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Autor(en): **Giovannini, N.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **50 (1977)**

Heft 3

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-114862

Elementary particles as representations of the covariance group in the presence of an external electromagnetic field

by **N. Giovannini**

Institute for Theoretical Physics, University of Nijmegen, the Netherlands

(26. XI. 1976)

Abstract. A complete description of the projective unitary/antiunitary representations of the general covariance group for a charged (relativistic) particle moving in an external (classical), e.m. field is given. This group was derived in a previous paper, independently of any equation of motion, on the basis of some simple physical assumptions. The physical consequences of these results are then discussed and it is shown how they open some new perspectives.

0. Introduction

In a previous paper [1], we have derived, independently of any equation of motion, a general covariance group for a charged relativistic particle moving in an (almost) arbitrary external electromagnetic field, essentially on the simple basis of the invariance of the Maxwell equations under Poincaré transformations. Let us briefly remind here how this group was constructed. The usual procedure, when one considers the problem of a charged particle moving in such a field is to modify the free equation of motion (Klein-Gordon, Dirac etc.) by so-called minimal coupling, introducing thus a potential. Since potentials are not observables as fields are, there is in fact a certain arbitrariness in the equation of motion, as for example the transformation law of a potential under a Poincaré group element is not unique and as there does not correspond one but an infinite set of potentials to a given field. One could on the other side try to avoid potentials and to construct a pure field dependent formalism [2].

In [1] we had chosen a compromise, using potentials, but getting rid of the arbitrariness mentioned above by defining group elements which, as acting on a potential, did not change the (arbitrary but fixed) gauge of the chosen potential and consisted then of coupled gauge and Poincaré transformations. The resulting group was shown to be independent of the reference frame and of the gauge chosen and also to contain the Poincaré group only as a (non-trivial) factor group (i.e. not as a subgroup).

In this paper we deduce systematically all projective unitary/antiunitary irreducible representations (short PUAIR) of this group and discuss their physical interpretation and some of the physical consequences of the results. In particular we show how our approach leads to a possible solution of the so-called a-causality troubles, for particles of spin equal to or larger than 1 [3–4]. The Klein-Gordon and Dirac equations, however, minimally coupled to the external field, are shown to correspond to representations of this group, i.e. to transform covariantly.

This paper will be organized as follows: in the first part we briefly remind the structure of our general covariance group and analyse it in some more detail. In part two we derive all irreducible unitary representations of a nilpotent normal subgroup of this group and, in part three, we induce these representations to the connected component to one of the whole group. In part four we include the so-called discrete transformations (related to space and time-reversals), too, and finally in part five we discuss some physical aspects and consequences of our results.

1. The structure of the general covariance group $\mathbb{M}$

Before we determine the PUAIR of $\mathbb{M}$, let us first recall its definition, as found in [1], and indicate in some more detail its structure. We write the elements $m \in \mathbb{M}$ in the form

$$m = \langle B, a, \Lambda \rangle \tag{1.1}$$

with $B \in T \wedge T$, the (antisymmetric) external Kronecker product of the Minkovski space $M(4)$ with itself ($B = B^{\mu\nu}E_{\mu\nu}$, $B^{\mu\nu} \in \mathscr{R}$, $\forall \mu, \nu = 0, 1, 2, 3$, $E_{\mu\nu} = e_\mu \wedge e_\nu$, $\{e_\mu\}$ a basis of $M(4)$, $(e_\mu)^\nu = \delta^\nu_\mu$ and $(a, \Lambda) \equiv g \in IO(3, 1)$, an element of the Poincaré group, $a \in U$, a space-time-translation, $\Lambda \in 0(3, 1)$, an element of the homogeneous Lorentz group [1]). The product in $\mathbb{M}$ is then given by

$$m \cdot m' = \langle B + \zeta(g)B' + A(g, g'), a + \Lambda a', \Lambda\Lambda' \rangle \tag{1.2}$$

with $(\zeta(g)B')^{\mu\nu} = \Lambda^{-1}{}^\mu_\rho \Lambda^{-1}{}^\nu_\sigma (B')^{\rho\sigma}$ and $A(g, g')$ a factor system, defined by

$$A(g, g')^{\sigma\rho} = \tfrac{1}{2}((\Lambda a')^\sigma \wedge a^\rho) = \tfrac{1}{4}((\Lambda a')^\sigma a^\rho - (\Lambda a')^\rho a^\sigma). \tag{1.3}$$

The inverse of (1.1) is then easily found from (1.2) and (1.3) to be equal to

$$m^{-1} = \langle -\zeta(g^{-1})B, -\Lambda^{-1}a, \Lambda^{-1} \rangle \tag{1.4}$$

because it follows from (1.3) that $A(g, g^{-1}) = 0$.

We also recall here that the physical representations of $\mathbb{M}$ were defined on a (separable) Hilbert space of (here scalar) functions ψ of x and $F(x)$ (the external field) by (see [1] sections 1 and 4)

$$V(\langle B, a, \Lambda \rangle)\psi(x, F(x)) \equiv \Phi(x, (gF)(x))$$

$$= \exp\left\{ -i \frac{e}{c\hbar} (B \cdot gF^{(0)} + \chi_g(\pi(gF), x)) \right\} \psi(g^{-1}x, F(g^{-1}x)) \tag{1.5}$$

with $B \cdot gF^{(0)} \equiv B^{\sigma\rho}(gF^{(0)})_{\sigma\rho} \in \mathscr{R}$, $\chi_g(gF), x)$ the compensating gauge and $F^{(0)}$ the c.u. part of the field as defined in [1]. The generalization to more components wave functions is then similar as in [1] (see also section 5 of the present paper).

Denoting now by $\mathbb{B}$ the subgroup of $\mathbb{M}$ consisting of all elements of the form $\langle B, 0, 1 \rangle$ ($\mathbb{B} \cong \mathscr{R}^6$) and by K the subgroup of $\mathbb{M}$ generated by all $\langle B, a, 1 \rangle$, $a \in U$ the following relations are easily verified:

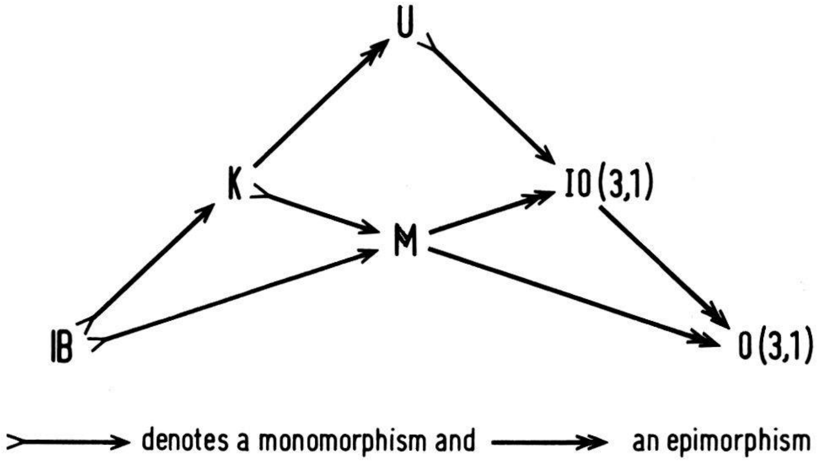
(i) $\mathbb{B} \lhd K$, ($\mathbb{B}$ normal in K) and $K/\mathbb{B} \cong U$, i.e. K appears as an extension of $\mathbb{B}$ by U; this extension is characterized by a factor set $A(a, a')$, given by the restriction to $U \times U$ of (1.3).

---

[1]) For elements of the Poincaré group $IO(3, 1)$, we use the same conventions as in [1], i.e. $(a, \Lambda)x = \Lambda x + a$ for x a 4-vector, with then $(a, \Lambda)(a', \Lambda) = (a + \Lambda a', \Lambda\Lambda')$; further $(\Lambda x)_\nu = \Lambda^\mu_\nu x_\mu$ for covariant, and $(\Lambda x)^\mu = \Lambda^{-1}{}^\mu_\nu x^\nu$ for contravariant vector components.

(ii) $\mathbb{B} \lhd \mathbb{M}$, $\mathbb{M}/\mathbb{B} \cong IO(3, 1)$, with as corresponding factor set $A(g, g')$ as in (1.3).
(iii) $K \lhd \mathbb{M}$, $\mathbb{M}/K \cong O(3, 1)$, giving rise to a semidirect product.
(iv) $U \lhd IO(3, 1)$, $IO(3, 1)/U \cong 0(3, 1)$, corresponding also to a semidirect product.

These properties are resumed in the following diagram of exact sequences.



(1.6)

We note further that, as can also easily be seen

(v) $\mathbb{B}$ and U are Abelian.
(vi) K is nilpotent, with a lower central series of length 2.

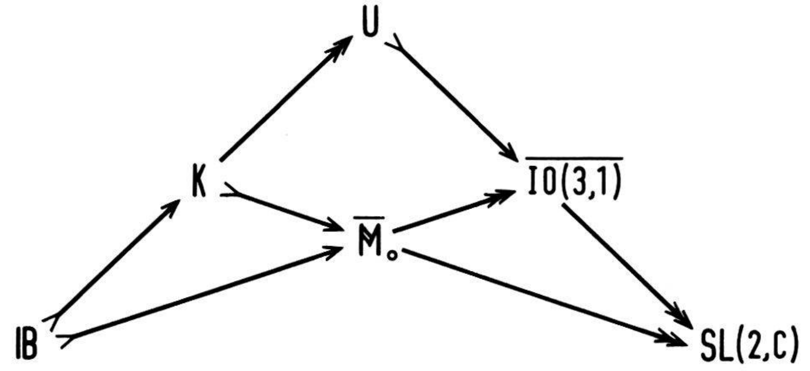
We are, because a physical state will be described by a ray in a separable Hilbert space $\mathscr{H}$ [1], not interested in the proper representations of $\mathbb{M}$ but in the projective (up to a factor) ones. These can be obtained, since $\mathbb{M}$ is separable and locally compact, by considering the ordinary representations of a larger group $\mathbb{M}^\sigma$, by the procedure of lifting [5, 6], where σ is a multiplier over $\mathbb{M}$ with respect to the UA decomposition $\mathbb{M}^\uparrow \cup \mathbb{M}^\downarrow$ ($\mathbb{M}^\uparrow$ being the subgroup of $\mathbb{M}$ corresponding to orthochronous Poincaré transformations only). This multiplier satisfies thus the relations

$$\sigma(m_1, m_2 m_3)\sigma^{m_1}(m_2, m_3) = \sigma(m_1, m_2)\sigma(m_1 m_2, m_3), \quad \forall m_1, m_2, m_3 \in \mathbb{M} \qquad (1.7)$$

with

$$\sigma^{m_1} = \begin{cases} \bar{\sigma}, & m_1 \in \mathbb{M}^\downarrow \\ \sigma, & m_1 \in \mathbb{M}^\uparrow \end{cases} \quad (\bar{\sigma} \text{ the complex conjugate of } \sigma).$$

Such a multiplier is sometimes also called a co-multiplier; we keep however the same nomenclature as in [6]. Because of the special role played by discrete elements, and in particular because of the (assumed) anti-unitary character of transformations containing time-inversion we consider in a first step the connected component $\mathbb{M}_0^\sigma$ of $\mathbb{M}^\sigma$ only (the remaining discrete transformations will be reintroduced later on). The group $\mathbb{M}_0^\sigma$ depends of course on σ and it is more useful, whenever possible, to consider a larger group $\mathbb{M}_0^\Sigma$ which is an extension of the multiplier group Σ (the group of all inequivalent possible σ) by $\mathbb{M}_0$, the connected component to one of $\mathbb{M}$, and whose ordinary representations describe all projective inequivalent representations of $\mathbb{M}_0$. It follows from theorems 3.2 and 4.1 of Bargmann [7] and from the fact that the second cohomology group $H^2(\underline{\mathbb{M}}, \mathscr{R}) = 0$ [2]), where $\underline{\mathbb{M}}$ is the Lie algebra of $\mathbb{M}_0$, that all projective inequivalent representations of $\mathbb{M}_0$ can be obtained from the ordinary ones of the covering group $\overline{\mathbb{M}}_0$ of $\mathbb{M}_0$, which has the following structure: it has elements $\langle B, a, \overline{\Lambda}\rangle$ with $\overline{\Lambda} \in SL(2, \mathbb{C})$, B and a as before, K being its own covering group. In an analogous way as in (1.6) we can illustrate the structure of $\overline{\mathbb{M}}_0$ with the following diagram of exact sequences:



(1.8)

As we can see from this diagram, a possible way for obtaining all irreducible unitary representations of $\overline{\mathbb{M}}_0$ will be by inducing, in the sense of Mackey [5], from the ones of K. The latter ones can be obtained following the theory of Kirillov [9, 10], since K is nilpotent.

