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STABILITY OF SOLITONS

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Phase transitions (PT's) are announced dynamically by a critical slowing-down phenomenon which manifests itself in general as a "softening" of a normal-mode frequency. Thus, bulk PT's both in lattice dynamics and in magnetism are associated with an extended soft mode (soft phonon and soft magnon, respectively), announcing the transition to a new ground state of the system. Similarly, the instabilities of topological solitons, refered to in this paper as PT's in domain walls (DW's), are connected in lattice dynamical systems (e.g. in ferroelectrics) with a soft localized mode signalling the transition to a new DW structure. In ferromagnetic systems, however, there occur DW-instabilities which are associated with an essentially different mechanism of critical slowing-down, termed "softening of the velocity change". The role of the soft eigenmode is taken over by the perturbation which carries the static DW into a moving one with infinitesimal velocity, and the role of the soft-mode frequency is taken over by the velocity change induced by the perturbation. The present contribution aims to survey by means of two typical examples the basic aspects of soliton (DW) instabilities in quasi 1D crystals and in Heisenberg ferromagnets, and to discuss their peculiar dynamics comparatively.

1. Domain Walls in Lattice Dynamical Systems /1,2/

1.1 Model and Domain Wall Solution

We consider an unbounded 3D crystal with two-component order parameter $\vec{Q} = (Q_1, Q_2)$, quartic anharmonicity and anisotropy in the on-site harmonic terms which may be realized by coupling to an external uniaxial strain. In classical continuum approximation the model is described by the Hamiltonian

$$H = \frac{1}{2} \int [\dot{Q}_1^{\,2}+\dot{Q}_2^{\,2}+(\nabla Q_1)^2+(\nabla Q_2)^2-g_1 Q_1^{\,2}-g_2 Q_2^{\,2}+ + \frac{1}{2}u(Q_1^{\,2}+Q_2^{\,2})^2]\, dxdydz \qquad (1.1)$$

giving rise to the equations of motion

$$\ddot{Q}_{1,2} - \nabla^2 Q_{1,2} = g_{1,2}\, Q_{1,2} - u(Q_1^2 + Q_2^2) Q_{1,2} \quad , \tag{1.2}$$

where $g_1 > g_2 > 0$, $u > 0$ and the dot denotes $\partial/\partial t$.

The system has a doubly degenerate ground state (GS)

$$Q_1 = \pm Q_{10} = \pm (g_1/u)^{1/2}, \; Q_2 = 0 \quad . \tag{1.3}$$

The linear excitations of these uniform field configurations consist of two phonon branches with frequencies

$$\omega_{1q}^2 = 2g_1 + q^2, \quad \omega_{2q}^2 = g_1 - g_2 + q^2 \tag{1.4}$$

showing the linear stability of the GS's for $g_2 < g_1$. At $g_1 = g_2$ there occurs a first-order PT to a state with Q_1 and Q_2 interchanged, associated with a soft phonon. The occurrence of a soft mode at a first- order PT is connected with the fact that the point $g_1 = g_2$ of the above model is a point of infinite degeneracy.

We are interested in this Section in a certain class of quasi 1D configurations of the displacement field $\vec{Q}$, namely in plane DW's $Q_{1,2}(x,y,z) = Q_{1,2}(z-vt)$, which are static ($v=0$) or uniformly moving ($v=\text{const}\neq 0$) permanent-shape field structures connecting the two GS's ("domains"). The dynamic equations (1.2) are invariant under a formal Lorentz transformation $(z,t) \to (\zeta,\tau)$, where $\zeta = \gamma(z-vt)$, $\tau = \gamma(t-vz)$, $\gamma = (1-v^2)^{-1/2}$ and $v^2 < 1$. Therefore it is sufficient to consider the static DW's only, since by the substitution $(z,t) \to (\zeta,\tau)$ all results may be transcribed directly for moving DW's.

The static wall structures are solutions of the system of coupled equations

$$Q''_{1,2} = -g_{1,2} Q_{1,2} + u(Q_1^2 + Q_2^2) Q_{1,2} \quad , \tag{1.5}$$

satisfying the boundary conditions

$$\begin{aligned} Q_1(\pm\infty) &= \pm Q_{10}\,, \quad Q_2(\pm\infty) = 0 \\ Q_1'(\pm\infty) &= Q_2'(\pm\infty) = 0\,, \end{aligned} \tag{1.6}$$

where the prime denotes d/dz.

