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Autor(en): **Pearson, D.B.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **52 (1979)**

Heft 4

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-115039

Helvetica Physica Acta, Vol. 52 (1979), Birkhäuser Verlag. Basel

Scattering theory for a class of oscillating potentials

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(13. VIII. 1979)

Abstract. Spectral and scattering theory for the operator $H=-(d^2/dr^2)+V$ in $L^2(0,\infty)(V=V_1+V_2,\ V_2=dW_2/dr,\ W_2=dU_2/dr$ with U_2 bounded) is closely related to that for $-(d^2/dr^2)+V_{\text{eff.}}$, where $V_{\text{eff.}}=(V_1-W_2^2)\exp(2U_2)$. This relationship is used to investigate the generalized eigenfunctions and scattering theory for a wide range of oscillating potentials, including $V(r)=\sin r/r$ and $V(r)=e^r\cos(e^r)$ (for which $\Omega_\pm(H;H_0-\frac{1}{2})$ exist and are complete), together with new classes of singular potentials.

1. Introduction

It has been shown by a number of authors [1–5] that the usual scattering theory for short range potentials may be extended to potentials which are not prima facie of short range, but which oscillate sufficiently strongly at infinity to compensate for this. In a similar way, one may deal with potentials such as $V=(\sin r^{-2})/r^3$ which are highly oscillating at the origin, for which eigenfunctions of $-(d^2/dr^2)+V$ behave very much like those of the free Hamiltonian $-d^2/dr^2$, and for which the usual scattering and spectral theory may be carried through almost without modification.

A characteristic feature of these potenials is that $\int_0^\infty V\,dr$ may exist (as an improper integral) whereas $\int_0^\infty |V|\,dr$ does not. Or that, with $V=dW/dr$, W may be L^1 whereas V need not be. Roughly, the *integral* of the potential behaves as well as, or better than, the potential itself.

In the present paper we study a different, but connected, class of potentials $V=V_1+V_2$, for which V_1 and V_2 are related in such a way that there may be a kind of interference between these two terms in the potential.

Suppose for example that $V_2=dW_2/dr$ and $W_2=dU_2/dr$, where now V_2 and W_2 may be unbounded and oscillating, but U_2 is bounded and converges to zero at infinity.

We have the identity

$$-\frac{d^2}{dr^2}+V_1+V_2=-\left(\frac{d}{dr}+W_2\right)\left(\frac{d}{dr}-W_2\right)+V_1-W_2^2.$$

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Since

$$\left(\frac{d}{dr} \pm W_2\right)f = P_\pm^{-1} \frac{d}{dr}\left(P_\pm f\right),$$

where $P_\pm = e \times p \pm U_2 \to 1$ as $r \to \infty$, we expect $-[(d/dr) + W_2][(d/dr) - W_2]$ to behave for large r very much like the free Hamiltonian $-d^2/dr^2$. In that case, the behaviour of $-(d^2/dr^2) + V$ should be governed by a kind of effective Hamiltonian $V_1 - W_2^2$; actually one has $V_{\text{eff.}} = (V_1 - W_2^2) \exp(2U_2)$.

In fact one can show (Section 3) that (for $V_{\text{eff.}} = 0$) the wave operators $\Omega_\pm\{-[(d/dr) + W_2][(d/dr) - W_2]; -(d^2/dr^2)\}$ exist and are complete under rather weak assumptions on U_2 (*e.g.* for $U_2 = (\sin r)/r$ or $U_2 = \sin(e^r)/r$). This result extends to $V_{\text{eff}} \neq 0$, provided $V_{\text{eff.}}$ (or rather the integral of $V_{\text{eff.}}$) is not too large. We may then require cancellations between V_1 and W_2^2 in order to prove the existence of $\Omega_\pm[-(d^2/dr^2) + V; -(d^2/dr^2)]$. These cancellations should not be regarded as exceptional; indeed if ϕ is a non-vanishing solution of $-(d^2\phi/dr^2) + V\phi = 0$ we can always write $V = V_1 + V_2$ with $V_1 = (\phi'/\phi)^2$ and $V_2 = (d/dr)(\phi'/\phi)$, such that $V_1 - W_2^2 \equiv 0$. The problem is that, for given V, we do not know ϕ in advance.

One may show that, in many other respects, V_{eff} is an effective potential for the Hamiltonian $-(d^2/dr^2) + V$. For example (unless $g = 1$ is a critical value for $-(d^2/dr^2) + gV_{\text{eff.}}$) one finds that $-(d^2/dr^2) + V$ is semi-bounded if and only if $-(d^2/dr^2) + V_{\text{eff}}$. is. We defer further consideration of spectral properties to a subsequent publication. An interesting feature is the non-linear dependence on the coupling constant; thus $-(d^2/dr^2) + g^2V_{\text{eff.}}$ is related more closely to $-(d^2/dr^2) + g^2V_1 + gV_2$ than to $-(d^2/dr^2) + g^2(V_1 + V_2)$.