Let us now determine the Lie algebra of $\mathbb{M}$. We use the following notation for the generators: let $M_{\mu\nu}$ generate a rotation in the μ–ν plane of space-time ($M_{\mu\nu} \in \mathfrak{sl}(2, \mathbb{C})$), Π_μ a translation and, for $\mathbb{B}$, let $\mathbb{F}_{\mu\nu}$ be the infinitesimal generator of the group element $E_{\mu\nu}$. The Lie algebra is then easily obtained, using the results concerning the group as obtained so far, as given by

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho}\\
[M_{\mu\nu}, \mathbb{F}_{\rho\sigma}] &= g_{\mu\rho}\mathbb{F}_{\nu\sigma} + g_{\nu\sigma}\mathbb{F}_{\mu\rho} - g_{\nu\rho}\mathbb{F}_{\mu\sigma} - g_{\mu\sigma}\mathbb{F}_{\nu\rho}\\
[\Pi_\mu, \Pi_\nu] &= \mathbb{F}_{\mu\nu}\\
[M_{\mu\nu}, \Pi_\sigma] &= g_{\mu\sigma}\Pi_\nu - g_{\nu\sigma}\Pi_\mu
\end{aligned}
\tag{1.9}
$$

where the metric $g_{\mu\nu}$ is given by $g_{00} = -g_{ii} = -1$, $i = 1, 2, 3$. All other commutators vanish.

This 16-dimensional Lie algebra is actually not unknown and has even a (very adapted) name, as proposed first by S. L. Glashow (see Stein [11]): the *Maxwell Lie algebra*. It has been quoted by Bacry et al. [12] and studied in some more detail by Schrader [8] in an actually quite different or, better said, much more specific context. First, all these authors consider only constant uniform (c.u.) e.m. fields whereas we are dealing with a very large class of (also inhomogeneous) fields (see [1]). Second the generators $\mathbb{F}_{\mu\nu}$ are identified by them with the eigenvalues they take in the presence of a given c.u. field (namely with the field components, as we shall see later on), and for representations generated from a given equation of motion (Klein-Gordon or Dirac

[2]) We omit here the proof since this result was already obtained by Schrader [8]. Since other results of this paper overlap some of the results to follow, we shall discuss it more in detail later on.

with minimal coupling). This indicates incidentally, since we have made no use of such an equation, that these equations will characterize representations of our group, i.e. are covariant under $\mathbb{M}$, as we shall see more in detail in section 5.

Our derivation and interpretation are clearly different and our goals somehow more ambitious. We shall discuss later on some interesting remarks made in [12]. More interesting at the moment are the results of Schrader: in his paper the irreducible unitary representations of the subgroup K of $\mathbb{M}$ are derived. Unfortunately, only the case of a c.u. *magnetic* field (and its Poincaré transforms) is then considered and only in two representations (as obtained in fact from the Klein-Gordon and Dirac equations).

Because we want to derive *all* PUAIR of $\mathbb{M}$, we shall, for completeness, calculate them from the beginning, including (in a somehow different and simpler way) those results obtained by Schrader concerning K that we shall need in the sequel, too.

2. The irreducible representations of the subgroup K of $\mathbb{M}$

These representations can be completely determined using the theory of Kirillov [9, 10] for connected (here simply connected) nilpotent Lie groups. Let us therefore first briefly recall the general procedure.

Let $\underline{k}$ be the Lie algebra of K, $\underline{k}'$ the dual space of $\underline{k}$ and $\mathrm{coAd}_{\underline{k}'}(K)$ the coadjoint representation of K on $\underline{k}'$ (the contragradient of the adjoint representation), which for $\omega \in \underline{k}'$ (i.e. for a linear form on $\underline{k}$) is defined by

$$(\mathrm{coAd}_{\underline{k}'}(k)\omega)(\xi) \stackrel{\mathrm{def}}{=} \omega(Ad_{\underline{k}}(k)^{-1}\xi) \tag{2.1}$$

with $k \in K$, $\xi \in \underline{k}$. Let then 0_ω denote the orbit of an element $\omega \in \underline{k}'$, i.e. the set of all images of ω under the action of K as defined by (2.1). Since (2.1) is a representation of K, the set $\{0_\omega\}$ is a partition of $\underline{k}'$. Consider now in each orbit one arbitrary (but fixed) element ω and consider a subalgebra $\underline{l} \subseteq \underline{k}$ satisfying

$$[\underline{l}, \underline{l}] \subseteq \mathrm{Ker}\ \omega \tag{2.2}$$

such a subalgebra $\underline{l}$ is called *subordinate to* ω. The following map T_ω on the Lie-subgroup L of K generated by $\underline{l}$ ($L = \exp \underline{l}$)

$$T_\omega(\exp x) \stackrel{\mathrm{def}}{=} \exp i\omega(x), \quad x \in \underline{l} \tag{2.3}$$

is then clearly a one dimensional unitary representation of L. This representation can then be induced, in the canonical way, to a representation $V_{\omega,\underline{l}}$ of K by

$$(V_{\omega,\underline{l}})(k)\varphi(\lambda) \equiv (T_\omega \uparrow K)(k)\varphi(\lambda) = T_\omega(\lambda k(\lambda')^{-1})\varphi(\lambda') \tag{2.4}$$

where λ, λ' are (fixed) representatives of the (right) coset decomposition K/L, with λ' determined by the condition $\lambda k(\lambda')^{-1} \in L$, and $\varphi(\lambda)$ is a measurable, quadratic integrable function on this coset space, with respect to some (quasi-)invariant (under the action of K) ergodic measure μ, and with values in the carrier space of T_ω. This measure is then unique, as a class. Since the coset space and a set of coset representatives are in one-to-one Borel correspondence with each other we use, or better said we abuse, as is usual, the parameter λ to describe both spaces. The following then holds

Theorem 2.1. (*Kirillov* [10]).

(i) $V_{\omega,\underline{l}}$ *is irreducible if and only if dim* $\underline{l}$ *is maximal.*

(ii) *any irreducible representation of K can be obtained in this way, up to equivalence.*

(iii) $V_{\omega,\underline{l}}$ *and* $V_{\omega',\underline{l}'}$, *both irreducible are equivalent, if and only if* $0_\omega = 0_{\omega'}$.

(iv) $\hat{K}$ *the dual of K is always of Type I.*

A subalgebra $\underline{l}$ subordinate to a linear form ω and of maximal dimension is also called a *real polarization at* ω. From now on only such subordinate subalgebras will be considered.

In the sequel, the following result will also be helpful:

Theorem 2.2. [13]. *Whenever* $\underline{l}$, *real polarization at* ω, *can be chosen ideal, then the following holds:*

(i) $\underline{l}$ *is his own centralizator with respect to* ω, *i.e. if* $\xi \in \underline{k}$,

$$\omega([\underline{l}, \xi]) = 0 \Leftrightarrow \xi \in \underline{l}$$

(ii) *the coset space K/L is Borel isomorphic with the orbit of* T_ω *under K and is 1-to-1 characterized by the classes* $\bar{\omega}$ *of elements* ω *in* 0_ω *which coincide with each other when restricted to* $\underline{l}$.

Let us now apply this theory to our group K: an element $\omega \in \underline{k}'$ can be characterized by an antisymmetric tensor $f_{\mu\nu}$ and a vector p_μ (μ, $\nu = 0, 1, 2, 3$), and will be denoted by (f, p). Its action on $\underline{k}$ is defined by

$$(f, p)(B^{\mu\nu}\mathbb{F}_{\mu\nu} + a^\mu\Pi_\mu) \stackrel{\text{def}}{=} B^{\mu\nu}f_{\mu\nu} + a^\mu p_\mu, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.5}$$

Using the commutation relations (1.9) one obtains after a short calculation that the adjoint representation is given, for $k = \langle B, a, 1\rangle \in K$, by

$$\mathrm{Ad}_{\underline{k}}(\langle B, a, 1\rangle)((B')^{\rho\sigma}\mathbb{F}_{\rho\sigma} + (a')^\sigma\Pi_\sigma) = (B'^{\mu\nu} + (a \wedge a')^{\mu\nu})\mathbb{F}_{\mu\nu} + a'^\mu\Pi_\mu \tag{2.6}$$

so that, by (2.1) and (2.5)

$$\mathrm{coAd}_{\underline{k}'}(\langle B, a, 1\rangle)(f, p) = (f, p + a\cdot f) \tag{2.7}$$

with $(a\cdot f)_\mu = a^\nu f_{\nu\mu} \stackrel{\text{def}}{=} (f(a))_\mu$, f being in this last expression considered as a linear map from U to $M^*(4)$, the dual Minkovski space. The orbits are thus given by $0_{(f,p)} = \{(f, p + a\cdot f) \mid \forall a \in U\}$ and are characterized by a tensor f and a manifold of vectors p modulo Im(f). The explicit form of Imf depends of course on f and needs to be investigated in more details, as we shall do now: because f is antisymmetric, it has, as a bilinear form, necessarily an image of even dimension. We may thus distinguish the following three cases, in turn:

(α) dim (Im(f)) = 0, then necessarily $f = 0$ and the corresponding orbits are completely characterized by a 4-vector $p \in M^*(4)$ and will therefore be denoted 0_p. This case will correspond of course also to free particles and will give rise to the well known PUA representations of the Poincaré group. It will thus only be shortly mentioned for completeness in the sequel.

(β) dim (Im(f)) = 4, then necessarily det $f \neq 0$. As a consequence, for any p, there exists an $a \in U$ such that $p = -a\cdot f$ so that each such orbit goes through each point in

$M^*(4)$ and the orbit is thus completely characterized by the tensor f (with $\det f \neq 0$). These orbits are denoted 0_f.

(γ) $\dim(\mathrm{Im}(f)) = 2$, then $\det f = 0, f \neq 0$, then there exists, because $\dim(\mathrm{Ker}(f)) = 4 - \dim(\mathrm{Im}(f)) = 2$, a 2-dimensional subspace of U generated by two linear independent (but of course not unique) translations $a^{(1)}$ and $a^{(2)}$ such that $\forall \mu_1, \mu_2 \in \mathscr{R}$

$$a(\mu_1, \mu_2) \cdot f \equiv (\mu_1 a^{(1)} + \mu_2 a^{(2)}) \cdot f = 0. \tag{2.8}$$

These orbits are thus completely characterized by an antisymmetric tensor f with $\det f = 0, f \neq 0$ and an element q of the 2-dimensional factor space $M^*(4)/\mathrm{Im}(f)$. They will be denoted $0_{f,q}$.