Equations (1.5) may be interpreted as the equations of motion of a point particle in the plane (Q_1,Q_2) under the influence of a potential

$$V(Q_1,Q_2) = \frac{1}{2}(g_1Q_1{}^2 + g_2Q_2{}^2) - \frac{1}{4}u(Q_1{}^2 + Q_2{}^2)^2. \tag{1.7}$$

This analogy permits to write down immediately a first algebraic integral of the system (1.5), namely the "energy" integral

$$I_1 = \frac{1}{2}(Q_1{}'^2 + Q_2{}'^2) + V(Q_1,Q_2) . \tag{1.8}$$

By using some special methods /3,4/ one finds surprisingly also a second algebraic integral /2/:

$$I_2 = (Q_1Q_2' - Q_2Q_1')^2 + u^{-1}(g_1-g_2)\,[Q_1{}'^2 - Q_2{}'^2 + g_1Q_1{}^2 - g_2Q_2{}^2 - \frac{1}{2}u(Q_1{}^4 - Q_2{}^4)] . \tag{1.9}$$

The existence of this integral offers the possibility to enumerate by qualitative analysis all the solutions of the system (1.5). This analysis shows /2/ that there are only two DW solutions satisfying the boundary conditions (1.6), namely:

(1) A one-component DW

$$Q_1{}^I(z) = Q_{10}\tanh[(\tfrac{1}{2}g_1)^{1/2}z] , \quad Q_2{}^I = 0 , \tag{1.10}$$

with energy per unit xy-area $E^I = (\gamma/3u)(2g_1)^{3/2}$, existing in the whole interval $0 < g_2 < g_1$, and

(2) A two-component DW

$$Q_1{}^{II}(z) = \pm Q_{10}\tanh[(g_1-g_2)^{1/2}z]$$
$$Q_2{}^{II}(z) = Q_{20}\,\mathrm{sech}[(g_1-g_2)^{1/2}z] , \tag{1.11}$$

with energy per unit xy-area $E^{II} = (2\gamma/3u)(g_1+2g_2)(g_1-g_2)^{1/2}$, where $Q_{20} = [(2g_2-g_1)/u]^{1/2}$, existing only in the interval $g_1/2 < g_2 < g_1$.

We notice that the system described by the Hamiltonian (1.1) has attracted considerable attention in recent years. The 1D-counterpart of (1.1) was also investigated extensively in various fields of the soliton theory /5-7/.

1.2 Energetic and Dynamic Stability

We discuss in this Section the linear stability of the DW's (1.10) and (1.11). To this end we first linearize Eqs. (1.2) according

If the LHS of (1.17) vanishes for both $z \to -\infty$ and $z \to +\infty$, the corresponding $\tilde{\chi}$ is termed a "localized mode" of the DW. Otherwise, $\tilde{\chi}$ is referred to as an "extended mode" of the DW.

We now immediately see that the DW is linearly stable if no negative eigenvalue ω^2 of Eqs. (1.15) exists (for $\omega^2 < 0$ the perturbation grows exponentially for $t \to \infty$). The case $\omega^2 = 0$ is of special importance because each solitary wave on a translationally invariant system has an $\omega^2 = 0$ mode, the well-known Goldstone mode (GM), which restores the broken translation symmetry. In the present context the GM of our DW is

$$\{\omega^2, \tilde{\chi}_{1,2}\}_{GM} = \{0, Q'_{1,2}(z)\} \quad . \tag{1.18}$$

According to the above considerations the GM is defectively degenerate, and one finds the accompanying "algebraic mode" (AM):

$$\{\omega^2, \tilde{\chi}_{1,2}\}_{AM} = \{0, t\, Q'_{1,2}(z)\} \quad . \tag{1.19}$$

Let us now examine the physical significance of the AM (1.19). If $Q_{1,2}(x)$ is a static DW solution of Eq. (1.2), then, as mentioned above, $Q_{1,2}(\zeta)$, $\zeta = \gamma(z-vt)$, is also a solution, describing a DW moving with velocity v. For infinitesimal v and finite times, $Q_{1,2}(\zeta) = Q_{1,2}(z) - vt\, Q'_{1,2}(z)$. Comparing this expression with Eq. (1.19), we see immediately that the AM (1.19) represents a perturbation which carries the static DW into a moving one with infinitesimal velocity. Similarly, the AM $\{\omega^2, \tilde{\chi}_{1,2}\}_{AM} = \{0, \tau\, Q'_{1,2}(\zeta)\}$, represents a perturbation which generates from the given DW moving with velocity v another DW moving with infinitesimally changed velocity $v + \delta v$, since for $v \to v + \delta v$, $Q_{1,2}(\zeta) \to Q_{1,2}(\zeta) - \gamma^2 \delta v \tau Q'_{1,2}(\zeta)$.