In Section 2 we investigate the asymptotic behaviour at $r = 0$ and at $r = \infty$ of solutions of $[-(d^2/dr^2) + V + k^2]f = 0$, with suitable assumptions on U_2 and $V_{\text{eff.}}$, using a non-linear integral equation from which the solutions may be constructed. (For derivation of the related differential equation see Ref. 6, p. 36). We show, for example, that solutions behave like those of the free differential equation provided $Q \in L^1$ and rQ is bounded, where $V_{\text{eff.}} = dQ/dr$. There is some reason for allowing U_2 to be complex (provided $\text{Re}(U_2)$ is bounded) or even to allow U_2 to contain logarithmic terms, since

$$\text{(i)} \quad -\frac{d^2}{dr^2} - k^2 = -\left(\frac{d}{dr} + ik\right)\left(\frac{d}{dr} - ik\right), \quad \text{and}$$

$$\text{(ii)} \quad -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} = -\left(\frac{d}{dr} + \frac{l+1}{r}\right)\left(\frac{d}{dr} - \frac{l+1}{r}\right).$$

Theorems 3 and 4 are devoted to extending the results to include potentials oscillating more slowly at infinity.

In Section 3 we consider applications to scattering theory. We recover the results of Dollard and Friedman, [5], for $V = \sin r/r$ and related potentials, and treat in addition a wide range of singular and long range potentials; including for example $V = e^r \cos(e^r)$, for which $-(d^2/dr^2) + V$ has absolutely continuous spectrum from $-\frac{1}{2}$ to ∞ and the wave operators $\Omega_\pm[-(d^2/dr^2) + V, -(d^2/dr^2) - \frac{1}{2}]$ exist and are complete.

2. Behaviour of eigenfunctions

We are interested in the asymptotic behaviour at $r=0$ and $r=\infty$ of solutions of the differential equation

$$-\frac{d^2f}{dr^2}+(V_1+V_2)f=0 \qquad (0<r<\infty) \tag{1}$$

where, to permit maximum generality we allow V_1 and V_2 to be complex.

Suppose

$$V_2=\frac{dW_2}{dr} \qquad (0<r<\infty) \tag{2}$$

and

$$W_2=\frac{dU_2}{dr} \qquad (0<r<\infty) \tag{2'}$$

where Re (U_2) is a bounded function of r for $r>0$. This assumption allows V_2 to be wildly oscillating either near $r=0$ or near $r=\infty$, and is satisfied for example by $V_2=r^{-N}\sin(r^{-M})(M>\frac{1}{2}(N-2)>0)$ or by $V_2=r^N\sin(r^M)(M>\frac{1}{2}(N+2)>0)$.

Define $Q(r)$, up to an additive constant, and $V_{\text{eff.}}$, by

$$V_{\text{eff.}}=(V_1-W_2^2)\exp(2U_2)=\frac{dQ}{dr} \qquad (0<r<\infty) \tag{2''}$$

We shall find that the asymptotic behaviour of solutions f of (1) may be determined, provided that Q is sufficiently small in a sense to be made precise. To this end, define an auxiliary function w by

$$w=\left(\frac{1}{f}\frac{df}{dr}-W_2\right)\exp 2U_2(r) \tag{3}$$

The function w will be defined in any interval which does not contain zeros of f. In particular, we are interested in solutions f which are non-vanishing either in a neighbourhood of zero or of infinity. Given w, the solution f may be determined up to a multiplicative constant by regarding (3) as a first order linear equation for f.

It may easily be verified that w satisfies the equation

$$\frac{dw}{dr}+w^2\exp(-2U_2)=\frac{dQ}{dr} \tag{4}$$

Equation (4) may often be converted into an integral equation which may then be solved by standard iterative techniques. Our first, and simplest, result obtained by this method is:

Theorem 1. *Suppose*

(i) Re U_2 *is bounded*
(ii) $|rQ(r)|$ *is bounded and*
(iii) $Q\in L^1(0,\infty)$.

Then equation (1) *has a solution having the asymptotic behaviour*

$$f(r)=\exp(U_2(r))(1+o(1)) \tag{5}$$

as $r\to 0$, *and a further solution having this asymptotic behaviour as* $r\to\infty$.

Proof. Consider first the behaviour near $r=0$, and write, for a particular solution of (4), the integral equation

$$w(r)=Q(r)+\int_r^R (w(t))^2 \exp(-2U_2(t))\,dt \tag{6}$$

Define the integral I_2 by

$$I_2(r)=\int_r^R |Q^2(t)\exp(-2U_2(t))|\,dt \tag{7}$$

and note that

$$\int_\varepsilon^R I_2(t)\,dt=[tI_2(t)]_\varepsilon^R+\int_\varepsilon^R |tQ^2(t)\exp(-2U_2(t))|\,dt$$

From the hypotheses of the Theorem, and noting that $\varepsilon I_2(\varepsilon)\geq 0$ for $\varepsilon<R$, we see that the r.h.s. is bounded as $\varepsilon\to 0$, so that $I_2\in L^1(0,R)$.