Let us now calculate subalgebras $\underline{l}_\omega$ subordinate to $\omega \in \underline{k}'$ and of maximal dimensions with ω an (arbitrary but fixed) representant of each of these orbits. We first have, because of (2.2) to calculate the commutator algebra of $\underline{k}$. It follows readily from (1.9) that

$$[\underline{k}, \underline{k}] = \{a^\mu a'^\nu \mathbb{F}_{\mu\nu}, \quad \forall a, a' \in U\}. \tag{2.9}$$

Considering again the different cases in turn, we find then:

(α) $f = 0$ so that trivially $\underline{l}_{(0,p)} = \underline{k}$, using (2.9).

(β) Because $\det f \neq 0$, there exist two, and no more (see [8, 14]), vectors $a^{(1)}$ and $a^{(2)}$ such that $(a^{(1)})^\mu (a^{(2)})^\nu f_{\mu\nu} = 0$, with $a^{(1)}$ arbitrary and $a^{(2)}$ determined by this condition. Denoting by $a(\lambda_1, \lambda_2)$ an element $\lambda_1 a^{(1)} + \lambda_2 a^{(2)}$ in the subspace generated by $a^{(1)}$ and $a^{(2)}$ we have then $\forall \lambda_i, \lambda_i' \in \mathscr{R}$, $i = 1, 2$

$$a^\mu(\lambda_1, \lambda_2) a^\nu(\lambda_1', \lambda_2') f_{\mu\nu} = 0. \tag{2.10}$$

Choosing now as representative of each orbit 0_f the element $(f, 0)$ we have $\mathrm{Ker}\, f = 0$, thus $\mathrm{Ker}\,(f, 0) = \{a^\mu \Pi_\mu \mid \forall a \in U\}$ so that, using (2.9) and (2.10), we obtain

$$\underline{l}_{(f,0)} = \{a^\mu(\lambda_1, \lambda_2)\Pi_\mu + B^{\mu\nu}\mathbb{F}_{\mu\nu} \mid \forall B^{\mu\nu} \quad \text{and} \quad \lambda_1, \lambda_2 \in \mathscr{R}\} \tag{2.11}$$

(γ) Because of (2.8) and because $f \neq 0$, it is always possible to choose one (and no more, see again [8, 14]) additional vector $a^{(3)}$ such that with $a^{(1)}$ and $a^{(2)}$ as in (2.8) we have $(a^{(i)})^\mu (a^{(j)})^\nu f_{\mu\nu} = 0$, $i, j \in 1, 2, 3$. Denoting by $a(\mu_1, \mu_2, \mu_3)$ an arbitrary element of the form $\mu_1 a^{(1)} + \mu_2 a^{(2)} + \mu_3 a^{(3)}$, $\mu_i \in \mathscr{R}$ we have then, $\forall \mu_i, \mu_i' \in \mathscr{R}$

$$a^\mu(\mu_1, \mu_2, \mu_3) \cdot a^\nu(\mu_1', \mu_2', \mu_3') \cdot f_{\mu\nu} = 0 \tag{2.12}$$

so that, for any representant $0_{f,q}$ of each of these orbits, we obtain, using once more (2.9),

$$\underline{l}_{(f,p)} = \{a^\mu(\mu_1, \mu_2, \mu_3)\Pi_\mu + B^{\mu\nu}\mathbb{F}_{\mu\nu} \mid \forall B^{\mu\nu} \quad \text{and} \quad \mu_i \in \mathscr{R}\}. \tag{2.13}$$

The corresponding subgroups L_ω of K are thus given, for the three cases separately

$$\begin{aligned} &(\alpha) \quad L_{(0,p)} = \{\langle B, a, 1\rangle\} = K \\ &(\beta) \quad L_{(f,0)} = \{\langle B, a(\lambda_1, \lambda_2), 1\rangle\} \lhd K \\ &(\gamma) \quad L_{(f,p)} = \{\langle B, a(\mu_1, \mu_2, \mu_3), 1\rangle\} \lhd K \end{aligned} \tag{2.14}$$

and are, by construction, of maximal dimension and normal in K.

The corresponding representations T_ω of these groups L_ω are then, using (2.3), given by

$$
\begin{aligned}
(\alpha)\ T_{(0,p)}(\langle B, a, 1\rangle) &= \exp i(0, p)(B^{\mu\nu}\mathbb{F}_{\mu\nu} + a^\mu \Pi_\mu) \\
&= \exp i(p \cdot a) \\
(\beta)\ T_{(f,0)}(\langle B, a, 1\rangle) &= \exp i(f, 0)(B^{\mu\nu}\mathbb{F}_{\mu\nu} + a^\mu(\lambda_1, \lambda_2)\Pi_\mu) \\
&= \exp i(B \cdot f) \\
(\gamma)\ T_{(f,p)}(\langle B, a, 1\rangle) &= \exp i(f, p)(B^{\mu\nu}\mathbb{F}_{\mu\nu} + a^\mu(\mu_1, \mu_2, \mu_3)\Pi_\mu) \\
&= \exp i(B \cdot f + a(\mu_1, \mu_2, \mu_3) \cdot p).
\end{aligned}
\tag{2.15}
$$

These representations can now be explicitly induced to K as described in (2.4). We again consider the different cases in turn

(α) $L_{(0,p)} = K$ so that the coset space consists of a single point and trivially

$$V_{(0,p)}(\langle B, a, 1\rangle) = \exp i(p \cdot a), \tag{2.16}$$

i.e. we get a *4-dimensional family of one-dimensional representations of K* parametrized by a vector $p \in M^*(4)$.

(β) The coset space K/L is two dimensional and can be identified with the translation subspace generated by two vectors $a^{(3)}$ and $a^{(4)}$, chosen so to complete $a^{(1)}$ and $a^{(2)}$ in (2.10) to a basis of U. We decompose thus any translation $a \in U$ as $a \equiv a(\lambda_1, \ldots, \lambda_4) = \sum_{i=1}^4 \lambda_i a^{(i)}$ and we parametrize the coset space by the same λ_3 and λ_4. The function φ of (2.4) is then a quadratic integrable measurable function on $\mathcal{R}^2$ with respect to the invariant ergodic (here Lebesgue) measure μ. Denoting then an element $k \in K$ by $\langle B, a(\lambda_1, \lambda_2, \lambda_3, \lambda_4), 1\rangle$ the coset condition reads

$$\langle 0, a(\lambda'_3, \lambda'_4), 1\rangle \langle B, a(\lambda_1, \ldots, \lambda_4), 1\rangle \langle 0, a(\lambda''_3, \lambda''_4), 1\rangle^{-1} \in L_{(f,0)}$$

i.e. using the group product (1.2) and the result (2.14)

$$a(\lambda''_3, \lambda''_4) = a(\lambda_3, \lambda_4) + a(\lambda'_3, \lambda'_4) = a(\lambda_3 + \lambda'_3, \lambda_4 + \lambda'_4).$$

We then obtain for the induced representations, using (2.4)

$$
\begin{aligned}
&V_{(f,0)}(\langle B, a(\lambda_1, \ldots, \lambda_4), 1\rangle)\varphi(\lambda'_3, \lambda'_4) \\
&= T_{(f,0)}(\langle B + A(a_1, a_2, a_3), a_2, 1\rangle)\varphi(\lambda_3 + \lambda'_3, \lambda_4 + \lambda'_4)
\end{aligned}
\tag{2.17}
$$

where $a_1 \equiv a(\lambda'_3, \lambda'_4)$, $a_2 \equiv a(\lambda_1, \lambda_2)$, $a_3 \equiv a(\lambda_3, \lambda_4)$. Furthermore $A(a_1, a_2, a_3)$ is given, using the group product rule and the linearity and antisymmetry properties of the factor set $A(a, a')$, the restriction to $U \times U$ of the factor set in (1.3), by

$$A(a_1, a_2, a_3) = A(a_1, a_2 + a_3) + A(a_1 + a_3, a_2). \tag{2.18}$$

With (2.17) we have now found a *6-dimensional family of infinite dimensional representations of K* parametrized by the 6-dimensional linear space of functions $f_{\mu\nu}$, where the hypersurface characterized by $\det f = 0$ is left out.

(γ) The coset space K/L is 1-dimensional and can be identified, completely analogously as in the previous case, with the subspace generated by a fourth translation $a^{(4)}$ completing $a^{(1)}\, a^{(2)}\, a^{(3)}$ of (2.12) to a basis. The functions φ are now (quadratic integrable and measurable) functions on $\mathcal{R}$ with respect to the invariant ergodic (Lebesgue) measure. We obtain similarly to (2.17), with now $a = a(\mu_1, \ldots, \mu_4) = \sum_{i=1}^4 \mu_i a^{(i)}$:

$$V_{(f,p)}(\langle B, a(\mu_1,\ldots,\mu_4), 1\rangle)\varphi(\mu'_4)$$
$$= T_{(f,p)}(\langle B + A(a_1, a_2, a_3), a_2, 1\rangle)\varphi(\mu_4 + \mu'_4) \tag{2.19}$$

where $A(a_1, a_2, a_3)$ is formally as in (2.18) with but now $a_1 = a(\mu'_4)$, $a_2 = a(\mu_1, \mu_2, \mu_3)$, $a_3 = a(\mu_4)$.

In (2.19) we have found a *7-dimensional family of infinite dimensional representations of K*, characterized by the 5-dimensional hypersurface in the $f_{\mu\nu}$ space with $\det f = 0$ (where the point $f = 0$ is obviously left out), and the 2-dimensional factor space $M^*(4)/\mathrm{Im}(f)$.

The fact that $\hat{K}$ is of Type I (from theorem 2.1) and thus smooth [15], allows us to apply the general theory of Mackey [5] concerning the representations by induction of group extensions. This is what we shall do in the next section.

3. Induction to the connected part $\overline{\mathbb{M}}_0$ of $\mathbb{M}$

(a) *Representations of group extensions*

The theory of Mackey for the induction of representations is a well known procedure, at least when applied to a (regular) semi-direct product $G = N\Lambda_\varphi H$, with G separable locally compact and N abelian [16]. Less known is the more general case where G, separable locally compact, is any (regular) extension of a group N, not necessarily abelian, by a group H, i.e. appears in the following exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1, m, \varphi \tag{3.1}$$

characterized by a factor set $m: H \times H \to N$ with $m(h_1, h_2h_3) + \varphi(h_1)m(h_2, h_3) = m(h_1, h_2) + m(h_1h_2, h_3)$ and a map $\varphi: H \to \mathrm{Aut}\, N$ satisfying the condition $\varphi(h_1)\varphi(h_2) = \mu(m(h_1, h_2))\varphi(h_1h_2)$, μ being the canonical epimorphism from N to In N, the group of inner automorphisms of N. The essential difference, in this more general case is that, as shown by Mackey [5], whether N is not abelian or whether the extension (3.1) does not split, no longer ordinary representations of the adequate subgroups of H have to be considered but certain projective ones. Since we shall need in the sequel the explicit formulas of the general theory, and in particular an explicit expression for the factor sets involved, we first indicate briefly and in a way convenient for our purposes, the general construction procedure. As, however, in our problem $m = 0$, we shall restrict ourselves here to this more special case.