After these general considerations let us examine the stability of the one-component DW (1.10) in detail. In this case the system (1.15) decouples into two Schrödinger-type equations:

$$\begin{aligned} &\left[\frac{d^2}{ds^2} + \frac{2}{g_1}(\omega^2 - 2g_1) + 6\,\mathrm{sech}^2 s \right] \chi_1 = 0 \\ &\left[\frac{d^2}{ds^2} + \frac{2}{g_1}(\omega^2 + g_2 - g_1) + 2\,\mathrm{sech}^2 s \right] \chi_2 = 0 \quad , \end{aligned} \tag{1.20}$$

where $s = (g_1/2)^{1/2} z$. These equations can be solved explicitly. One finds three localized modes: the GM

$$\omega^2 = 0; \quad \chi_1 = \mathrm{sech}\, s \;, \quad \chi_2 = 0 \quad , \tag{1.21}$$

a thickness vibration mode

$$\omega^2 = \frac{3}{2} g_1 \;; \quad \chi_1 = \mathrm{sech}\, s \tanh s, \quad \chi_2 = 0 \quad , \tag{1.22}$$

and an internal oscillation mode of the DW

$$\omega^2 = \frac{1}{2} g_1 - g_2 \;; \quad \chi_1 = 0 \;, \quad \chi_2 = \mathrm{sech}\, s \quad . \tag{1.23}$$

(The extended modes of the DW (1.10) can also be written down explicitly. They are, however, of no relevance for the stability, to the ansatz

$$Q_{1,2} = Q_{1,2}(z) + \tilde{\chi}_{1,2}(z,t) \qquad (1.12)$$

and obtain

$$\ddot{\tilde{\chi}}_{1,2} - \nabla^2\tilde{\chi}_{1,2} = [g_{1,2} - u(3Q_{1,2}^2 + Q_{2,1}^2)]\tilde{\chi}_{1,2} - 2uQ_1Q_2\tilde{\chi}_{2,1} \qquad (1.13)$$

where $\tilde{\chi}_{1,2}$ are small deviations from the components $Q_{1,2}(z)$ of the DW under consideration (the superscripts I and II were omit-) ted). We are interested in solutions of the form

$$\tilde{\chi}_{1,2}(z,t) = \chi_{1,2}(z)\, T(t) \ . \qquad (1.14)$$

Equations (1.13) and (1.14) lead to the following system of ordinary differential equations

$$\left[\frac{d^2}{dz^2} + \omega^2 + g_{1,2} - u(3Q_{1,2}^2 + Q_{2,1}^2)\right]\chi_{1,2} = 2uQ_1Q_2\ \chi_{2,1} \qquad (1.15)$$

$$\ddot{T} + \omega^2 T = 0 \qquad (1.16)$$

where $-\omega^2$ denotes the constant of separation. These equations are of basic importance since (1) they determine the possible linear excitations of the DW, and (2) the linear excitations yield complete information about the linear stability.

Equation (1.16) is very simple. For $\omega \neq 0$ its general solution is a linear combination of the exponentials $T_{1,2} = \exp(-i\omega_{1,2}t)$ with $\omega_{1,2} = \pm\omega$, whereas for $\omega = 0$ one has T_1 = const. and $T_2 = t$. We see, therefore, that in the function space $\{\tilde{\chi}\}$ of separable perturbations, each eigenvalue ω^2 of Eqs. (1.15) is doubly degenerate with respect to the time-dependent part of the perturbation, i.e. each eigensolution $\{\omega^2; \chi_{1,2}\}$ of (1.15) is associated with two linearly independent time-evolution functions T. For nonvanishing eigenvalues ω^2, the corresponding T's are purely exponential functions of t, but for $\omega^2 = 0$ the two characteristic frequencies become coincident, $\omega_1 = \omega_2 = 0$, and both the purely exponential T's reduce to a constant. In analogy to comparable situations occurring in algebraic eigenvalue problems, this type of degeneracy may be termed "defective" /8/. Obviously, it is only apparently "defective", since the role of the missing exponential T is taken over in fact by a solution with an "algebraic" t dependence, T = t .

The separation constant ω^2 was referred to above as eigenvalue parameter of Eqs. (1.15), but the corresponding eigenvalue problem was not fully defined. This problem is specified by the physical requirement that the configurational parts $\chi_{1,2}(z)$ of the normal modes (1.14) be bounded functions of x, i.e.

$$\lim_{|z|\to\infty} \chi_{1,2}(z) < \infty \ . \qquad (1.17)$$

since these solutions correspond to the continuous spectrum of two Eqs.(1.20) extending to the regions $\omega^2 > 2g_1$ and $\omega^2 > g_1-g_2 > 0$, respectively). Therefore, the one-component DW (1.10) is stable for $g_2 < g_1/2$. At $g_2 = g_1/2$ there occurs an instability against a soft localized oscillation (1.23) and a bifurcation /6,7/ of the two-component DW (1.11), signalizing a second-order PT within the DW. For $g_2 > g_1/2$ the two-component DW must represent the stable DW up to the bulk stability limit $g_2 = g_1$, because no other solution connecting the two GS's exists /2/. It does not appear simple, however, to confirm this stability by solving analytically the corresponding eigenvalue problem (1.15) & (1.17) for the linear excitations of the DW (1.11). Indeed, Eqs. (1.15) do not decouple in this case, and except for the zero-frequency mode (1.18) no other closed-form solution seems to exist.