Equation (6) may be iterated by taking $w_1=Q$ and

$$w_{n+1}(r)=Q(r)+\int_r^R (w_n(t))^2\exp(-2U_2(t))\,dt \tag{8}$$

Then $|w_2-w_1|\leq I_2(r)$, and supposing

$$|w_{n-1}|\leq AI_2(r)+|Q(r)|, \qquad |w_n|\leq AI_2(r)+|Q(r)|$$

and $|w_n-w_{n-1}|\leq A_n(I_2(r))$, we have

$$\begin{aligned}|w_{n+1}-w_n|&\leq\int_r^R |w_n-w_{n-1}|\cdot|w_n+w_{n+1}|\,|\exp(-2U_2)|\,dt\\ &\leq 2A_n\int_r^R I_2(t)(AI_2(t)+|Q(t)|)\exp(-2U_2(t))|\,dt\\ &\leq 2CA_nI_2(r)\int_r^R (AI_2+|Q|)\,dt,\end{aligned}$$

where $C=\sup|\exp(-2U_2(t))|$. Hence we can write

$$|w_{n+1}-w_n|\leq A_{n+1}(I_2(r)) \qquad (0<r<R), \quad \text{where} \quad A_{n+1}=KA_n$$

and K may be made arbitrarily small by choosing R sufficiently small. The constant A may be chosen such that $A=\sum_0^\infty K^n=(1-K)^{-1}$, and the constant K may be taken arbitrarily close to $2C\int_0^R (I_2+|Q|)\,dt$. Then the iteration converges for $0<r<R$, and the iterative solution satisfies

$$|w|\leq AI_2+|Q|.$$

In particular, $w \in L^1(0, R)$, and we may use (3) to construct the solution

$$f(r) = \exp\left\{U_2(r) + \int_0^r w(t)\exp(-2U_2(t))\,dt\right\} \tag{9}$$

of equation (1), which satisfies the estimate (5) as $r \to 0$. For the behaviour at infinity, replace (6) by

$$w(r) = Q(r) + \int_r^\infty (w(t))^2 \exp(-2U_2(t))\,dt \tag{10}$$

and set $R = \infty$ in the definition (7) of I_2. Then $rI_2(r) \le \text{const} \int_r^\infty t\,|Q(t)|^2\,dt$ and is bounded for large r, and again we find $I_2 \in L^1(a, \infty)$. We may then prove convergence of the iteration for sufficiently large r, and the solution

$$f(r) = \exp\left\{U_2(r) - \int_r^\infty w(t)\exp(-2U_2(t))\,dt\right\} \tag{11}$$

satisfies the estimate (5) as $r \to \infty$.

Remark 1. Potentials $V = V_1 + V_2$ satisfying conditions (i)–(iii) of the theorem are a generalization of the so-called W-class of potentials first studied by Baetman and Chadan [3]. The W-class corresponds to the special case $V_2 \equiv 0$.

Remark 2. A second solution of equation (1), which is $o(r)$ as $r \to 0$, is $f \int_0^r 1/f^2\,dt$. If $\lim_{r\to 0} U_2(r)$ exists, we obtain in this way two solutions f_1, f_2, such that $f_1 \sim 1$ and $f_2 \sim r$ as $r \to 0$. In that case, solutions behave near $r = 0$ like the solutions of the Schrödinger equation with a potential which is regular at the origin. However, df/dr may well be unbounded, since in equation (3) W_2 may be unbounded and oscillating. Similar comments apply to the behaviour at infinity.

Example 1. Let

$$V_1 = 1/(18r^4) \quad \text{and} \quad V_2 = r^{-6}\sin(r^{-3}).$$

Then $U_2 = -\frac{1}{9}r^2\sin(r^{-3}) + O(r^5)$ as $r \to 0$ (taking $U_2(0) = 0$) and $Q = \frac{1}{108}\sin(2r^{-3}) + O(r)$.

(From (2″), we need to evaluate $\int_0^r (V_1 - W_2^2)\exp(2U_2)\,dt$, which can be done by writing $\exp(2U_2) = 1 + 2U_2 + O(U_2^2)$.)

Theorem 1 applies, and we obtain two solutions f_1, f_2 of equation (1), such that $f_1 \sim 1$ and $f_2 \sim r$ as $r \to 0$.