Let therefore $\hat{N}$ be the dual of N in (3.1), $[\hat{n}] \in \hat{N}$, $\hat{n}$ a representant of the corresponding class $[\hat{n}]$ of irreducible representations. One defines from φ a map $\hat{\varphi}$ on H with

$$\hat{\varphi}(h): \hat{N} \longrightarrow \hat{N}, \qquad \hat{\varphi}(h)[\hat{n}] \equiv [\hat{n}_h]$$

in the canonical way, i.e. by

$$\hat{n}_h(n) \overset{\mathrm{def}}{=} \hat{n}(\varphi(h)^{-1} \cdot n). \tag{3.2}$$

The set of all classes $\{[\hat{n}_h]\}$ generated by $\hat{\varphi}(H)$ from a class $[\hat{n}]$ is called the orbit of $[\hat{n}]$ and will be denoted $0_{[\hat{n}]} \subseteq \hat{N}$. The action in (3.2) defines at the same time an *homogeneous little group* $H_{\hat{n}} \subseteq H$ by the invariance condition

$$h \in H_{\hat{n}} \Leftrightarrow [\hat{n}_h] = [\hat{n}] \tag{3.3}$$

i.e. $h \in H_{\hat{n}}$ if and only if there exists a unitary operator $S(h)$ in $\mathscr{H}(\hat{n})$, the representation space of $\hat{n}$, such that

$$\hat{n}_h(n) = S(h)^{-1}\hat{n}(n)S(h), \qquad \forall n \in N. \tag{3.4}$$

The map $S: h \mapsto S(h)$ is in general not a homomorphism, but can easily be shown from (3.4) (because of the irreducibility of $\hat{n}$ and Schur's Lemma) to be a projective map, satisfying thus

$$S(h_1)S(h_2) = \tau(h_1, h_2)S(h_1h_2) \tag{3.5}$$

for some factor set $\tau(h_1, h_2) \in U(1)$, the unit circle of the complex plane. The isotropy (or little) group of $[\hat{n}]$ $G_{\hat{n}} \subseteq G$ is defined as the subgroup of G that leaves $[\hat{n}]$ invariant under the action

$$g: \hat{n}(n) \longrightarrow \hat{n}(g^{-1}ng)$$

N being identified with its image as a subgroup of G. This group $G_{\hat{n}}$ appears then as an extension (here trivial because so is G) of N by $H_{\hat{n}}$ as is shown in the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G_{\hat{n}} & \xrightarrow{\pi} & H_{\hat{n}} & \longrightarrow & 1, \qquad 0, \varphi \\ & & \| & & \downarrow (\iota) & & \downarrow (\iota)' & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi'} & H & \longrightarrow & 1, \qquad 0, \varphi \end{array} \tag{3.6}$$

(ι) and $(\iota)'$ denoting the natural injection monomorphisms and π, π' the canonical epimorphisms. One can now construct, in a first step, a (projective) representation of $G_{\hat{n}}$ as follows: let L be a projective representation of $H_{\hat{n}}$ with factor set ω and carrier space $\mathscr{H}(L)$ and let $(\hat{n}_S L)$ be defined on $(n, h_{\hat{n}}) \in G_{\hat{n}}$ as follows

$$(\hat{n}_S \cdot L)(n, h_{\hat{n}}) \stackrel{\text{def}}{=} \hat{n}(n)S(h_{\hat{n}}) \otimes L(h_{\hat{n}}). \tag{3.7}$$

One obtains in this way a *projective* representation of $G_{\hat{n}}$ with factor set σ, where σ is obtained from (3.4), (3.5) and (3.7) to be equal to

$$\sigma(g_1, g_2) = \tau(\pi g_1, \pi g_2)\omega(\pi g_1, \pi g_2) \tag{3.8}$$

so that by choosing $\omega = \tau^{-1}$ (as we shall do in the sequel) one obtains an *ordinary* representation of $G_{\hat{n}}$ with carrier space $\mathscr{H}(\hat{n}) \otimes \mathscr{H}(L)$. The last step is now the following: one decomposes H in (right) cosets with respect to $H_{\hat{n}}$, with coset representatives $\{h_i \mid i \in I\}$, where I is some index set (Borel) isomorphic to $H/H_{\hat{n}}$ and similarly G with respect to $G_{\hat{n}}$, choosing now as coset representatives the images of $\{h_i\}$ under a fixed section $r: H \to G$. This set of representatives is thus given by $\{(0, h_i) | i \in I\}$. Let now $\mu_{\hat{n}}$ be the (unique, as a class) invariant ergodic measure on $H/H_{\hat{n}}$, and let us also assume that his measure is right and left invariant (since $\overline{\mathrm{M}}_0$ is unimodular, we assume thus G to be such). It follows then from the assumed regularity of (3.1) that this measure is also transitive (i.e. concentrated on the orbit). We may then identify the coset spaces $H/H_{\hat{n}}$ and $G/G_{\hat{n}}$ with $0_{[\hat{n}]}$ via the 1-to-1 Borel isomorphism $[\hat{n}_{h_i}] \leftrightarrow h_i$, and $(0, h_i)$ respectively, so that we may use the same parametrization $\{h_i\}$ to describe all these spaces. Let us then consider a vector valued function $\varphi: \{h_i\} \to \mathscr{H}(\hat{n}) \otimes \mathscr{H}(L)$ satisfying the 2 conditions

$$\text{(i) } (\varphi(h_i), \varphi') \text{ is } \mu_{\hat{n}}\text{-measurable}, \ \forall \varphi' \in \mathscr{H}(\hat{n}) \otimes \mathscr{H}(L) \tag{3.9}$$

$$\text{(ii)}\ \|\varphi\|^2 \equiv \int_I \|\varphi(h_i)\|^2\, d\mu_{\hat{n}}(h_i) < \infty. \tag{3.9}$$

These vector-valued functions span a separable Hilbert space [5] on which the induced representation is defined as follows, $\forall (n, h) \in G$

$$(\hat{n} \uparrow G)^L(n, h)\varphi(h_i) \stackrel{\text{def}}{=} (\hat{n}_S \cdot L)((0, h_i)(n, h)(0, h_j)^{-1})\varphi(h_j) \tag{3.10}$$

where h_j is the (unique) coset representative satisfying the condition $h_i h h_j^{-1} \in H_{\hat{n}}$. The following then holds, and is a consequence of [5] for this special case:

Theorem 3.1. (*Mackey* [5]). *Consider an orbit* $0_{[\hat{n}]}$ *and a transitive ergodic measure* $\mu_{\hat{n}}$ *concentrated on it as described above. Then the representation* (3.10) *is unitary and irreducible if and only if L is. Two such representations* $(\hat{n}_1 \uparrow G)^{L_1}$ *and* $(\hat{n}_2 \uparrow G)^{L_2}$ *are equivalent if and only if* $\hat{n}_1$ *and* $\hat{n}_2$ *are in the same orbit and* $L_1 \sim L_2$. *Moreover all irreducible unitary representations of G are obtained in this way, up to equivalence, when one induces once per orbit and for each orbit one considers all inequivalent projective ω-representations L of the corresponding homogeneous little groups, with ω satisfying* (3.8), *for* $\sigma = 1$, *and* τ *determined by* $\hat{n}$ *through* (3.4) *and* (3.5).

Let now N be nilpotent. It follows from theorem 2.1 and from (2.14) that in each class $[\hat{n}]$ we may choose as representative $V_{\nu,\underline{l}}$, defined as in (2.4) with $\underline{l}$ now ideal. Let $L = \exp \underline{l}$, then $L \lhd N$ and we may construct the following exact sequence of groups:

$$1 \longrightarrow L \longrightarrow N \xrightarrow{\ \pi\ } N/L \longrightarrow 1, \qquad \rho, \psi. \tag{3.11}$$

We have shown elsewhere [13] that in this case, the operators $S(h)$ and the factor set ω can be calculated explicitly and are respectively given by

$$S(h) = V_{\nu,\underline{l}}(n(h))$$
$$\omega(h_1, h_2) = \rho(\pi n(h_1), \pi n(h_2)) \tag{3.12}$$

where ρ is defined by (3.11) and $n(h)$ is the unique (up to L) element of N satisfying the condition

$$(\text{coAd}(h)\cdot \nu)(\xi) = (\text{coAd}(n(h))\cdot \nu)(\xi), \quad \forall \xi \in \underline{l}. \tag{3.13}$$

(b) *Induction to* $\overline{\mathrm{M}}_0$

Let us now apply the general theory just mentioned to our group $\overline{\mathrm{M}}_0$, which can, as seen in section 1, be written as a semi-direct product of K by $SL(2, \mathbb{C})$. The action (3.2) of $\overline{\mathrm{M}}_0$, resp. $\overline{\mathrm{M}}_0/K$ on K is obtained from (2.5) and (2.7) by the coadjoint action on the dual algebra $\underline{k}'$ of K, as shown in [13]. We obtain, for $\overline{m} = \langle B, a, \overline{\Lambda}\rangle \in \overline{\mathrm{M}}_0$

$$\hat{\varphi}(\overline{m})(f, p) = (\Lambda^{-1}f, \Lambda^{-1}(p + a\cdot f)) \tag{3.14}$$

where the action of $\overline{\Lambda} \in SL(2, \mathbb{C})$ on p and f is given, via the covering map, by the action of the corresponding Lorentz transformation on covariant vectors and tensors respectively. For the homogeneous part we obtain thus with $\overline{\Lambda} \in SL(2, \mathbb{C})$

$$\hat{\varphi}(\overline{\Lambda})(f, p) = (\Lambda^{-1}f, \Lambda^{-1}p). \tag{3.15}$$

Since a class of irreducible representations of K was characterized by an orbit $0_{(f,p)} = \{(f, p + a\cdot f)) \mid \forall a \in U\}$ the corresponding homogeneous little group is then given by the subgroup of $SL(2, \mathbb{C})$ which leaves the orbit invariant under (3.15), i.e. whose elements Λ fulfill the two conditions

$$\text{(i) } \Lambda^{-1}f = f$$
$$\text{(ii) } \Lambda^{-1}p = p + a'f, \quad \text{for some } a' \in U. \tag{3.16}$$

It is useful at this point, because of (3.16) (i) to parametrize f (as in [8]) as a formal constant uniform (c.u.) e.m. field with $f_{0i} = e_i, f_{ij} = \varepsilon_{ijk}b_k, i, j, k \in 1, 2, 3$ and ε_{ijk} the totally antisymmetric tensor of order 3, so that the first condition of (3.16) reduces to the well known problem of the symmetry group of such a field (see e.g. [12], [17] or [18]). These symmetry groups can be classified with the help of the following two Lorentz invariants of the tensor f:

$$i_1(f) = (f \cdot f) \equiv f_{\mu\nu}f^{\mu\nu} = 2(\mathbf{b}^2 - \mathbf{e}^2)$$
$$i_2(f) = (f \cdot f^*) \equiv f_{\mu\nu}(f^*)^{\mu\nu} = 4(\mathbf{b} \cdot \mathbf{e}) \tag{3.17}$$

where $(f^*)^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}f_{\rho\sigma}$ is the dual of f ($\varepsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric tensor of order 4). It is easy to see that, because $\det f = (\mathbf{b} \cdot \mathbf{e})^2$, the previous classification in cases (α) (β) and (γ) corresponds in this language to $f = 0$, $i_2(f) \neq 0$, and $i_2(f) = 0$, $f \neq 0$ respectively.