As an important conclusion we emphasize that the instability at $g_2 = g_1/2$ represents a second-order PT in a DW with order parameter Q_{20}, which is associated with a soft localized mode, and which anticipates the bulk PT occurring at $g_2 = g_1$ (Fig. 1). Such behaviour (DW instability as precursor of a bulk instability) appears to be a general phenomenon in lattice dynamics as discussed by Lajzerowicz and Niez /9/.

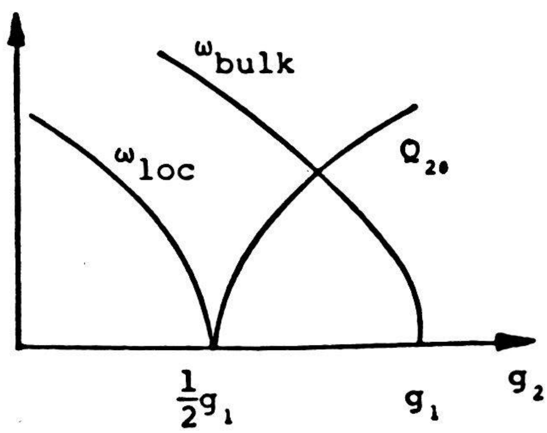

Fig. 1 Phase transition in a lattice-dynamical DW. The bulk PT at $g_2 = g_1$ is associated with a soft phonon of frequency $\omega_{q=0}^{bulk} = (g_1-g_2)^{1/2}$; the domain-wall PT at $g_2 = \frac{1}{2}g_1$ is associated with a soft localized mode of frequency $\omega_{loc} = (\frac{1}{2}g_1-g_2)^{1/2}$

We close this Section with three remarks.

(i) We see that at the bifurcation point $g_2 = g_1/2$ of the two-component DW, the (soft) eigenvalue ω^2 corresponding to the internal oscillation mode (1.23) of the one-component DW becomes defectively degenerate in the function space $\{\tilde{\chi}(z,t)\}$ (and regularly degenerate with the GM (1.21) in the function space $\{\chi(z)\}$). Hence, at $g_2 = g_1/2$ the soft mode (1.23), $\tilde{\chi}^S_{1,2} = \{0, \text{sech } s\}$ is accompanied by an algebraic mode $\tilde{\chi}^A_{1,2} = \{0, t \text{ sech } s\}$. What is the significance of the defective degeneracy of the soft mode? At the bifurcation point, the linear part of the restoring force of the internal oscillation mode vanishes. Therefore, an initial perturbation $\tilde{\chi}^S_{1,2}(0) = a\,\{0, \text{sech } s\}$, given an initial velocity $\dot{\tilde{\chi}}^S_{1,2}(0) = v\,\{0, \text{sech } s\}$, will evolve in time according to a linear law $\tilde{\chi}^S_{1,2}(t) = (a + vt)\ \{0, \text{sech } s\}$ corresponding to a linear superposition of the soft mode and the associated AM, until the motion is reversed as a result of anharmonic contributions not considered in linear stability analysis. In other words, the AM describes in this case the initial stage of the nonlinear oscillation in the purely anharmonic potential at the bifurcation point / 8/.

(ii) Our second remark concerns the space-dependence of the perturbations of the DW's. By linearizing Eqs. (1.2) according to the ansatz (1.12) we have considered "uniform" perturbations, i.e. perturbations depending only on the coordinate z normal to the plane of the DW's. Obviously, "nonuniform" perturbations, i.e. perturbations depending also on the coordinates x and y in the plane of the wall, $\tilde{\chi} = \tilde{\chi}(x,y,z,t)$, may not be excluded a priori. Their (undramatic) effect, however, may be evaluated easily. Indeed, the linearized dynamic equation (1.13) is invariant under x- and y-translations which implies that the (x,y)-dependence of the non-uniform perturbations may be taken of the form $\exp[i(q_1x+q_2y)]$ where q_1 and q_2 are the components of a wave vector $\vec{q}$ lying in the (x,y)-plane. Thus, the only effect of nonuniform perturbations on the eigenvalue equations (1.15) is the replacement of d^2/dz^2 by $(d^2/dz^2)-q^2$, where $q^2 = q_1{}^2 + q_2{}^2$. This is equivalent to a shift of the eigenvalues ω^2 by a positive quantity q^2. Therefore, the nonuniform perturbations have no influence on the stability limit of the one-component DW.