Notice that this example depends on a cancellation in $V_1 - W_2^2$. The behaviour is quite different if, e.g., we change the 'coupling constant' in V_1 from its value of $\frac{1}{18}$. Analysis of examples such as this with variation of the coupling constant will be treated in a subsequent publication. A family of potentials which generalize Example 1 is given by

Example 2

$$V = V_1 + V_2 = \frac{B^2}{2M^2 r^{2(N-M-1)}} + \frac{B\sin(r^{-M})}{r^N} \qquad (M > \tfrac{2}{3}(N-2) > 0)$$

This example is most conveniently analysed by means of the separation

$$V_1 = \frac{B^2}{2M^2 r^{2(N-M-1)}} + \frac{B(N-M-1)}{M r^{N-M}} \cos (r^{-M})$$

$$W_2 = \frac{B \cos (r^{-M})}{M r^{N-M-1}}$$

Again Theorem 1 may be applied to obtain two solutions near $r=0$, $f_1 \sim 1$ and $f_2 \sim r$.

Example 3

$$V_1 = \tfrac{1}{2}, \qquad V_2 = e^r \cos (e^r), \quad \text{so that} \quad W_2 = \sin (e^r)$$

and

$$Q = \int \tfrac{1}{2} \cos (2e^r)\, dr + O(e^{-r}) \quad \text{as} \quad r \to \infty \qquad \therefore Q = O(e^{-r}).$$

Also $U_2 = O(e^{-r})$ and we obtain 2 solutions with $f_1 \sim 1$ and $f_2 \sim r$ as $r \to \infty$.

In order to apply the present analysis to potential scattering, we need to consider the effect on the behaviour of f of the inclusion of a term $l(l+1)/r^2$ in the potential $V = V_1 + V_2$. The following result, while strengthening the hypotheses of Theorem 1, applies in practise to almost as wide a class of potentials, and provides the necessary extension of the Theorem for behaviour near $r=0$.

Theorem 2. *Let Q and U_2 be given by equations* (2), (2′), (2″) *and suppose that, for some $\varepsilon > 0$,*

$$U_2 = O(r^{\varepsilon}), \qquad Q = O(r^{-1+\varepsilon}) \quad \text{as} \quad r \to 0$$

Then the differential equation

$$-\frac{d^2 f}{dr^2} + \left(V_1 + V_2 + \frac{l(l+1)}{r^2} \right) f = 0 \qquad (0 < r < \infty) \tag{12}$$

has 2 solutions f_1, f_2, satisfying respectively

$$\begin{aligned} f_1 &= r^{-l}(1 + o(1)) \\ f_2 &= r^{l+1}(1 + o(1)) \end{aligned} \tag{13}$$

as $r \to 0$,

Proof. Since

$$\frac{l(l+1)}{r^2} = \frac{(l+1)^2}{r^2} + \frac{d}{dr}\left\{ \frac{(l+1)}{r} \right\}$$

we can follow through the proof of Theorem 1, replacing everywhere V_1 by $V_1 + [(l+1)^2/r^2]$ and W_2 by $W_2 + [(l+1)/r]$. Hence U_2 must be replaced by $\tilde{U}_2 = U_2 + (l+1) \log r$. In $(V_1 - W_2^2)$ there will be an additional term

$-[2(l+1)/r]W_2$. Hence Q must be replaced by

$$\begin{aligned}\tilde{Q} &= \int r^{2(l+1)}\left(V_1 - W_2^2 - \frac{2(l+1)}{r} W_2\right) \exp(2U_2)\, dr \\ &= \int (r^{2(l+1)} \frac{dQ}{dr} - (l+1)r^{2l+1} \frac{d}{dr} (\exp(2U_2) - 1)\, dr \\ &= O(r^{2l+1+\varepsilon})\end{aligned}$$

on integrating by parts and noting $\exp(2U_2) = 1 + O(U_2)$.

Defining w by equation (3) with appropriate modifications where now f satisfies equation (12), equation (4) becomes

$$\frac{dw}{dr} + w^2 r^{-2(l+1)} \exp(-2U_2) = \frac{d\tilde{Q}}{dr} \tag{14}$$

which, for a particular solution w, may be replaced by the integral equation

$$w(r) = \tilde{Q}(r) - \int_0^r w^2(t) t^{-2(l+1)} \exp(-2U_2(t))\, dt \tag{14'}$$

This equation may now be iterated as in the proof of Theorem 1, starting with $w_1(r) = \tilde{Q}(r) = O(r^{2l+1+\varepsilon})$, and it is straightforward to verify that the iteration converges to $w(r) = O(r^{2l+1+\varepsilon})$.

Replacing U_2 by $\tilde{U}_2$ in equation (9) immediately gives the estimate (13) for f_2. The estimate for f_1 follows on writing

$$f_1 = f_2 \int \frac{1}{f_2^2}\, dr.$$

Remark 3. If we assume only $U_2 \to 0$, $rQ \to 0$ as $r \to 0$, we find

$$\left.\begin{aligned} f_1 &= r^{-l} \exp(o(\log r)) \\ f_2 &= r^{l+1} \exp(o(\log r)) \end{aligned}\right\}$$

Equation (14′) may also be iterated if we assume only that $\operatorname{Re}(U_2)$ and $|rQ|$ are bounded, provided l is sufficiently large.

