_2} = \frac{\hbar_2}{c^2}$$

In fact, the form of the constraints (12) and (13) has been suggested by the free particle case.

A class of internal Hamiltonians to consider are those of the form

$$\hbar(p^\mu, q^\mu, t) = \frac{(p^\mu - A^\mu(q^\mu, t))(p_\mu - A_\mu(q^\mu, t))}{2m} + V(q^\mu, t).$$

In particular,

$$\hbar(p^\mu, q^\mu, t) = \frac{(\mathbf{p} - \mathbf{A}(q^\mu, t))^2}{2m} - \frac{(p^\circ - V(q^\mu, t))^2}{2m} + V(q^\mu, t) \tag{14}$$

is assumed to describe a system of two charged particles interacting via the electromagnetic field ($V(q^\circ + ct, \mathbf{q})$, $\mathbf{A}(q^\circ + ct, \mathbf{q})$ while Hamiltonians of the form

$$\begin{aligned} \hbar(p^\mu, \mathbf{q}, t) &= \frac{(\mathbf{p} - \mathbf{A}(\mathbf{q}, t))^2}{2m} - \frac{p^{\circ 2}}{2m} + V(\mathbf{q}, t) \\ &= h(\mathbf{p}, \mathbf{q}, t) - \frac{p^{\circ 2}}{2m} \end{aligned} \tag{15}$$

relates to a Galilean model described by h an Einsteinian model.

4. The restricted two-particle theory

When α and β are constants of motion ($\partial q^\circ \hbar = 0$, $\partial_t \hbar = 0$), it is sufficient to impose the constraints on the initial conditions. In this case thus, we can formulate an alternative and in many respects equivalent classical theory by the following construction.

In terms of the coordinates ($Y^\mu, X^\mu, p^\mu, q^\mu$), the phase space ($\Gamma_{12}, \omega$) is characterized by

$$\Gamma_{12} = \Big\{(Y^\mu, X^\mu, p^\mu, q^\mu, t) \in \mathbb{R}^{16} \mid \Delta M > -M \quad \text{and} \quad \frac{1}{2M}(\Delta M + M)^2 c^2 - \frac{p^\mu p_\mu}{2m} > 0 \text{ and } \frac{m_1}{M}(\Delta M + M)c > p^\circ > -\frac{m_2}{M}(\Delta M + M)c\Big\}$$

$$\omega = dY_\mu \wedge dX^\mu + dp_\mu \wedge dq^\mu.$$

Now, Y^μ, $\mathbf{X}$, p^μ, $\mathbf{q}$ transforms independently of X° and q° under an inhomogeneous Lorentz transformation. Thus, the quotient $\Gamma_{12}/\sim$ obtained by identifying points which differ only with respect to the values of the coordinates X° and q° is Lorentz invariant. Moreover, by introducing the constraints $\alpha = 0$ and $\beta = 0$ we obtain a Lorentz invariant phase space ($\bar{\Gamma}_{12}, \bar{\omega}$),

$$\Gamma_{12} = \{(\mathbf{Y}, \mathbf{X}, \mathbf{p}, \mathbf{q}) \in \mathbb{R}^{12}\}$$
$$\omega = dY_i \wedge dX^i + dp_i \wedge dq^i$$

on which the dynamics is described by Hamiltonians of the form

$$\mathscr{H}(\mathbf{Y}, \mathbf{X}, \mathbf{p}, \mathbf{q}) = c\sqrt{\mathbf{P}^2 + (\Delta M(\mathbf{p}, \mathbf{q}) + M)^2 c^2} - Mc^2$$

where $\Delta M(\mathbf{p}, \mathbf{q})$ is determined by the constraints $\alpha = 0$, $\beta = 0$. For example (15) gives

$$\Delta M(\mathbf{p}, \mathbf{q}) = M\left(\left(1 + \frac{4mh}{Mc^2} + 2\frac{m}{M}\left(\left(1 + \frac{M - 2m}{mMc^2}h + \frac{4h^2}{M^2c^4}\right)^{-1/2} - 1\right)\right)^{-1/2} - 1\right).$$

In this formalism however, any trace of the one-particle coordinates are lost, except in the free case. Moreover, it becomes impossible to give explicitly the action of $SO(3,1)$ on the internal coordinates; it becomes highly non-linear.

The restricted theory is equivalent to the theory constructed by Bakjamin and Thomas [6].

5. Particle in an external field

A particle in an external field is by definition a system of two 'particles' in the limit where the mass of one of the 'particles' (the 'field-generating device') becomes infinite.

Consider the description of the two-particle system in the coordinates $(Y^\mu, X^\mu, p^\mu, q^\mu)$. The objects p^μ, q^μ, ΔM, $\mathbf{X}$ and $\hbar$ are uneffected by the process of taking the limit $m_1 \to \infty$; moreover, the objects $\mathbf{P}/M$, α, β, $\mathbf{P}c/(P^\circ + Mc)$ are well-defined in this limit,

$$\lim_{m_1 \to \infty} \frac{\mathbf{P}}{M} = \Gamma \mathbf{U}, \quad \Gamma = (1 - \mathbf{U}^2/c^2)^{-1/2}$$

$$\lim_{m_1 \to \infty} \alpha = \alpha' = \Delta M - \frac{\hbar}{c^2}$$

$$\lim_{m_1 \to \infty} \beta = \beta' = p^\circ - \frac{\hbar}{c}$$

$$\lim_{m_1 \to \infty} \frac{\mathbf{P}}{P^\circ + Mc} = \mathbf{U}.$$

A particle in an external field is thus described by the state-space

$$\Omega = \{(p^\mu, q^\mu, t) \in \mathbb{R}^9\}$$

on which the inhomogeneous Lorentz group acts by the rotations $\boldsymbol{\theta}_w$ defined by the velocity $\mathbf{U}$ of the center of mass, i.e. the field-generating device.

The dynamics is described by the Hamiltonian $\hbar$, and the motion is submitted to the constraint

$$\alpha' = p^\circ - \frac{\hbar}{c} = 0.$$

When α' is a constant of motion, we can consider the corresponding restricted theory, and introduce the constraint $\alpha' = 0$ in the Hamiltonian. For $\hbar$ of the form (14), this gives

$$\hbar = c\sqrt{(\mathbf{p}^2 - \mathbf{A}(\mathbf{q}))^2 + m^2c^2} - mc^2 + V(\mathbf{q}).$$

Remark

The theories constructed in the last two paragraphs are not subtheories of the original theory, but appear as independent theories, some of whose predictions coincide with those of the original theory. In particular, one should notice that while the position $\mathbf{q}$ of the original theory is a space-time position, the $\mathbf{q}$ in the restricted theories is a Newton–Wigner-like position.

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