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Classical local fields for hadrons in chromodynamics

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Abstract. Gauge invariant local fields which are polynomial functions of classical gauge and quark fields are studied as candidate classical fields for hadrons. Using invariant-theoretic methods, a complete set of fields for 'elementary' hadrons and a complete set of algebraic constraints among them are identified. The elementary hadrons are either pure gluon composites (glueballs) or quark composites (mesons, baryons and antibaryons), without any constraint relating the two types. The constraint-free configuration space is, in general, that of a nonlinear field theory with values in an algebraic variety. The constraints include a complete set of bosonisation identities expressing every baryon current as a polynomial in mesons, and the mesonic fields take values in a manifold having the same topology as a complex Grassmannian. Preliminary results for anticommuting quarks, including modified bosonisation formulae, are also given.

1. Introduction

Chromodynamics, the theory of a nonabelian gauge field with structure group $SU(3)$ in interaction with a set of fermion fields, is generally accepted to underlie the strong interactions of hadrons. The geometric richness that goes with a nonabelian structure group makes such a theory qualitatively different from an abelian gauge theory like electrodynamics. Already at the classical level, the true (constraint-free) configuration or phase space of nonabelian gauge theory is topologically more complex than that of the Maxwell field and has only recently begun to be understood [1–3, see also 4]. The constraints implied by gauge invariance are too complicated to be solved [5, 1, 2], for example by fixing a gauge. Consequently, little progress has so far been made in understanding the conceptual features of the corresponding quantum theory.

On the other hand, the general expectation is that the only asymptotic states of chromodynamics are hadrons, the colourless composites of gluons and quarks, and it is reasonable to ask whether a local field description of hadrons can be attempted without first quantising chromodynamics. It was established long ago that, in a conventional quantum field theory, local interpolating fields for bound states can be constructed as suitably regularised polynomials in the elementary fields and their space–time derivatives of bounded order [6–8, see also 9]. Motivated by this, we wish to study in this paper gauge-invariant local polynomial functions of *classical* gluon and quark fields as candidate *classical* fields for hadrons. The space of such fields is 'smaller' than the orbit space studied (in a pure gauge theory) in [1–4], but may be sufficient for the description of hadrons by local fields (see later).

The natural mathematical framework for this study is the theory of polynomial invariants [10, 11] in a set of variables (here, quark and gluon fields) transforming in a given way under the action of a group (the gauge group). The algebra of invariant polynomial fields is generated by a set of irreducible invariant fields which will be seen to correspond to 'elementary' hadrons. In general, there are polynomial identities among the irreducible invariants and these give residual algebraic constraints among the elementary fields. The true classical configuration space of hadrons, namely the space of irreducible invariants subject to these constraints, will thus define a nonlinear classical field theory, with the fields taking values in an algebraic variety.

The main part of this paper is organised as follows. After some preliminary simplification, we study pure gauge theories with $SU(2)$ and $SU(3)$ as structure groups (the 'glueball' configuration space). While for $SU(2)$ our results are essentially complete, the $SU(3)$ case is somewhat less explicitly worked out because the mathematical information available is less detailed. Next we add spinor fields representing classical commuting quark and antiquark fields in an arbitrary number of flavours interacing with the gauge field. The new irreducible invariants are mesons, baryons and antibaryons. The constraints among them are fully and explicitly listed and include a complete set of 'bosonisation formulae' expressing every baryon-antibaryon bilinear as a known polynomial in mesons. The origin of these formulae has nothing in common with those of 2-dimensional theories of fermions [12]. But they have qualitatively similar consequences: the baryons can be eliminated from the true configuration space. The effective mesonic space is that of a non-linear model with the fields having values in a manifold.

We conclude by indicating how these results can be expected to change when classical quark fields anticommute, as seems to be required. A few heuristic remarks are also offered on how we may approach the quantisation of nonlinear models such as we have for hadrons here so as to have them fulfil such general requirements as TCP invariance and the spin-statistics relation.

2. Preliminary considerations

2.1. *Classical local fields for hadrons*

Given the physical picture of hadrons as colourless composites of gluons and quarks, it is to the quantum theory of local fields describing bound states that we look to for motivation. Whenever a local quantum field theory of an irreducible field has a bound state, it is possible [6–9] to construct a local interpolating field for the bound state as a regularised *local polynomial* in the 'elementary' field and its derivatives of *finite* order. Depending on its quantum numbers, the composite field is itself irreducible in an appropriate subspace of the state space. If there existed a quantum theory of chromodynamics we would, by analogy, expect to be able to construct a local interpolating field ϕ_{had} for every colourless composite as a regularised polynomial in the gauge and quark fields and their finite order derivatives, invariant under gauge transformations. The set $\{\phi_{\text{had}}\}$ would be a complete set of fields in the hadronic state space H_{had} in the sense that regularised local polynomials in $\{\phi_{\text{had}}\}$ acting on an assumed hadronic vacuum Ω_{had} would approximate every state in H_{had}. It is then possible that a proper subset of $\{\phi_{\text{had}}\}$ could itself be a complete set, in which case a minimal subset of $\{\phi_{\text{had}}\}$ can be identified as an irreducible set $\{\phi_{\text{irr}}\}$, with respect to which Ω_{had} is a cyclic vector for H_{had}. In other words, we would have a standard local quantum field theory for the hadronic composites in which the standard 'axiomatic' theorems would hold [13].