Remark 4. A similar result to that of Theorem 2 applies to behaviour at infinity. In that case we assume $U_2 = O(r^{-\varepsilon})$, $Q = O(r^{-1-\varepsilon})$ as $r \to \infty$. The proof is essentially the same, except that we replace V_1 by $V_1 + (l^2/r^2)$ and W_2 by $W_2 - (l/r)$.

Example 4. Define V_1 and V_2 as in Examples 1 or 2. Then we can find solutions f_1, f_2 of equation (12), satisfying (13) as $r \to 0$. For example 3, we can find solutions of equation (12) satisfying (13) as $r \to \infty$.

Theorem 1 may readily be used to investigate asymptotic behaviour near $r = 0$ of solutions of

$$-\frac{d^2 f}{dr^2} + (V_1 + V_2 - k^2) f = 0 \tag{15}$$

for $k^2>0$, by including in the definition of V_1 the contribution $-k^2$. At $r=\infty$ the situation is different, since $-k^2$ may in no sense be regarded as a small perturbation. However, Theorem 1 may still be used if we suitably redefine V_1 and V_2, and we have

Theorem 3. *Suppose that, as $r\to\infty$, and for some $\varepsilon>0$*

(i) $U_2=O\left(\frac{1}{r^{1+\varepsilon}}\right),\qquad Q=O\left(\frac{1}{r^{1+\varepsilon}}\right)$

(ii) $\int_r^\infty U_2(t)\exp(2ikt)\,dt=O\left(\frac{1}{r^{1+\varepsilon}}\right),$

$\int_r^\infty Q(t)\exp(2ikt)\,dt=O\left(\frac{1}{r^{1+\varepsilon}}\right),$

where k is real and non-zero.

Then equation (15) has a solution f having the asymptotic behaviour, as $r\to\infty$.

$$f(r)=e^{ikr}\left(1+O\left(\frac{1}{r^{1+\varepsilon}}\right)\right) \tag{16}$$

Proof. In the proof of Theorem 1, replace V_1 by V_1-k^2 and W_2 by W_2+ik. Then U_2 becomes $\tilde{U}_2=U_2+ikr$ and Q becomes

$$\begin{aligned}\tilde{Q}&=\int(V_1-W_2^2-2ikW_2)\exp(2U_2)\exp(2ikr)\,dr\\&=\int\left(\frac{dQ}{dr}-ik\frac{d}{dr}(\exp(2U_2)-1)\right)e^{2ikr}\,dr\\&=O\left(\frac{1}{r^{1+\varepsilon}}\right)\end{aligned}$$

on integrating by parts and noting that $\exp(2U_2)=1+2U_2+O(U_2^2)$.

Hence the conditions of Theorem 1 are satisfied. Defining w by equation (3) with appropriate modifications the resulting integral equation for w may be iterated, to give a solution $w=O(1/r^{1+\varepsilon})$. We also have:

$$\begin{aligned}\int_r^\infty we^{-2\tilde{U}_2}\,dt&=\int_r^\infty we^{-2ikt}\,dt+O\left(\frac{1}{r^{1+2\varepsilon}}\right)\\&=(2ik)^{-1}\int_r^\infty\frac{dw}{dt}e^{-2ikt}\,dt+O\left(\frac{1}{r^{1+\varepsilon}}\right)\end{aligned}$$

on integrating by parts,

$$=(2ik)^{-1}\int_r^\infty\frac{d\tilde{Q}}{dt}e^{-2ikt}\,dt+O\left(\frac{1}{r^{1+\varepsilon}}\right)$$

on using equation (4) (suitably modified).

$$\begin{aligned}&=-(2ik)^{-1}(Q(r)-ik(\exp(2U_2(r))-1))+O\left(\frac{1}{r^{1+\varepsilon}}\right)\\&=O\left(\frac{1}{r^{1+\varepsilon}}\right).\end{aligned}$$

On using the formula corresponding to equation (11), the Theorem is proven.

Example 5. Define V_1 and V_2 as in Example 3. Then equation (15) has 2 solutions, satisfying respectively

$$f(r) = e^{\pm ikr}(1+O(e^{-r})) \quad \text{as} \quad r \to \infty$$

It may happen that equation (10) may be iterated even in cases for which Q and U_2 converge to zero much less rapidly that $1/r$. The following Lemma gives sufficient conditions for this to happen.