(iii) We would like to comment briefly also on the title of the present Subsection 1.2. The linear stability analysis carried out above is a "dynamic" stability analysis, since it yields complete information about the time evolution of the perturbations. On the other hand, the structure equations (1.2) of static ($\dot{Q} \equiv 0$) field configurations are just the Euler-Lagrange equations corresponding to the static part H_o of the energy functional (1.1), i.e. for static structures the functional H_o is stationary. Therefore, the linear stability of static DW's could have been analyzed also "energetically". The energetic stability analysis only requires to expand $H_o\,[Q_{1,2}]$ to second order in the static deviations $\chi_{1,2}(z)$ from the DW $Q^W_{1,2}$ (where W stands for I or II), $H_o\,[Q_{1,2}] = H_o\,[Q^W_{1,2}] + \Delta H_o\,[\chi_{1,2}]$, and to determine the eigenvalues ε of ΔH_o as function of g_1 and g_2. The DW under consideration is energetically stable if no eigenvalue ε is negative. The limit of stability occurs at critical values of g_1 and g_2 where an eigenvalue ε vanishes. This "diagonalization" procedure of the second variation ΔH_o of H_o leads to the eigenvalue equations

$$\frac{\delta(\Delta H_o)}{\delta \ \chi_{1,2}} = \varepsilon \ \chi_{1,2} \qquad (1.24)$$

where the LHS represents the functional derivative of ΔH_o with respect to $\chi_{1,2}$. An energetic stability analysis is in general less powerful than a dynamic one, since it yields no information about the time-evolution of the perturbations (in particular on the dynamics of possible instabilities), but the corresponding eigenvalue equations may be substantially simpler than the linearized dynamic eigenvalue equations. The spin-dynamic system discussed in Sect. 2 yields an explicit example in this sense. In the present lattice-dynamical context, however, the "energetic" eigenvalue equations (1.24) coincide with the dynamical ones (1.15), with $\varepsilon \equiv \omega^2$.

2. Domain Walls in Magnetic Systems /10-12/

2.1. Model and Domain Wall Solutions

We consider in this Section a 1D biaxial Heisenberg ferromagnet described by Hamiltonian

$$H = \Sigma_i \{-J\vec{S}_i \cdot \vec{S}_{i+1} - A(S_i^z)^2 - C(S_i^x)^2\} \quad , \qquad (2.1)$$

where J and C are positive constants and A is a parameter satisfying $A < C$ such that the S_x axis is the easy axis. The hard axis is along S_y for $A > 0$ and along S_z for $A > 0$. In the classical continuum approximation one obtains the energy functional

$$E[s] = \frac{1}{2} \int_{-\infty}^{+\infty} (s'^2 - as_z^2 - s_x^2 + 1)dz \qquad (2.2)$$

and the equation of motion

$$\dot{\vec{s}} = \vec{s} \times (\vec{s}'' + s_x \vec{e}_x + as_z \vec{e}_z) \quad , \qquad (2.3)$$

where $\vec{s}(z,t) = \vec{S}(z,t)/S$, $\vec{s}^2 = 1$, $a = A/C < 1$, the prime indicates $\partial/\partial z$, the dot denotes $\partial/\partial t$, and the units of z,t and E are $[z] = (J/2C)^{1/2}d$ (d is the lattice constant), $[t] = (2SC)^{-1}$ and $[E] = S^2(2JC)^{1/2}$, respectively.

For $a < 1$, the ground states of the system are the domains $\vec{s} = (\pm 1,0,0)$ while for $a > 1$ this role would be taken over by uniform solutions $\vec{s} = (0,0,\pm 1)$. The dispersion relation for the small-amplitude oscillations (spin waves or magnons) in the domains $(\pm 1,0,0)$,

$$\omega_q^2 = (1-a+q^2)\ (1+q^2) \quad , \qquad (2.4)$$

shows that these configurations are indeed linearly stable for $a < 1$, and that the first-order PT ("bulk instability") to the new ground states $\vec{s} = (0,0,\pm 1)$ occurring at a = 1 is associated with a soft magnon.

We now consider solitary waves $\vec{s}(z,t) = \vec{s}(Z)$, $Z = z - vt$, connecting the two ground states $(\pm 1,0,0)$ of the system. These are 180^o DW's satisfying the boundary conditions $\vec{s}(\pm\infty) = (\pm 1,0,0)$. There are two static (v=0) DW's (plus their trivially symmetric counterparts) satisfying these boundary conditions. One of them is the Bloch wall

$$s_x = \tanh z, \quad s_y = \text{sech}\, z, \quad s_z = 0 \qquad (2.5)$$

with energy $E_B = 2$, and the other is the Néel wall

$$s_x = \tanh[(1-a)^{1/2} z], \quad s_y = 0, \quad s_z = \text{sech}[(1-a)^{1/2} z] \qquad (2.6)$$

with energy $E_N = 2(1-a)^{1/2}$. The widths of these DW's are given (in physical units) by the expressions $\delta_B = (J/2C)^{1/2} d$ and $\delta_N = (J/2C)^{1/2}(1-a)^{-1/2} d$, respectively. Therefore, the Bloch and Néel walls are long-wavelength profiles of the spin field satisfying the validity requirement of the continuum approximation ($\delta \gg d$) only if $J \gg C$ or $J \gg C-A$, respectively.

For $a \neq 0$, any "intermediate" wall between the Bloch and Néel wall is a moving DW. These moving DW's may also be given in simple analytic form /12/. It turns out /12/ that for fixed $a \neq 0$ the possible wall velocities are restricted to the interval $|v| < |1-(1-a)^{1/2}|$. For a = 0, on the other hand, there exists a family of intermediate static DW's which is described by

$$\vec{s} = (\tanh z,\; k_2\, \text{sech}\, z,\; k_3\, \text{sech}\, z), \qquad (2.7)$$

where the constants $k_{2,3}$ are subjected to the condition $k_2^2 + k_3^2 = 1$.