Before we determine the homogeneous little groups of the classes $[V_{(f,p)}]$ of irreducible representations of K, let us analyze more precisely what is the action of $SL(2, \mathbb{C})$ on them, for the three cases we had in turn. For this purpose, we can, as seen before, look equivalently at the action on the orbits $0_{(f,p)}$. This action follows from (3.15). Let us thus consider again the three cases separately

(α) The orbits 0_p are mapped onto the collection $\{0_{\Lambda^{-1}p} \mid \forall \Lambda \in 0_0(3,1)\}$,

(β) The orbits 0_f are mapped onto the collection $\{0_{\Lambda^{-1}f} \mid \forall \Lambda \in 0_0(3,1)\}$,

(γ) The orbits $0_{f,q}$ (with $q \in M^*(4)/\mathrm{Im}(f)$) are mapped onto the collection $\{0_{\Lambda^{-1}f, \Lambda^{-1}q}\}$. This follows from the fact that

$$\mathrm{Im}(\Lambda^{-1}f) = \{a^\mu(\Lambda^{-1})^\rho_\mu(\Lambda^{-1})^\sigma_\nu f_{\rho\sigma} \mid \forall a \in U\}$$
$$= \{(\Lambda^{-1})^\sigma_\nu(a')^\rho f_{\rho\sigma} \mid \forall a' = \Lambda a \in U\} = \Lambda^{-1}(\mathrm{Im}(f)). \tag{3.18}$$

We shall denote in the sequel superorbits these sets (orbits of orbits) in order to avoid confusion. Let us remark here that all these superorbits are connected, since $0_0(3, 1)$ is so [19]. Further, since the class of an induced representation of $\overline{\mathbb{M}}_0$ does not depend on the element of the superorbit it is based on, we shall fix in each case the coordinates in such a way that the formal field f has a simple structure (note that this will imply that it will in general not be possible to put at the same time the other parameter $p \in M^*(4)$ in a form as simple as in the free field case). It is well known (see e.g. [12]) that for f as in case (β) it is always possible to choose a reference frame in such a way that $\mathbf{e} \| \mathbf{b} \|$ z-axis and for f as in case (γ) there are three possibilities depending on the first invariant of f in (3.17): if $i_1(f) > 0$, there exists a frame with $\mathbf{e} = 0$, and $\mathbf{b} \|$ z-axis; if $i_1(f) < 0$ there exists a frame with $\mathbf{b} = 0$, $\mathbf{e} \|$ z-axis, and if $i_1(f) = 0$ there exists a frame with $\mathbf{e} \|$ z-axis, $\mathbf{b} \|$ x-axis. We denote for obvious reasons (and as in [8]) these cases by (γ mag) (γ el) and (γ rad) respectively. With this choice of representatives we have made on the superorbits we have in fact also constructed, as is easily seen, a Borel cross-section, i.e. a Borel set in K^c intersecting per definition each superorbit once and only once. This ensures us that the semi-direct product of K by $SL(2, \mathbb{C})$ is regular in the sense of

Mackey [5] and thus that all ergodic measures on the superorbits will be transitive so that any possible pathology is excluded (on the contrary of certain physically relevant subgroups of $\overline{\mathbf{M}}_0$, see [20]). The theorem 3.1 may thus be applied.

With these particular choices on the superorbits, it is now an easy matter to determine the homogeneous little groups which are the subgroups of the symmetry groups of the various (formal) fields $f_{\mu\nu}$ fulfilling (3.16) (ii) too. This last condition may split of course in function of p each case in more subcases. In case (β) we know that p can always be chosen 0 so that (3.16) (ii) is always fulfilled. In case (γ) we have, with the above choices of coordinates, in $M^*(4)$

$$\begin{aligned} &(\gamma\ \text{mag}): \operatorname{Im}(f) = \{(0, \lambda, \mu, 0) \mid \forall \lambda, \mu \in \mathscr{R}\} \\ &(\gamma\ \text{el})\ \ : \operatorname{Im}(f) = \{(\lambda, 0, 0, \mu) \mid \forall \lambda, \mu \in \mathscr{R}) \\ &(\gamma\ \text{rad}): \operatorname{Im}(f) = \{(\lambda, 0, \lambda, \mu) \mid \forall \lambda, \mu \in \mathscr{R}\} \end{aligned} \qquad (3.19)$$

so that the quotient spaces $M^*(4)/\operatorname{Im}(f)$ can be parametrized by (p_0, p_3), (p_1, p_2), and $(p_1, \sqrt{\tfrac{1}{2}}(p_0 - p_2) \stackrel{\text{def}}{=} p_-)$ respectively. We do not go into details of all these calculations, since they are tedious but straightforward: we just give the complete results in the form of Tables. The homogeneous little groups are listed in Table I where the elements of $SL(2, \mathbb{C})$ are characterized by their generators in $\mathfrak{sl}(2, \mathbb{C})$. We also already include in this Table the elements of the Poincaré group which are not in the connected component and whose action is defined in the next section on p and f. We denote these last elements m_i for a mirror perpendicular to the i-axis; an accent means time-inversion and a bar space-inversion. For the case (α) we just include the well known results of Wigner for completeness (see e.g. [19], [21]). In Table II we list other useful results for each case, in particular the isomorphism classes of the homogeneous little groups (including again discrete symmetries and denoting C_2' a discrete generator containing time-inversion), the structure of the coset spaces and a set of generators of

Table I
Homogeneous little groups of the representants $0_{(f,p)}$ of the superorbits

Case	Subcases	Infinitesimal generators	Discrete transformations: unitary	Discrete transformations: anti-unitary
(α)	(i) $p = p_0\,(1, 0, 0, 0), p_0 \neq 0$	M_{12}, M_{13}, M_{23}	$\overline{1}$	$1'$
	(ii) $p = p_0\,(1, 0, 1, 0), p_0 \neq 0$	$M_{13}, M_{03} + M_{23}, M_{12} - M_{01}$	m_1	$\overline{m_2'}$
	(iii) $p = p\,(0, 1, 0, 0)$	M_{23}, M_{02}, M_{03}	$\overline{m_2}$	$\overline{1'}$
	(iv) $p = 0$	$M_{\mu\nu}, \forall \mu, \nu \in 0, \ldots, 3$	$\overline{1}$	$1'$
(β)	$p = 0$	M_{12}, M_{03}		m_1'
(γm)	(i) $p_3 \neq 0$	M_{12}		
	(ii) $p_3 = 0 \neq p_0$	M_{12}	$\overline{1}$	m_1'
	(iii) $p_3 = p_0 = 0$	M_{12}, M_{03}	$\overline{1}$	m_1'
(γe)	(i) $p_1 \neq 0$ or $p_2 \neq 0$	M_{03}	$m_{(-2,1)}$	
	(ii) $p_1 = p_2 = 0$	M_{03}, M_{12}	m_1	$\overline{1'}$
(γr)	(i) $p_1 \neq 0 \neq p_-$	$M_{03} + M_{23}$		
	(ii) $p_1 = 0 \neq p_-$	$M_{03} + M_{23}$	m_1	m_2'
	(iii) $p_1 \neq 0 = p_-$	$M_{03} + M_{23}, M_{12} - M_{01}$		
	(iv) $p_1 = 0 = p_-$	$M_{03} + M_{23}, M_{12} - M_{01}$	m_1	m_2'

Table II
Isomorphy classes of homogeneous little groups and coset spaces

Case	Isomorphy classes	Coset spaces: Dimension	Parameter spaces	Coset representants
(α)	(i) $SU(2) \otimes (C_2 \otimes C_2')$	3	$\mathscr{R}^3$	M_{01}, M_{02}, M_{03}
	(ii) $\Delta(2) \otimes (C_2 \otimes C_2')$ [1]	3	$\mathscr{R}^3$	M_{12}, M_{23}, M_{02}
	(iii) $SL(2, \mathscr{R})$ Ⓢ $(C_2 \otimes C_2')$	3	$\mathscr{R}^3$	M_{01}, M_{12}, M_{13}
	(iv) $SL(2, \mathbb{C})$ Ⓢ $(C_2 \otimes C_2')$	0	—	—
(β)	$(\mathscr{R}/4\pi\mathbb{Z} \otimes \mathscr{R})$ Ⓢ C_2'	4	$\mathscr{R}^4 \times 2$	$M_{01}, M_{02}, M_{13}, M_{23}, \bar{1}$
(γm)	(i) $\mathscr{R}/4\pi\mathbb{Z}$	5	$\mathscr{R}^5 \times 4$	$M_{01}, M_{02}, M_{03}, M_{13}, M_{23}, \bar{1}, 1'$
	(ii) $\mathscr{R}/4\pi\mathbb{Z}$ Ⓢ $(C_2 \otimes C_2')$	5	$\mathscr{R}^5$	$M_{01}, M_{02}, M_{03}, M_{13}, M_{23}$
	(iii) $(\mathscr{R}/4\pi\mathbb{Z} \otimes \mathscr{R})$ Ⓢ $(C_2 \otimes C_2')$	4	$\mathscr{R}^4$	$M_{01}, M_{02}, M_{13}, M_{23}$
(γe)	(i) $\mathscr{R} \otimes C_2$	5	$\mathscr{R}^5 \times 2$	$M_{01}, M_{02}, M_{12}, M_{13}, M_{23}, \bar{1}'$
	(ii) $(\mathscr{R}/4\pi\mathbb{Z} \otimes \mathscr{R})$ Ⓢ $(C_2 \otimes C_2')$	4	$\mathscr{R}^4$	$M_{01}, M_{02}, M_{13}, M_{23}$
(γr)	(i) $\mathscr{R}$	5	$\mathscr{R}^5 \times 4$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, \bar{1}, 1'$
	(ii) $\mathscr{R}$ Ⓢ $(C_2 \otimes C_2')$	5	$\mathscr{R}^5$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{13}$
	(iii) $\mathscr{R}^2$	4	$\mathscr{R}^4 \times 4$	$M_{02}, M_{12}, M_{13}, M_{23}, \bar{1}, 1'$
	(iv) $\mathscr{R}^2$ Ⓢ $(C_2 \otimes C_2')$	4	$\mathscr{R}^4$	$M_{02}, M_{12}, M_{13}, M_{23}$

[1]) For this notation, see [19].

the corresponding coset decompositions. Finally $\otimes$ denotes a direct product and Ⓢ a semidirect product.

The last step of the induction procedure to $\overline{\mathbf{M}}_0$, as described in the first part of this section, is now the following: from each of the superorbits one takes the irreducible representation $V_{(f,p)}$ of K as given by (2.17), (2.18) or (2.19), with (f, p) as just chosen and one constructs all unitary irreducible representations of $\overline{\mathbf{M}}_0$ as in (3.10). The result is then as follows

$$(V_{(f,p)\cdot S}\uparrow\overline{\mathbf{M}}_0)^L\langle B, a, \overline{\Lambda}\rangle\varphi(\overline{\Lambda}_i) = (V_{(f,p)\cdot S}\cdot L)(\langle 0, 0, \overline{\Lambda}_i\rangle\langle B, a, \overline{\Lambda}\rangle\langle 0, 0, \overline{\Lambda}_j^{-1}\rangle)\varphi(\overline{\Lambda}_j) \quad (3.20)$$

with again the usual identifications, with $\overline{\Lambda}_j$ determined by the condition that $\overline{\Lambda}_i\overline{\Lambda}\overline{\Lambda}_j^{-1}$ is in the homogeneous little group of $V_{(f,p)}$ and with $\varphi(\overline{\Lambda}_i)$ defined as in (3.9). In (3.20), the representation $(V_{(f,p)\cdot S}\cdot L)$ of the little group (here the semidirect product of K by the corresponding homogeneous little group) is given, as in (3.7), by

$$(V_{(f,p)\cdot S}\cdot L)\langle B, a, \overline{\Lambda}\rangle = V_{(f,p)}(\langle B, a, 1\rangle)S(\overline{\Lambda}) \otimes L(\overline{\Lambda}) \quad (3.21)$$

for some ω-representation L of the homogeneous little group, with ω as in (3.12), and which has now to be calculated explicitly. We first observe therefore in Table I that, except in case (α) of course, all homogeneous little groups are Abelian (we consider now again the connected component of $\overline{\mathbf{M}}$ to unity only), and their Lie-algebra is isomorphic either to $\mathscr{R}$ or to $\mathscr{R}^2$, so that, using the well known result of Bargmann [7] on the relevant cohomology groups

$$H^2(\mathscr{R}^n, U(1)) \cong \mathscr{R}^{n(n-1)/2} \quad (3.22)$$

we know already that only in the two dimensional cases the factor set ω could be non-trivial. Let us now use the more precise result (3.12) for the induction from a normal nilpotent subgroup: it shows that ω, as a class, is also an element of $H^2(N/L, U(1))$, since ρ is a factor set on N/L and $T_{(f,p)}$ is unitary and one-dimensional (in fact $[\omega] \in H^2(N/L(H_{\hat{n}}), U(1))$ where $N/L(H_{\hat{n}})$ is the subgroup of N/L generated by the classes of all $n(h)$, $h \in H_{\hat{n}}$). Let us now then consider again the various cases in turn, with now $N = K$:

Case (α) $K/L \cong 1$ (from (2.14)) so that $\omega = 1$. This is of course trivial and well known, but it shows how our result works.