Lemma. *Suppose that* $Re(U_2)$ *is bounded and that functions* w_1 *and* F *can be found, such that*

$$\text{(i)} \quad w_1(r) = Q(r) + F(r) + \int_r^\infty w_1^2(t) \exp(-2U_2(t))\, dt \tag{17}$$

$$\text{(ii)} \quad w_1(r) = O\left(\frac{1}{r^\beta}\right) \quad \textit{and} \quad \int_r^\infty w_1(t) \exp(-2U_2(t))\, dt = O\left(\frac{1}{r^{\beta'}}\right)$$

as $r \to \infty$, *for some* $\beta > \frac{1}{2}$, $\beta' > 0$, $\beta + \beta' > 1$, *the integral being defined in the improper Riemann sense.*

$$\text{(iii)} \quad \int_r^\infty F(t) w_1(t) \exp(-2U_2(t))\, dt = O\left(\frac{1}{r^{1+\varepsilon}}\right) \quad \textit{and}$$

$$F(r) = O\left(\frac{1}{r^{1+\varepsilon}}\right) \quad \textit{as} \quad r \to \infty, \quad \textit{for some} \quad \varepsilon > 0.$$

Then there is a solution w *of equation* (10), *satisfying*

$$w = w_1 + O\left(\frac{1}{r^{1+\varepsilon}}\right) \quad \textit{as} \quad r \to \infty.$$

Proof. Set $w = w_1 + Z$ in equation (10)
Then

$$Z = -F + \int_r^\infty Z^2(t) \exp(-2U_2(t))\, dt + \int_r^\infty 2Z(t) w_1(t) \exp(-2U_2(t))\, dt \tag{18}$$

In the final integral on the r.h.s., write

$$I(r) = \int_r^\infty w_1(t) \exp(-2U_2(t))\, dt$$

and integrate by parts, substituting for dZ/dt from (18). This gives

$$\begin{aligned} Z = -F + \int_r^\infty Z^2(t) \exp(-2U_2(t))\, dt + 2Z(r)I(r) \\ + \int_r^\infty 2I(t)\left(-\frac{dF}{dt} - Z^2(t)\exp(-2U_2(t))\right. \\ \left. -2Z(t)w_1(t)\exp(-2U_2(t))\right) dt \end{aligned} \tag{19}$$

Equation (19) is completely equivalent to (18), since if θ denotes the difference between the r.h.s. of (18) and Z then (19) implies $\theta+\int_r^\infty 2I(d\theta/dt)\,dt=0$, of which the only solution is $\theta\equiv 0$. On the other hand, equation (19) may be iterated starting with $Z_1\equiv 0$. The inhomogeneous term is $-(F+\int_r^\infty 2I(dF/dt)\,dt=O(1/r^{1+\varepsilon})$ on using assumptions (ii) and (iii). One readily finds that the iterative solution satisfies $Z=O(1/r^{1+\varepsilon})$, from which the conclusion of Lemma follows.

Remark 5. Condition (i) of the Lemma asserts that F *is the difference* between some estimate w_1 for the solution of equation (10) and the estimate w_2 obtained from w_1 by iteration. Conditions (ii) and (iii) imply that w_1 and F are not too large. The Lemma means, roughly, that once $|w_1-w_2|<\text{const.}/r^{1+\varepsilon}$, the iteration will continue to converge.

We now have, as an application of the Lemma,

Theorem 4. *Suppose that, for some real* $k\neq 0$, *and for some* $1\geq\beta>\frac{2}{3}$,

(i) $U_2(r)=O\left(\frac{1}{r^\beta}\right)$

(ii) $\int_r^\infty U_2(t)e^{2ikt}\,dt=O\left(\frac{1}{r^\beta}\right)$

(iii) $\int_r^\infty (U_2(t))^2e^{2ikt}\,dt=O\left(\frac{1}{r^{2\beta}}\right)$

(iv) $\int_r^\infty \frac{dQ(t)}{dt}\,e^{2ikt}\,dt=O\left(\frac{1}{r^{2\beta}}\right)$

Then equation (15) *has a solution f having the asymptotic behaviour*

$$f(r)=e^{ikr}\left(1+O\left(\frac{1}{r^{2\beta-1}}\right)\right) \tag{20}$$

Proof. Apply the lemma with V_1 replaced by V_1-k^2 and W_2 by W_2+ik. Then (c.f. proof of Theorem 3), U_2 becomes $\tilde{U}_2=U_2+ikr$ and Q becomes

$$\begin{aligned}\tilde{Q}&=O\left(\frac{1}{r^{2\beta}}\right)-ik(\exp(2U_2)-1)e^{2ikr}\\&\quad-2k^2\int(\exp(2U_2)-1)e^{2ikr}\,dr\\&=-2ikU_2e^{2ikr}-4k^2\int U_2e^{2ikr}\,dr+O\left(\frac{1}{r^{3\beta-1}}\right)\\&=2ik\int_r^\infty\frac{dU_2(t)}{dt}\,e^{2ikt}\,dt+O\left(\frac{1}{r^{3\beta-1}}\right)\end{aligned}$$