3.2 Energetic and Dynamic Stability

Our aim in this Section is to examine the linear stability of the Bloch and Néel walls in detail. We start with the Bloch wall, by representing the spin field in spherical coordinates $\vec{s} = (\cos\Theta \cos\phi, \cos\Theta \sin\phi, \sin\Theta)$, where $|\Theta| \leq \pi/2$ and $0 \leq \phi \leq 2\pi$. Thus, the wall (2.5) becomes $\Theta_B = 0$, $\sin\phi_B = \text{sech}\, z$. Linearizing now the dynamic equations (2.3) around (Θ_B, ϕ_B) according to the ansatz

$$\begin{aligned} \Theta(z,t) &= \Theta_B(z) + \alpha(z) \exp(-i\omega t) \\ \phi(z,t) &= \phi_B(z) + \beta(z) \exp(-i\omega t), \end{aligned} \qquad (2.8)$$

we obtain for infinitesimal deviations (α,β) from the wall configuration the following system of coupled eigenvalue equations:

$$(L+a)\alpha = i\omega\beta, \quad L\beta = -i\omega\alpha, \qquad (2.9)$$

where $L = -1 + 2\,\mathrm{sech}^2 z + d^2/dz^2$. Before we examine the physical content of these equations, i.e. the dynamical aspects of the stability, we discuss the stability problem from the energetic point of view. This is possible, since for static spin configurations the energy functional (2.2) is stationary, i.e. for $\dot{\vec{s}} \equiv 0$ the structure equations (2.3) are obtained as Euler-Lagrange equations corresponding to the variational functional (2.2). Therefore, as described at the end of Sect. 1.2, we expand $E[\Theta,\phi]$ to second order in the static deviations $(\alpha(z), \beta(z))$ from (Θ_B, ϕ_B) and determine the eigenvalues ε of $\Delta E[\alpha,\beta]$ as function of a. This leads us to the eigenvalue equations

$$(L+a+\varepsilon)\alpha = 0 \;, \quad (L+\varepsilon)\beta = 0 \;. \tag{2.10}$$

The Bloch wall is stable as long as all $\varepsilon_n(a)>0$. The limit of stability occurs at a critical value a_c found as the root of $\varepsilon_0(a_c) = 0$, where ε_0 is the lowest eigenvalue. The corresponding eigenmodes (α_n, β_n) are for $\varepsilon_n = 0$ solutions of the linearized equations of motion (2.9) with $\omega = 0$.
We see now explicitly that in the case of the magnetic DW's (in contrast to the lattice dynamical ones) the energetic stability analysis is operationally much simpler than a dynamic analysis, since the energetic eigenvalue equations (2.10) are decoupled, unlike their dynamical counterparts (2.9)

The system (2.10) admits two localized solutions: (1) $\varepsilon=0$, $\alpha=0$, $\beta=\mathrm{sech}\,z$, which is the Goldstone mode reflecting the marginal stability of the Bloch wall against translations, and (2) $\varepsilon=-a$, $\alpha=\mathrm{sech}\,z$, $\beta=0$. Therefore, the Bloch wall is energetically stable against Θ perturbations for any $a<0$, but at $a = a_c = 0$ it becomes unstable with respect to a perturbation connecting it to the family of static 180° DW's given by (2.7). A similar analysis of the Néel wall (2.6) shows that this static spin configuration is energetically stable for $0<a<1$ and becomes unstable also at $a = 0$ with respect to a perturbation connecting it to the same family of static 180° DW's.

Let us now return to Eqs. (2.9) with the aim to elucidate the normal-mode dynamics of the DW instability at a=0. The dispersion relation (2.4) shows that the domains connected by the walls under consideration are stable above the bulk-PT point a=-1. Having in mind the lattice-dynamical example discussed in Sect. 1 one might expect (compare Figs. 1 and 2) that the instability of the Bloch wall at a=0 is associated for $a\neq 0$ with a localized dynamic spin mode $(\alpha(z,a),\beta(z,a))$ corresponding to a frequency $\omega(a)$ whose square is positive for $a<0$ and negative for $a>0$, such that for $a\to 0$, $\omega(a)$ goes to zero and $(\alpha(z,a),\beta(z,a))$ approaches the static instability mode $(\mathrm{sech}\,z, 0)$. This is, however, not the case. In the present model, the non-existence of such a soft localized dynamic spin mode can explicitly be proven, since the system (2.9) may be reduced to trivially coupled equations *). Indeed, by the ansatz $\alpha=(1-k)W_o-(1+k)W_1$, $\beta=(1+k)W_o+(1-k)W_1$, where $k=(a+2i\omega)^{1/2}(a-2i\omega)^{-1/2}$, the system (2.9) goes over into