In the absence of a satisfactory formulation of *quantum* chromodynamics, our starting point is the recognition that the above notions and constructions have exact and well-definited *classical* counterparts. The classical fields which we shall continue to denote by ϕ_{had} consist of all gauge invariant local polynomials (there is no need for any regularisation) in the classical gauge and quark fields including their finite order derivatives. $\{\phi_{\text{had}}\}$ is thus the polynomial algebra in a set of variables which are functions on space–time (for pointwise or local multiplication), invariant under gauge transformations. $\{\phi_{\text{irr}}\}$ is a set of generators of this invariant polynomial algebra: it is a minimal subset of $\{\phi_{\text{had}}\}$ such that every ϕ_{had} is a polynomial function of $\{\phi_{\text{irr}}\}$.

The space $C_{\text{irr}} = \{\phi_{\text{irr}}\}$ is not the same as the space of gauge orbits, and contains less information. In a pure gauge theory, it is known that a knowledge of the field strength and its covariant derivatives of all orders determines, locally and except for certain singular configurations, the orbits of the gauge group in the space of potentials [14–16]. Polynomials in finite order derivatives, on the other hand, do not split points in the orbit space. It should be stressed therefore that if hadrons are to have local fields, our variables are necessarily restricted to derivatives of arbitrary but finite order and our functions to polynomials of some arbitrary degree – in a standard quantum field theory, neither monomials in unbounded derivatives of local fields, nor infinite series in local fields, are in general local fields. It is also not known that points of the orbit space are classical local fields, i.e., that they are actually functions on space–time.

In any invariant-theoretic problem the term 'first fundamental theorem' denotes a characterisation of the irreducible invariants. The 'second fundamental theorem' is a statement about possible polynomial relations among them which hold by virtue of their being specific functions of the same set of variables [10], in our context, relations of the form $Q(\{\phi_{\text{irr}}\}) = 0$, Q a polynomial. The set of all such polynomials will in turn be generated by an irreducible set $\{Q_{\text{irr}}\}$. It is evident that the irreducible relations define (algebraic) constraints on the configuration space C_{irr} and that the 'true' configuration space is

$$C = \{\phi_{\text{irr}} \mid \{Q_{\text{irr}}(\{\phi_{\text{irr}}\})\} = 0\}, \tag{1}$$

The space C is the object of our study here.

2.2. *The variables*

A priori, we are interested in local polynomial functions in the elementary fields of chromodynamics, namely the potential A and the quarks $\{\psi\}$ and $\{\psi^*\}$, and their derivatives, invariant under the action of the gauge group. An elementary first result is that there are no non-trivial invariant polynomials in variables which transform affinely, such as A_μ, $\partial_\mu\psi$, etc. Consider, for example, a polynomial P in A, i.e., a real valued function on $\mathbb{R}$ whose value at x can be expressed as a polynomial in the real variables $\{A_\mu^a(x)\}$ which are the components of $A_\mu(x)$ in a hermitian basis for the Lie algebra of $SU(N)$. For an infinitesimal gauge transformation $g = 1 + i\varepsilon h$, $h(x) \in \text{Lie}\, SU(N)$, ε small, we have

$$A_\mu^{(g)}(x) = A_\mu(x) + i\varepsilon([h(x), A_\mu(x)] - \partial_\mu h(x))$$

and

$$P(A^{(g)}(x)) = P(A(x)) + i\varepsilon([h(x), A_\mu(x)]^a - \partial_\mu h^a(x)) \frac{\partial P}{\partial A_\mu^a}(x)$$

to order ε. For P to be invariant, the coefficient of ε has to vanish for arbitrary $h(x)$. First specialise to a constant h, $h(x) = h_0$; then

$$[h_0, A_\mu(x)]^a \frac{\partial P}{\partial A_\mu^a}(x) = 0.$$

Since h_0 is an arbitrary element of $\text{Lie}\, SU(N)$, we may substitute $h(x)$ for it so that the two terms in the coefficient of ε vanish separately. From the second term,

$$\partial_\mu h^a(x) \frac{\partial P}{\partial A_\mu^a}(x) = 0,$$

we conclude that P must be the constant polynomial.