We therefore take

$$w_1(r)=2ik\int_r^\infty\frac{dU_2(t)}{dt}\,e^{2ikt}\,dt \tag{21}$$

in the lemma. From (i) and (ii) above, $w_1(r)=O(1/r^\beta)$. Hence

$$\begin{aligned}\int_r^\infty w_1^2(t)\exp(-2\tilde{U}_2(t))\,dt &= O\left(\frac{1}{r^{3\beta-1}}\right)+\int_r^\infty w_1^2(t)e^{-2ikt}\,dt\\ &= O\left(\frac{1}{r^{3\beta-1}}\right)+(2ik)^{-1}\int_r^\infty 2w_1\frac{dw_1}{dt}e^{-2ikt}\,dt\\ &= O\left(\frac{1}{r^{3\beta-1}}\right)+\int_r^\infty 2\left[-2ikU_2e^{+2ikt}+4k^2\int_t^\infty U_2e^{2iks}\,ds\right]\frac{dU_2}{dt}\,dt\end{aligned}$$

on using equation (21). Note $3\beta-1=1+\varepsilon$, $\varepsilon>0$.

Using

$$2U_2\frac{dU_2}{dt}=\frac{d}{dt}(U_2^2)$$

for the first term, both terms of the integrand may be integrated by parts, giving

$$\int_r^\infty w_1^2(t)\exp(-2\tilde{U}_2(t))\,dt=O\left(\frac{1}{r^{3\beta-1}}\right)$$

Noting that $w_1-\tilde{Q}=O(1/r^{3\beta-1})$ we find $F=O(1/r^{3\beta-1})$ in the equation corresponding to (17).

Using (21), it may readily be verified that

$$\int_r^\infty w_1\exp(-2\tilde{U}_2(t))\,dt=O\left(\frac{1}{r^{2\beta-1}}\right).$$

It remains to estimate the integral in (iii) of the Lemma, for which it suffices to consider $\int_r^\infty F(t)w_1(t)e^{-2ikt}\,dt$. We first calculate

$$\int_r^\infty \tilde{Q}(t)w_1(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^{3\beta-1}}\right)+\int_r^\infty Q_1(t)w_1(t)e^{-2ikt}\,dt$$

where

$$Q_1(r)=-2ik\int W_2\exp(2U_2)e^{2ikr}\,dr=O\left(\frac{1}{r^\beta}\right) \tag{22}$$

Writing

$$Q_1e^{-2ikt}=\frac{d}{dt}\{-(2ik)^{-1}Q_1e^{-2ikt}-\tfrac{1}{2}(\exp(2U_2)-1)\}$$

and integrating by parts, we have

$$\int_r^\infty Q_1(t)w_1(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^{2\beta}}\right)-\int_r^\infty\left\{Q_1\frac{dU_2}{dt}+ik(\exp(2U_2)-1)\frac{dU_2}{dt}e^{2ikt}\right\}dt$$

But the first contribution in the integral is just

$$Q_1(r)U_2(r)-2ik\int_r^\infty U_2W_2\exp(2U_2)e^{2ikt}\,dt=O\left(\frac{1}{r^{3\beta-1}}\right)$$

on writing

$$Ze^{2Z}=\frac{d}{dZ}\left(\frac{Z^2}{2}+O(Z^3)\right)$$

and integrating again by parts. The second contribution may be estimated similarly and we have

$$\int_r^\infty \tilde{Q}(t)w_1(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^{3\beta-1}}\right)$$

Substituting for $F(t)$ from (17) and noting that we have already shown that

$$\int_r^\infty w_1^2(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^{3\beta-1}}\right),$$

in order to verify (iii) of the Lemma we need only consider

$$\int_r^\infty dtw_1(t)e^{-2ikt}\left[\int_t^\infty w_1^2(s)\exp(-2U_2)\,ds\right]$$

But, noting that

$$\int_r^\infty w_1(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^\beta}\right)$$

from (21) and integrating by parts, it is readily checked that the remaining integral is $O(1/r^{3\beta-1})$.

Hence the conditions of the Lemma are satisfied, and a solution f of equation (15) may be constructed by using the formula corresponding to (11).