Case (β) $K/L \cong \mathscr{R}^2$ (from (2.14)) thus non-trivial multipliers could occur. The elements $n(h)$ of (3.12) are given by the translations a' in (3.16) (ii) and thus the multiplier ω by $T_{(f,p)}(A(a'_1, a'_2))$ where A is the factor system (1.3) as restricted to $U \times U$. We had however seen that, choosing $p = 0$ (as it is always possible in this case) $a' = 0$ satisfies (3.16) (ii) $\forall \overline{\Lambda} \in SL(2, \mathbb{C})$, so that ω is necessarily trivial, A being then equal to zero.

Case (γ) $K/L \cong \mathscr{R}$ (from (2.14) thus by (3.22) ω is, also in this case, trivial.

These results make the situation quite easier, because we then only have to consider ordinary representations of the homogeneous little groups and these are of so simple structure (except in case (α) they are all Abelian) that this problem is easily solved: the irreducible unitary representations of $\mathscr{R}/4\pi\mathbb{Z}$ are given by $\{e^{ijr}, r \in \mathscr{R}/4\pi\mathbb{Z}, 2j \in \mathbb{Z}\}$ and the ones of $\mathscr{R}$ by $\{e^{i\lambda r}, r \in \mathscr{R}, \lambda \in \mathscr{R}\}$. We shall call *spins*, similarly as in the free case, the labels j resp. λ of these representations and *spinors* the square μ-integrable functions on the corresponding coset spaces, with μ the ergodic (here transitive) measures on these spaces under the action of $SL(2, \mathbb{C})$. A complete list of these spins and of the dimension (i.e. number of independent components) of the corresponding spinors will be given at the end of the next section and we shall discuss in Section 5 the physical meaning of these spins.

4. Inclusion of the discrete transformations

So far we have obtained all projective unitary representations of the connected component $\mathbb{M}_0$ of $\mathbb{M}$ (by means of the ordinary unitary irreducibles of $\overline{\mathbb{M}}_0$). If we want to reintroduce the discrete transformations and calculate all PUAIR of $\mathbb{M}$ we first have to calculate all multipliers (1.7) on $\mathbb{M}$, with respect to the *UA* decomposition $\mathbb{M}^{\uparrow} \cup \mathbb{M}^{\downarrow}$, $\mathbb{M}^{\uparrow}$ being, as seen in Section 1, the subgroup of $\mathbb{M}$ corresponding to orthochronous Poincaré transformations only). Let us therefore consider the following exact sequence of groups

$$1 \to \overline{\mathbb{M}}_0 \to \overline{\mathbb{M}} \to V_4 \to 1, \varphi \tag{4.1}$$

where V_4 is the Klein Vierergruppe. Since $\mathbb{M}$ is a split extension of K by 0(3, 1) (see Section 1), V_4 can be identified with the subgroup of $\mathbb{M}$ (and of $\overline{\mathbb{M}}$) generated by $\langle 0, 0, 1\rangle$, $\langle 0, 0, 1'\rangle$, $\langle 0, 0, \overline{1}\rangle$ and $\langle 0, 0, \overline{1}'\rangle$ thus (4.1) is also split. The action φ of V_4 on $\overline{\mathbb{M}}_0$ follows then from (1.2) and from the corresponding action of the Poincaré group. Using (4.1) we may write each element of $\overline{\mathbb{M}}$ as a pair $(\overline{m}, h)$ with $\overline{m} \in \overline{\mathbb{M}}_0$ and $h \in V_4$. The product reads then

$$(\overline{m}, h) \cdot (\overline{m}', h') = (\overline{m} \cdot \varphi(h)\overline{m}', hh').$$

The problem of the multipliers on $\mathbb{M}$ can now be solved, using the following:

Proposition 4.1. Let σ be a multiplier on $\overline{\mathbb{M}} \times \overline{\mathbb{M}}$ (w.r. to the above UA decomposition), then there exists a multiplier $\sigma_1 \sim \sigma$ with

$$\sigma_1((\bar{m}, h), (\bar{m}', h')) = \delta(h, h') \tag{4.2}$$

where δ is a multiplier on $V_4 \times V_4$ (w.r. to the UA decomposition $\{1, \bar{1}\} \cup \{1', \bar{1}'\}$).

Proof. Since $\overline{\mathbb{M}}$ is a semidirect product in (4.1) the same argumentation as in theorem 9.4 of [5b] applies (see also [6] for this extension of Mackey's proof) so that $\sigma \sim \sigma_1$ with

$$\sigma_1((\bar{m}, h), (\bar{m}', h')) = \tau(\bar{m}, \varphi(h)\bar{m}')\,\delta(h, h')\psi(\bar{m}', h) \tag{4.3}$$

where ψ is a Borel function from $\overline{\mathbb{M}}_0 \times V_4$ on $U(1)$, τ and δ are multipliers on $\overline{\mathbb{M}}_0 \times \overline{\mathbb{M}}_0$ and $V_4 \times V_4$ (w.r. to the given UA decomposition) respectively, and τ and ψ satisfy, $\forall \bar{m}, \bar{m}' \in \overline{\mathbb{M}}_0$, $h, h' \in V_4$

$$\text{(i)}\ \tau(\varphi(h)\bar{m}, \varphi(h)\bar{m}') = \tau(\bar{m}, \bar{m}')\psi(\bar{m}\bar{m}', h)\psi(\bar{m}, h)^{-1}\psi(\bar{m}', h)^{-1}$$
$$\text{(ii)}\ \psi(\bar{m}, hh') = \psi(\varphi(h')\bar{m}, h)\psi(\bar{m}, h'). \tag{4.4}$$

Since any multiplier on $\overline{\mathbb{M}}_0 \times \overline{\mathbb{M}}_0$ is trivial, τ is necessarily so. Using this fact and the equations (4.4) it is straightforward to verify that τ may be then chosen equal to one, as giving rise to an equivalent factor system σ_1 in (4.3) It follows then from (4.4) (i) that for any h, ψ is a 1-dimensional unitary representation of $\overline{\mathbb{M}}_0$ and because the only 1-dimensional unitary representation of $\overline{\mathbb{M}}_0$ is the identity (from our previous results), $\psi \equiv 1$. The proof then follows from (4.3).

The multipliers on $V_4 \times V_4$ (w.r. to the given UA decomposition) are well known (see e.g. [7]). There are 4 inequivalent classes with representants $\delta^{\alpha\beta}$, where $\alpha, \beta = \pm 1$ and

$\delta^{\alpha\beta}$	1	$\bar{1}$	$\bar{1}'$	$1'$
1	1	1	1	1
$\bar{1}$	1	1	1	1
$\bar{1}'$	1	$\alpha\beta$	α	β
$1'$	1	$\alpha\beta$	α	β

(4.5)

The last step for obtaining all PUAIR of $\mathbb{M}$ would thus now be to induce the unitary representations of $\overline{\mathbb{M}}_0$ to $\overline{\mathbb{M}}$. The theory of Mackey has for this purpose to be slightly generalized in order to take the antiunitary character into account. This generalization has recently been achieved by Shaw and Lever [22] and we refer to their paper for a detailed description of the theory, whose application is quite straightforward in our problem and will therefore just briefly be sketched here.

Let us first determine the action of the discrete transformations on the dual $\hat{\overline{\mathbb{M}}}_0$ of $\overline{\mathbb{M}}_0$. This action is obviously given by the corresponding action on an element (f, p) of the dual algebra $\underline{k}'$ of $\underline{k}$, and similarly as in the free particle case, we assume that elements of $\mathbb{M}$ containing time-reversal are represented by antiunitary operators. We obtain in this way

$$\hat{\varphi}(h)\cdot p = \varepsilon(h)\Lambda(h)^{-1}\cdot p$$
$$\hat{\varphi}(h)\cdot f = \varepsilon(h)\Lambda(h)^{-1}\cdot f \tag{4.6}$$

with $h \in V_4$, $\Lambda(h)$ the corresponding Lorentz transformation and

$$\varepsilon(h) = \text{sign}\,(\Lambda_0^0(h)).$$

This action determines, for any irreducible representation of $\overline{\mathbb{M}}_0$, and similarly as before, little groups as the groups that leave this representation (as a class) invariant. Because of the unitary/antiunitary character of the representations, these groups are termed *generalized little groups* (respectively *generalized homogeneous little groups*). Using the action in (4.6) and the explicit form of the representations we have found previously, these generalized little groups can be straightforwardly computed. The results are listed in Table I.

Once these groups are known one may further induce the corresponding representations to the whole of $\mathbb{M}$. Since this procedure of 'generalized inducing' has already been applied [23] for the PUAIR of the Poincaré group (our case (α)), we indicate briefly for the case (β), and for illustration, the essential steps of the procedure:

In case (β), the generalized homogeneous little group G is given (see Tables I and II) by $G = (\mathscr{R}/4\pi\mathbb{Z} \otimes \mathscr{R})\,Ⓢ\,C_2'$ with generators M_{12}, M_{03} and $h = m_1'$ respectively. The UA decomposition of G reads then

$$G = G^+ \cup G^- = \{M_{12}, M_{03}\} \cup m_1'\{M_{12}, M_{03}\}. \tag{4.7}$$

The unitary irreducible representations $D^{j,\lambda}$ of G^+ are 1-dimensional and labelled by (j, λ) with $2j \in \mathbb{Z}$ and $\lambda \in \mathscr{R}$. Since h is in the same connected component as $\overline{1}'$ we have by (4.2) and (4.5)

$$\sigma^{\alpha\beta}((0, h), (0, h')) = \delta^{\alpha\beta}(\overline{1}', \overline{1}') = \alpha$$

We now, using Theorem B of [22], define the following $\bar{\sigma}$-representation of G^+, with $\bar{\sigma}$ the factor system obtained from σ by complex conjugation:

$$E^{j,\lambda}(g) = \overline{\sigma(g, h)}/\overline{\sigma(h, \varphi(h)^{-1}g)}\,D^{j,\lambda}(\varphi(h)^{-1}g) \tag{4.8}$$

with $g \in G^+$ and $\sigma(g, h)$ a shorthand notation for $\sigma((\bar{m}, 1), (1, h))$.

The following possibilities can now occur: If E and D are antiunitarily equivalent (in the sense of ordinary (and not projective) equivalence) by means of an antiunitary operator K, then the induced PUAIR will be of Wigner-Type I respectively of Wigner-Type II if $K^2 = \sigma(h, h)D(h^2)$, and respectively $K^2 = -\sigma(h, h)D(h^2)$. If no such K exists then it will be of Wigner-Type III[3]).