$$(2L+a+\Omega)W_o=0, \qquad (2L+a-\Omega)W_1=-4i\omega W_o \quad , \qquad (2.11)$$

where $\Omega=(a^2+4\omega^2)^{1/2}$. It is easy to show that this system admits localized solutions only for $\Omega=-a$, i.e. for $\omega=0$, /12/. Thus, the only localized solutions of Eqs. (2.9) are identical to the static eigenmodes, i.e. the GM $(\alpha,\beta)=(0, \text{sech } z)$ for any a, and at a=0 in addition the static instability mode $(\alpha,\beta)=(\text{sech } z, 0)$. Therefore, the energetic instability of the Bloch wall at a=0 is not associated with a localized soft dynamic spin mode. The normal-mode analysis of the Néel wall leads to a similar result.

How can the absence of a soft mode be reconciled with the existence of an instability? As we have shown, the family of intermediate static DW's (2.7) bifurcate from the Bloch wall at the critical field a=0. On the other hand, any infinitesimal Θ deviation from the Bloch wall leads for any value of a to a DW moving with infinitesimal velocity /12/, except for a=0 where it connects to a static DW with infinitesimal k_3. This result suggests that in the present problem the role of the soft eigenmode is taken over by the Θ perturbation which carries the static DW into a moving one, and the role of the soft-mode frequency is taken over by the resulting velocity change. The "softening" of the velocity change δv at the critical value $a_c=0$ may be seen explicitly by calculating δv produced by a perturbation with maximum deviation $\delta\Theta_o$ as function of a. The simple and convincing result is $\delta v/\delta\Theta_o = a$ (Fig. 2).

This picture is confirmed by a more careful stability analysis. Instead of making the exponential ansatz (2.8), we linearize the equation of motion (2.3) for deviations $\tilde{\alpha}(z,t)$ and $\tilde{\beta}(z,t)$ from the Bloch wall, with arbitrary time dependence. In this way, in addition to the Goldstone mode $(\tilde{\alpha}_1,\tilde{\beta}_1)=(0, \text{sech } z)$ we find a nonexponential solution $\tilde{\alpha}_2=(v/a)\,\text{sech } z$, $\tilde{\beta}_2= -vt\,\text{sech } z$, the "algebraic mode" of the DW. For $a\neq 0$, $v\neq 0$, this mode describes a DW moving with infinitesimal velocity v. For $a\to 0$, $v\to 0$, v/a finite, it reduces to the static instability mode $(\alpha,\beta)=(\text{sech } x, 0)$.

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*) We want to emphasize that the possibility of decoupling of Eqs. (2.9) is an accidental property of the model discussed. In other cases of physical interest, e.g. for a planar ferromagnet in an external field /13,14/, the dynamic eigenvalue equations can not be decoupled, and thus the energetic stability analysis represents the only source of analytic information about the soliton stability.

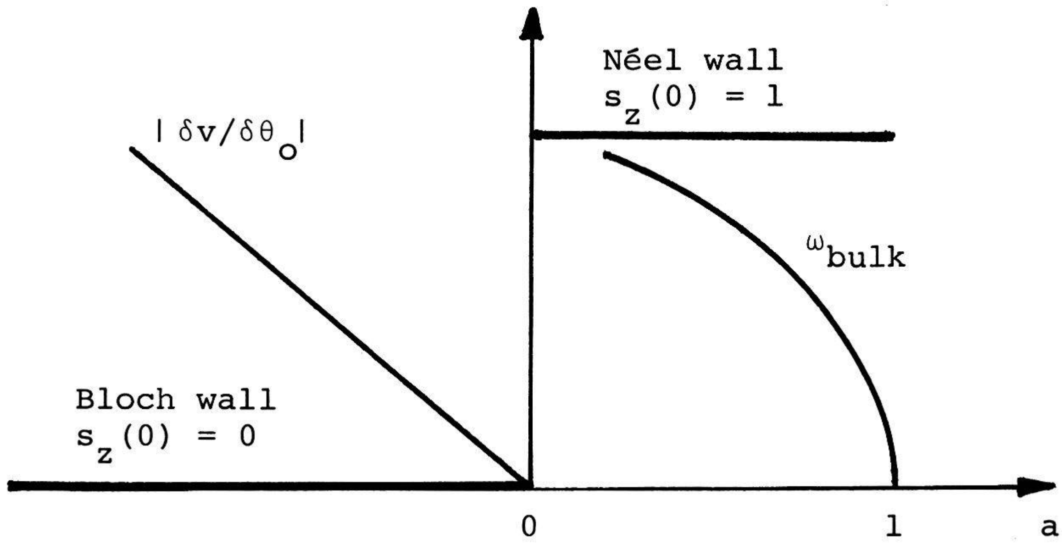

Fig.2. Phase transition in a magnetic DW. The bulk PT at a = 1 is associated with a soft magnon of frequency $\omega_{q=0}^{bulk} = (1-a)^{1/2}$; the domain-wall PT at a = 0 is associated with a soft velocity change $\delta v/\delta\theta_o = a$