From (11), to complete the proof of Theorem 4 it remains to show that

$$\int_r^\infty w(t)\exp(-2U_2(t))e^{-2ikt}\,dt=O\left(\frac{1}{r^{2\beta-1}}\right)\tag{23}$$

The l.h.s. of (23) is $O(1/r^{2\beta-1})+\int_r^\infty w(t)e^{-2ikt}\,dt$. Writing, $w=w_1+Z$, we have already noted that

$$\int_r^\infty w_1(t)e^{-2ikt}\,dt=O\left(\frac{1}{r^\beta}\right)$$

Substituting for Z from (18) and integrating by parts we have

$$\int_r^\infty Z(t)e^{-2ikt}\,dt=-\int_r^\infty F(t)e^{-2ikt}\,dt+O\left(\frac{1}{r^\beta}\right),$$

and substituting for $F(t)$ from (7) and integrating by parts we have

$$-\int_r^\infty F(t)e^{-2ikt}\,dt = \int_r^\infty \tilde{Q}(t)e^{-2ikt}\,dt + O\left(\frac{1}{r^{2\beta-1}}\right)$$
$$= \int_r^\infty Q_1(t)e^{-2ikt}\,dt + O\left(\frac{1}{r^{2\beta-1}}\right),$$

where Q_1 is given by (22). Using (22), this is just

$$O\left(\frac{1}{r^{2\beta-1}}\right) - \int_r^\infty W_2 \exp(2U_2)\,dt = O\left(\frac{1}{r^{2\beta-1}}\right) + \tfrac{1}{2}(\exp(2U_2)-1) = O\left(\frac{1}{r^{2\beta-1}}\right)$$

This completess the proof of (23) and of Theorem 4.

3. Applications to Scattering Theory

Let $\hat{H} = -(d^2/dr^2) + V_1 + V_2$ acting on C^∞ functions having compact support in $(0, \infty)$, where V_1 and V_2 are supposed real and

$$V_2 = \frac{dW_2}{dr}, \qquad W_2 = \frac{dU_2}{dr}$$

Let H be a self-adjoint extension of $\hat{H}$, acting in $L^2(0, \infty)$. We rely on a result of Dollard and Friedman [5], itself based on a paper of Green and Lanford [7], which implies the existence and completeness of the wave operators $\Omega_\pm(H, H_0)(H_0 = -d^2/dr^2)$, provided solutions of (15) can be found satisfying (20) with $\beta > \frac{3}{4}$, where the error term is uniform in k in finite closed subinterval of $(0, \infty)$. The subintervals may be chosen to exclude any given finite set of exceptional values of k, at which (20) need not hold.

This result allows us to deduce the existence and completeness of wave operators, provided U_2 and Q satisfy the conditions of Theorem 3, or of Theorem 4 with $\beta > \frac{3}{4}$, where $dQ/dr = (V_1 - W_2^2)\exp(2U_2)$.

Example 6. Let

$$V = V_1 + V_2 = \frac{\sin r}{r^\beta} \quad \text{with} \quad \beta > \tfrac{3}{4}$$

(Take $V_1 = 0$.) Then $\Omega_\pm(H, H_0)$ exist and are complete

Example 7. Let

$$V = \frac{1}{2r} - 2r^{1/2}\sin(r^2).$$

Then $\Omega_\pm(H, H_0)$ exist and are complete.

Example 8. Let

$$V_1 = \frac{g}{r^{2\beta}} \quad \text{and} \quad W_2 = \frac{\cos r^M}{r^\beta}$$

for $0 < \beta < \frac{1}{2}$ and M sufficiently large.

Then $\Omega_\pm(H, H_0)$ exist and are complete for $g = \frac{1}{2}$. On the other hand, if β is close to $\frac{1}{2}$ and $g \neq \frac{1}{2}$, the wave operators do *not* exist. In that case wave operators $\Omega_\pm\{H, H_0 + [(g - \frac{1}{2})/r^{2\beta}]\}$ exist, so that *modified* wave operators $\Omega_\pm(H, H_0)$ exist.

Example 9. Let

$$V = \tfrac{1}{2} + e^r \cos(e^r)$$

Then $\Omega_\pm(H, H_0)$ exist and are complete. (Note that $\Omega_\pm(H_0 + gV, H_0)$ in this case exist *only* if $g = 0$ or 1!)

Equivalently, if $V = e^r \cos(e^r)$, H has absolutely continuous spectrum from $-\frac{1}{2}$ to ∞, and $\Omega_\pm(H, H_0 - \frac{1}{2})$ exist and are complete.

The potential $V(r) = e^r \cos(e^r)$ is also interesting in that it gives rise to a non-trivial example in the algebraic theory of scattering of a potential requiring renormalization of the unperturbed (kinetic) energy in order to define wave operators. This possibility is included in the theory of Jauch, Misra and Gibson [9] and corresponds to Case 3, p. 330, of the approach of Amrein, Martin and Misra [10].

Example 10. Let V be as in Examples 1 and 2. Then $\Omega_\pm(H, H_0)$ exist and are complete (c.f. for example 8); one has the limit circle case at the origin, but $\hat{H}$ has a 'natural' self-adjoint extension.

Acknowledgements

The author wishes to thank Professor K. Chadan for his kind hospitality at the Theoretical Physics Department, Orsay, for many helpful discussions, and for his support and encouragement of this work.

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