Using now the commutation relations

$$\begin{aligned} \Lambda(m_1')\Lambda(M_{03}) &= \Lambda(-M_{03})\Lambda(m_1') \\ \Lambda(m_1')\Lambda(M_{12}) &= \Lambda(-M_{12})\Lambda(m_1') \end{aligned} \tag{4.9}$$

we get for (4.8)

$$E^{j,\lambda}(g) = D^{j,\lambda}(g^{-1}) = D^{-j,-\lambda}(g). \tag{4.10}$$

Hence the representations E and D are antiunitarity equivalent by means of the complex conjugation operator K. As $K^2 = 1$ the induced PUAIR will be, from the

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3) The (Wigner) Type we refer to have of course nothing to do with the (Murray-von Neumann) Types we referred to previously.

criterion just given, of (Wigner-) Type I if $\alpha = (-)^{2j}(K^2 = \delta^{\alpha\beta}(h, h)D^{j,\lambda}(h^2)$ with $D^{j,\lambda}(h^2) = D^{j,\lambda}((m'_1)^2) = (-)^{2j})$ and of (Wigner-) Type II if $\alpha = (-)^{2j+1}$. In the first case there is no doubling of states and the restriction of the inducing representation on G^+ is irreducible. In the second case the carrier space is doubled and the representation is given by

$$U^{j,\lambda}(g) = \begin{pmatrix} D^{j,\lambda}(g) & 0 \\ 0 & D^{j,\lambda}(g) \end{pmatrix}; \quad U^{j,\lambda}(m'_1) = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \tag{4.11}$$

the restriction to G^+ being thus then reducible in two equivalent irreducible subrepresentations. Taking into account the doubling of states occurring from the fact that the coset-space of the complete superorbits has two connected components we get finally, inserting also the dimension of the corresponding spinors, the following possibilities

	α	β	Wigner type	Dimension inducing representation	Dimension spinors	
Case (β)	$(-)^{2j}$	$\pm$	I	1	2	(4.12)
	$(-)^{2j+1}$	$\pm$	II	2	4	

Denoting then $\Delta^{j,\lambda}_{(\beta)}$ the representation of $\overline{\mathbb{M}}_0$ based on an orbit $0_{(f,p)}$ with $\det f \neq 0$ and with spin values j and λ, the whole UAIR of $\overline{\mathbb{M}}$ (and hence, via the covering map of $SL(2, \mathbb{C})$ onto $0_0(3, 1)$), the whole PUAIR of $\mathbb{M}$ is finally given by

$$(\Delta^{j,\lambda}_{(\beta)} \uparrow \mathbb{M})(\bar{m}, h)\varphi_r(\overline{\Lambda}_i)$$
$$= \frac{\delta^{\alpha\beta}(h, r)}{\delta^{\alpha\beta}(s, rhs^{-1})} (V_{(f,p)\cdot s} \cdot U^{j,\lambda})(\overline{\Lambda}_{i,r}(\bar{m}, h)\overline{\Lambda}^{-1}_{j,s})\varphi_s(\overline{\Lambda}_j) \tag{4.13}$$

where r, s label the different connected components of the complete superorbits (and the corresponding (discrete) elements of V_4), $\overline{\Lambda}_i$, $\overline{\Lambda}_j$ are as before, with the usual identifications, $\overline{\Lambda}_{i,r}(\bar{m}, h)\overline{\Lambda}^{-1}_{j,s} \in G$, the generalized homogeneous little group (as e.g. in (4.7)) and $(V_{(f,p)\cdot s} \cdot U^{j,\lambda})$ is given as in (3.21) with L replaced by U (this last representation being then as constructed explicitly in (4.11) for the illustrative example we have discussed).

More important than this formula (which essentially illustrates how the factor set and the antiunitary operators are introduced in the generalized induction procedure), are of course the characterizations of the various PUAIR of $\mathbb{M}$ by means of the spins and of the dimensions of the corresponding spinors. The latter are defined, in analogy with the free particle case, by the number of independent states with definite $f_{\mu\nu}$ and p_μ (modulo $Im(f)$) in an irreducible representation. In other words, calling states the carrier space functions (i.e. the μ-measurable square integrable functions on the superorbits as in (3.9) with values in the carrier space of the representations U of the generalized homogeneous little groups G), the dimension of the spinors is then given by the dimension of this representation of G times the number of connected components in the superorbit.

The case (γ) splits in more subcases (see Tables I and II) but does not present very different situations than the one we have just sketched. Since the calculations are quite straightforward applications of the general theory, we again drop the details and list only the results (Table III). We include for completeness in this Table the free particle case, too (as obtained in [23]), but for some 'physical' representations only.

As it can be seen from this Table, as soon as an external e.m. field (with non-zero c.u. part, see [1] and Section 5) is present, the dimension of the spinors, as well as the spin-labels, may change discontinuously, together in fact with the discontinuous change which has occurred in the representation class of the covariance group. Another fact is that one or more continuous spins occur for some PUAIR of the new covariance group $\mathbb{M}$. These continuous spins are however of quite a different type than the well known continuous spins of $m^2 \leqslant 0$ free particles, since the corresponding spinors have here only finitely many components.

We are thus naturally led to a first conclusion: except in case (α), the (usual) concept of spin has, as a matter of fact, lost the group theoretical meaning it had in the free case and, as a consequence, it is no longer necessarily related to the number of independent states nor to a characterization of these states. However, new spin label(s) appear in a natural way as a consequence of a *covariance principle* in the *actual*

Table III
Characterization of the PUAIR of $\mathbb{M}$

Case	Spins of the PUAIR	Subcases	Factor sets α	Factor sets β	Wigner type	Dimension repr. G	Dimension spinors
(α)	(i) j		$(-)^{2j}$	$(-)^{2j}$	I	$2j+1$	$2j+1$
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	$2(2j+1)$	$2(2j+1)$
			$\pm$	$\mp$	III	$2(2j+1)$	$2(2j+1)$
	(ii) $\lambda_1, \lambda_2, \pm j$	$\lambda_1 = \lambda_2 = 0$, $j \neq 0$	$\pm(-)^{2j}$	$\pm(-)^{2j}$	I	2	2
			$\pm$	$\mp$	II	4	4
		$\lambda_1 = \lambda_2 = 0$, $j = 0$	$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2
		else	$\pm$	$\pm, \mp$		∞	∞
(β)	$\lambda, \pm j$		$(-)^{2j}$	$\pm$	I	1	2
			$(-)^{2j+1}$	$\pm$	II	2	4
(γm)	(i) $\pm j$		$\pm$	$\pm, \mp$	I	1	4
	(ii) $\pm j$		$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2
	(iii) $\lambda, \pm j$	$\lambda = 0$	$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2
		$\lambda \neq 0$	$(-)^{2j}$	$(-)^{2j}$	I	2	2
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	4	4
			$\pm$	$\mp$	III	4	4
(γe)	(i) λ		$\pm$	$\pm, \mp$	I	1	2
	(ii) $\lambda, \pm j$	$j \neq 0$	$\pm$	$\pm, \mp$	III	4	4
		$j = 0$	$\pm$	$\pm, \mp$	III	2	2
(γr)	(i) λ		$\pm$	$\pm, \mp$	I	1	4
	(ii) λ		$+$	$+$	I	1	1
			$-$	$-$	II	2	2
			$\pm$	$\mp$	III	2	2
	(iii) λ_1, λ_2		$\pm$	$\pm, \mp$	I	1	4
	(iv) λ_1, λ_2		$+$	$+$	I	2	2
			$-$	$-$	II	4	4
			$\pm$	$\mp$	III	4	4

situation, in presence thus of an external e.m. field, and as a characterization of representations of the new covariance group. Consider for illustration the case of a c.u. magnetic field: the corresponding representations are still characterized by a discrete half-integer spin label that has however a different and more profound significance than simply a kind of 'remembering' of the (free) spin as transposed in the new situation. This spin has an *intrinsic new meaning*, and is then of 'helicity' type, i.e. corresponds to only two polarization states (see Table III), not along an arbitrary axis as in the free case, but along the magnetic field axis (this corresponding in fact to what is physically observed).

In this respect, the external approximation is conceptually more far reaching than perhaps expected and it can be useful to extend the (group-theoretical) definition of an elementary particle, as adapted to this new situation and as in a new 'world' where the external field is present. This is what we shall do in the next section, together with a discussion of our results and of some, we think important, consequences of them.

5. Discussion

(*a*) *Elementary particles in external e.m. fields*

A real (physical) particle can be identified, as a set of states, with the solutions of some covariant equation of motion, i.e. with a representation space of the general covariance group. Since the external field is an approximation, it is however not necessary that the corresponding representation is irreducible. Nevertheless it is useful to introduce the following group theoretical.

Definition An elementary (relativistic) particle in an external e.m. field is a quantum mechanical system which, as a set of states, spans the carrier space of a PUAIR of the general covariance group $\mathbb{M}$.

It is clear from (1.5) and [1] that when the c.u. part of the external field is zero this definition reduces to the well known group theoretical definition of relativistic free particles[4]) of Wigner. Note however that even then the covariance operator group may be only isomorphic to the ordinary Poincaré action (see e.g. below the formulas (5.5) or (5.6) with compensating gauges χ_g possibly non zero).

As a consequence of its definition, an elementary particle in an external e.m. field may be characterized by the labels of the corresponding representation, i.e. by its spins (see Table III), and, in addition, by the values of the invariants of the Lie algebra of $\mathbb{M}$ in this representation. These invariants can be obtained from the Lie algebra (1.9) (see [12]) and are given explicitly by

$$\begin{aligned} Q_1 &= \mathbb{F}_{\mu\nu}\cdot\mathbb{F}^{\mu\nu} \\ Q_2 &= \mathbb{F}_{\mu\nu}(\mathbb{F}^*)^{\mu\nu} \\ Q_3 &= \Pi_\mu\Pi^\mu - M_{\mu\nu}\mathbb{F}^{\mu\nu} \\ Q_4 &= 2\Pi_\mu(\mathbb{F}^*)^{\mu\nu}\Pi^\rho(\mathbb{F}^*)_{\rho\nu} - (\mathbb{F}^*)^{\mu\nu}M_{\mu\nu}\cdot Q_2 \end{aligned} \tag{5.1}$$

[4]) By this definition we clearly leave open the problems of the physical interpretation and existence of the discrete symmetry operators [24]. We assume here that they represent exact and rigorously valid symmetries of space-time.

where $(\mathbb{F}^*)^{\mu\nu}$ is the 'dual' of $\mathbb{F}_{\mu\nu}$, defined in a similar way as in (3.17) for a field, i.e.

$$(\mathbb{F}^*)^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathbb{F}_{\rho\sigma}.$$

A basis element of the carrier space of a PUAIR of $\mathbb{M}$ being, as we saw characterized by an element $(f, p) \in \underline{k}'$ on some superorbit and by a set $\{s_i\}$ of spins, we denote it $|f, p, s_i\rangle$. It follows from (4.13), with (2.15), that in all representations we have

$$\mathbb{F}_{\mu\nu}|f, p, s_i\rangle = f_{\mu\nu}|f, p, s_i\rangle \tag{5.2}$$

so that, comparing with (1.5), we see that the eigenvalues of these operators are equal (up to a factor $-e/c\hbar$) to the c.u. part of the field, as expected. The interpretation of the first two invariants in (5.1) is then quite obvious: they specify so to say the 'world' in which the corresponding particle is 'elementary', by means of the invariants of the corresponding (c.u. part of the) field. The two last invariants Q_3 and Q_4 correspond then to the particle itself and will thus be related to the equation of motion. Let us note that we can now explain why we do not share the point of view in [12] by which the Lie algebra (1.9) is rejected as 'non-informative': these authors observe that this Lie algebra gives rise to too many invariants in (5.1) in order to characterize the degrees of freedom of an elementary particle, because they want to characterize such a particle in terms of its *free* quantum numbers, whereas these are in our opinion no longer a good characterization of the states of that particle, as a consequence of the presence of the external field and thus as a consequence of the change occurred in the covariance.