We close this Section with a brief discussion of the nonuniform modes of plane DW's in 3D magnets. Similarly to the case of the lattice-dynamical DW's the (x,y)-dependence of the nonuniform perturbations of the Bloch wall may be taken of the form $\exp[i(q_1x + q_2y)]$, where q_1 and q_2 are the components of a wave vector $\vec{q}$ lying in the (x,y)-plane. Thus the only effect of the nonuniform perturbations on the eigenvalue equations (2.9) and (2.11) is the replacement of L by*) $L-q^2$, where $q^2=q_1{}^2+q_2{}^2$. As a consequence, the system (2.11) admits localized solutions only for $\Omega = 2q^2-a$, i.e. for $\omega^2 = q^2(q^2-a)$. We see, therefore, that in a real 3D ferromagnet the Goldstone mode is connected to a branch of corrugating modes which for small q are unstable for $a > 0$ and stable for $a < 0$ (Fig. 3).

*) This is now only valid if the contribution of the magnetostatic interaction to the total energy density may be replaced by a term proportional to the hard-axis part as_3^2 of the anisotropy energy. This is exact for the unperturbed DW and for uniform perturbations, and is a good approximation also for nonuniform perturbations with wave lengths large compared to the DW thickness /15/.

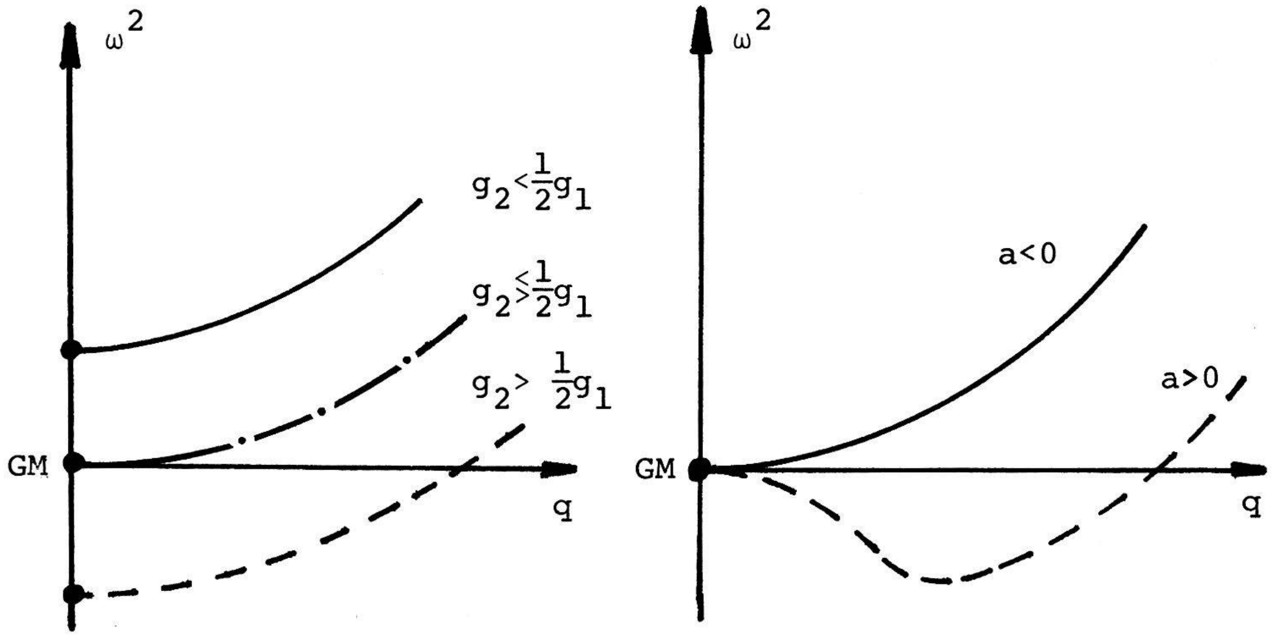

Fig.3 Frequencies of the nonuniform wall modes as functions of the wave vector in the wall plane for a lattice-dynamical DW (a) and a magnetic DW (b)

Fig.3 demonstrates the different dynamical behaviour at a DW phase transition in a lattice-dynamical DW and a magnetic DW. In the lattice-dynamical case there exist two low-lying branches, the Goldstone branch $\omega^2 = q^2$ and an internal - oscillation branch $\omega^2 = q^2 + \frac{1}{2}g_1 - g_2$, and the DW phase transition is associated with the softening of the latter. In the magnetic case there exists only a single branch, the Goldstone branch $\omega^2 = q^2(q^2 - a)$,which becomes destabilized at $q \neq 0$ for $a > 0$. At $q=0$, the DW phase transition shows only in the algebraic mode, giving rise to the softening of the velocity change.

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