(*b*) *Covariant equations of motion*

Irreducible PUA representations of symmetry groups describe the properties of *solutions*, i.e. give information on the classification of the possible states of a quantum-mechanical system and on their properties (invariants of these states, matrix elements, selection rules and so on). Irreducible PUA representations of covariance groups describe properties of the *equations of motion*. Let us therefore consider first the Klein-Gordon and the Dirac operators, with minimal coupling:

$$0_{KG}(x, \pi F) = (\hat{p}_\mu - \frac{e}{c}(\pi F)_\mu(x))^2 - m^2c^2 \tag{5.3}$$

$$0_D(x, \pi F) = -i\gamma^\mu(\hat{p}_\mu - \frac{e}{c}(\pi F)_\mu(x)) - mc \tag{5.4}$$

where $\hat{p}_\mu = -i\hbar\partial_\mu$ and πF is some uniquely chosen potential (as fixed by a map π). Remember that, as we had shown in [1], the choice of π is not essential. Let us then consider the following operators, $\forall m = \langle B, g\rangle \in \mathbb{M}$, $B \in \mathbb{B}$, $g \in IO(3, 1)$

$$V_{KG}(\langle B, g\rangle) \overset{\text{def}}{=} \exp\left\{-i\frac{e}{c\hbar}((B\cdot gF^{(0)}) + \chi_g(\pi(gF), x))\right\}C_g \cdot P_g \tag{5.5}$$

and

$$V_D(\langle B, g\rangle) \overset{\text{def}}{=} \exp\left\{-i\frac{e}{c\hbar}((B\cdot gF^{(0)}) + \chi_g(\pi(gF), x))\right\}S_g \cdot P_g \tag{5.6}$$

where $\chi_g(\pi(gF), x)$ are the compensating gauge functions as for example obtained in (3.32) of [1], $F^{(0)}$ is the c.u. part of the field (see (3.24)–(3.28) of [1]), P_g is the substitution operator in the x-coordinate. Furthermore S_g is given by $S_g^0 \cdot C_g$ where

$$C_g = \begin{cases} 1 & \text{for } g \text{ orthochronous} \\ C & \text{for } g \text{ antichronous} \end{cases} \tag{5.7}$$

with C the charge conjugation operator and S_g^0 is determined by the condition that for Λ the homogeneous part of g we have

$$(S_g^0)^{-1}\gamma^\mu S_g^0 = \Lambda_\nu^\mu \gamma^\nu. \tag{5.8}$$

A straightforward calculation, similar as in [25], shows now that for the Klein-Gordon operator we have

$$\begin{aligned} &V_{KG}(\langle B, g\rangle)0_{KG}(x, \pi F)(V_{KG}(\langle B, g\rangle))^{-1} \\ &\quad = 0_{KG}(x, g\pi F - \partial\chi_g(\pi(gF), x)) \\ &\quad = 0_{KG}(x, \pi(gF)) \end{aligned} \tag{5.9}$$

by definition of the gauge transformation $\chi_g(\pi(gF), x)$. Similarly, for the Dirac operator we obtain with (5.6)

$$V_D(\langle B, g\rangle)0_D(x, \pi F)(V_D(\langle B, g\rangle))^{-1} = 0_D(x, \pi(gF)). \tag{5.10}$$

This means that, by definition of the covariance in the presence of an external e.m. field (see Section 4 of [1]), both the Klein-Gordon and the Dirac equations are covariant under $\mathbb{M}$ and for the whole class of (almost) arbitrary external fields we have considered. On the other side, the operators defined in (5.5) and in (5.6) are homomorphic on $\overline{\mathbb{M}}$, i.e. generate (projective) representations of $\mathbb{M}$. With the usual definitions of the scalar products of the free equations of motion in these both cases, these representations are also obviously unitary (on $\mathbb{M}^\uparrow$).

As follows then also from the above equations (5.9) and (5.10) if $\psi(x, F)$ is a solution of the Klein-Gordon or of the Dirac equation in presence of the field F, i.e. satisfies for the respective operator (5.3) or (5.4) the equation

$$0(x, \pi F)\psi(x, F) = 0 \tag{5.11}$$

we have, with $\Phi(x, gF) \equiv (V\langle B, g\rangle\psi)(x, F)$ as given by (5.5), respectively (5.6):

$$0(x, \pi(gF))\Phi(x, gF) = 0 \tag{5.12}$$

i.e. $\Phi(x, gF)$ is a solution in the field gF. Note furthermore that, as another consequence of our approach, the *invariance operator groups* [20, 25] corresponding to a given field, and that are by definition generated by the covariant operators in the Klein-Gordon or Dirac representations that in addition leave the external field invariant, will appear now *in a natural way as subgroups* of the operator groups defined by (5.5) and (5.6) respectively.

The Klein-Gordon and Dirac equations transform thus covariantly under $\mathbb{M}$ and for the whole class of fields considered. This will however not necessarily be true for higher spins equations. in the presence of c.u. fields, as an arbitrary Poincaré transformation in the corresponding (free) representations will in general mix up wave components of different spin values along a given axis and possibly belonging thus to different irreducible representations of $\mathbb{M}$, this in contradiction with the covariance statement. The difficulty of the group theoretical interpretation of these higher spin equations in the presence of an external c.u. field lies in fact even deeper: as is well known, these equations are characterized by the fact that unwanted additional (free) spin components are eliminated by the so-called constraints. In the presence of a c.u. field, the very existence of the constraints becomes questionable as they have in fact lost this group theoretical meaning.

Let us, for illustration, consider a (free) spin 3/2 particle with positive mass and moving in a *constant uniform magnetic* field. Since it is a priori not possible to choose in general a reference frame where the field is along the z-axis and at the same time $p_z = 0$ and since the case (γm) (iii) (see Tables I and III) can be seen to describe tachyons states, we are in case (γm) (ii), so that the (free) 3/2 – spin representation of the extended Poincaré group will split in two irreducible sub-representations of the new covariance group, characterized by the (helicity) spins $\pm 3/2$ and $\pm 1/2$ respectively. Now, the Dirac equation with minimal coupling describes spin $\pm 1/2$ particles and is covariant with respect to $\mathbb{M}$, as we have shown in (5.10). We can thus construct, assuming a gyromagnetic factor of $1/s$ [26], with s the spin, an equation for the (reducible) new 'particle', by means of a *direct sum of two Dirac equations* with minimal coupling, one for each irreducible component. Since the Dirac equation is covariant with respect to $\mathbb{M}$ so is trivially the new one. This splitting of a representation in two subrepresentations, as a consequence of the presence of the field, can somehow be compared to the mass zero limit in the case of a free particle: the Poincaré representations describing massive particles of spin s split in this limit in disjoint subrepresentations for massless particles of (helicity) spin $s, s - 1, \ldots$ (see e.g. [27] §16). That the above constructed equation is not equivalent to the usual known equations of motion can be seen as follows: it has been shown by Velo and Zwanziger [3–4] that the 3/2 (free-) spin equations with minimal coupling in an external *c.u. magnetic field* (such as the *Rarita-Schwinger* equation for instance) are *a-causal* in the sense that the propagators do not vanish for space-like vectors, whereas the Dirac (as the Klein-Gordon) equation is free of a-causality and thus so is our equation, too.

The above argumentation strongly indicates that these pathological difficulties that have given rise in the last few years to an abundant literature (because of their importance with respect to fully quantized theories e.g.) can possibly be reduced to a covariance problem, and this is the reason for mentioning this problem here.

It is perhaps also interesting to note here in this respect that tachyons in a c.u. field (see Table III (γm) (iii) for example) are necessarily described by finite dimensional spinors whereas in the field 0 limit all these representations will join together to give the known infinite dimensional spinors. This fact could also be used as a trick to avoid infinite spinors when one wants to introduce (virtual) tachyons states.

It is quite clear that these last remarks are not meant as a conclusion but merely as a sketch of some applications of our results and of some new open possibilities we have been led to, by considering this problem of covariance in external e.m. fields. We have however been able to relate the Klein-Gordon and the Dirac equations, minimally coupled to the potential of an (almost) arbitrary external e.m. field, to representations of a well defined covariance group, i.e. to reintroduce and identify a relationship, well known in the free case, but which was in fact no longer necessarily present because of the introduction of the external field.

Acknowledgements

For important suggestions and remarks, I am very thankful to A. Janner. Fruitful discussions with M. Boon, T. Janssen and U. Cattaneo are also gratefully acknowledged.

REFERENCES

[1] N. GIOVANNINI, Helv. Phys. Acta *50*, 337, 1977.
[2] See for example Kursunoglu in Reference [4], p. 21.
[3] G. VELO and D. ZWANZIGER, Phys. Rev. *186*, 1337 (1969), *188*, 2218 (1969).
[4] *Troubles in the external field problem for invariant wave equations*, Tracts in Math. and Nat. Sc. Vol. 4, rev. A. S. Wightman, (Gordon and Breach, 1971).
[5] G. W. MACKEY, Trans. Amer. Math. Soc. *85*, 134 (1957), Acta Math. *99*, 265 (1958).
[6] K. R. PARTHASARATHY, Comm. Math. Phys. *15*, 305 (1969).
[7] V. BARGMANN, Ann. of Math. *59*, 1 (1954).
[8] R. SCHRADER, Fort. der Phys. *20*, 701 (1972).
[9] A. GUICHARDET, Séminaire Bourbaki 1962/63, no. 249.
[10] A. KIRILLOV, Uspekhi Nat. Nauk. *17*, 57 (1962).
[11] See E. M. Stein in *High energy and elementary particles*, Trieste (1965), IAEE Vienne, pp. 563–584.
[12] H. BACRY, PH. COMBE and J. L. RICHARD, Nuovo Cim. *70A*, 267 (1970).
[13] N. GIOVANNINI, Ann. Inst. H. Poincaré *26*, 181 (1977).
[14] A. ARTIN, *Geometrical Algebra* (Interscience N.Y., 1957).
[15] See for instance G. A. Pozzi, Suppl. al Nuovo Cim. *4*, 37 (1966).
[16] For a review, see G. W. Mackey, *Group representations*, Lecture notes, Oxford University (1971).
[17] A. JANNER and E. ASCHER, Physica *48*, 425 (1970).
[18] L. C. CHEN and T. JANSSEN, Physica *77*, 290 (1974).
[19] D. SIMMS, *Lie groups and quantum mechanics*, Lecture notes in Math. 52, Ed. Springer (1968).
[20] N. GIOVANNINI, *Proc. 3rd Int. Coll. on Group Theoretical Methods in Physics, Marseille* (1974), p. 336 and Ph.D. Thesis Nijmegen (1976).
[21] E. P. WIGNER, Ann. of Math. *40*, 149 (1939).
[22] R. SHAW and J. LEVER, Comm. Math. Phys. *38*, 257 (1974).
[23] R. SHAW and J. LEVER, Comm. Math. Phys. *38*, 279 (1974).
[24] E. P. WIGNER, in *Group Theor. Concepts and Methods in Elementary Particle Physics*, Istanbul Summer School 1962, Ed. Gürsey.
[25] A. JANNER and T. JANSSEN, Physics *53*, 1 (1971).
[26] See e.g. F. J. Belinfante, Phys. Rev. *92*, 997 (1953) and P. A. Moldauer and K. M. Case, Phys. Rev. *102*, 279 (1956).
[27] L. LANDAU, E. LIFSHITZ, Cours de Physique Théorique, tome 4, (Editions Mir, Moscou, 1972).