We may therefore restrict our variables to the field strength F and its covariant derivatives of bounded order, $\{F_r = F_{\mu\nu}, D_\lambda F_{\mu\nu}, \ldots\}$, transforming as

$$F_r^{(g)}(x) = g(x)F_r(x)g(x)^{-1} \tag{2}$$

for all r, and the quark and antiquark fields and their covariant derivatives of bounded order, $\{\psi_s = \psi_{\alpha f}, D_\lambda\psi_{\alpha f}, \ldots\}$ and $\{\psi_s^*\}$, with α the Dirac index and f a finite flavour index, transforming as

$$\psi_s^{(g)}(x) = g(x)\psi_s(x);\ \psi_s^*(x) = \psi_s^*(x)g(x)^{-1} \tag{3}$$

for all s. On these variables, the action of the gauge group does not involve derivatives and is local. Therefore the space of local polynomials in the set of functions $\{F_r, \psi_s, \psi_s^*\}$ invariant under the gauge group coincides with the space of functions on $\mathbb{R}^4$ taking values in the set of invariants in constant variables (which we shall denote by the same symbols) under constant gauge transformations. The original ∞-dimensional problem is thus replaced by a finite dimensional one: determine the irreducible invariants and their irreducible relations in a finite number K of $N \times N$ traceless hermitian matrices, a finite number L of complex

N-vectors and th same number of N-covectors [10] under simultaneous conjugation and linear transformation by $SU(N)$.

2.3. *Reduction to pure vector and pure matrix invariants*

A form of the first fundamental theorem for invariants simultaneously in vectors and matrices has recently been proved by Procesi [11] which allows a further simplification of our problem.

First fundamental theorem for vectors and matrices. The algebra of invariant polynomials in K matrices $\{F_r\}$, L vectors $\{\psi_s\}$ and L co-vectors $\{\psi_s^*\}$ in $\mathbb{C}^N$ for the natural action of $SU(N)$ is generated by

i) the scalar products $\langle \psi_{s'}^*, \psi_s \rangle$;
ii) the determinants $[\psi_{s_1}, \ldots, \psi_{s_N}]$ and $[\psi_{s_1}^*, \ldots, \psi_{s_N}^*]$;
iii) the mixed scalar products $\langle \psi_{s'}^*, \phi\psi_s \rangle$;
iv) the mixed determinants $[\phi_1\psi_{s_1}, \ldots, \phi_N\psi_{s_N}]$ and $[\psi_{s_1}^*\phi_1, \ldots, \psi_{s_N}^*\phi_N]$;

and

v) the monomial traces tr ϕ.

Here the ϕs are arbitrary monomials in $\{F_r\}$ bounded in degree by an integer independent of K and L, $\langle , \rangle$ is the usual scalar product of vectors in $\mathbb{C}^N$ considered as a map from $\mathbb{C}^{N^*} \times \mathbb{C}^N \to \mathbb{C}$, $[, \ldots ,]$ is the determinant of components of vectors

$$[\psi_{s_1}, \ldots, \psi_{s_N}] = \det\begin{pmatrix} \psi_{s_1}^1, & \ldots, & \psi_{s_N}^1 \\ \vdots & & \\ \psi_{s_1}^N, & \ldots, & \psi_{s_N}^N \end{pmatrix}; \tag{4}$$

and the indices s, s', $s_1, \ldots, s_N$ range independently over 1 to L.

In the present context, the mixed invariants iii) and iv) are actually contained in the set i) and ii). This is because of the Ricci identity:

$$D_\mu D_\nu \psi_s - D_\nu D_\mu \psi_s = F_{\mu\nu}\psi_s \tag{5}$$

holding for all ψ_s; in every monomial in $\{F_r\}$ multiplying ψ_s, we may systematically replace $F_{\mu\nu}$, $D_\lambda F_{\mu\nu}$, etc., by $[D_\mu, D_\nu]$, $D_\lambda[D_\mu, D_\nu]$, etc. It follows that the set of all invariant polynomials is generated by an irreducible set of invariants in vectors alone, of type i) and ii) and an irreducible set of invariants in matrices alone, of type v). Moreover, no relation can exist involving both these general types simultaneously. The 'glueball' sector and the meson and baryon sector can therefore be studied independently.

The Ricci identity also makes it clear that there is no gauge-invariant distinction possible between the orbital excitation of a constituent field in a composite and a gluonic admixture.

3. Glueball fields for $SU(2)$ and $SU(3)$

3.1. *General*

Given the K basic variables $\{F_r\}$ which are Lie $SU(N)$ matrices, it is evident that every polynomial function of the form of a monomial trace,

$$T^{(n)}_{r_1r_2\cdots r_n} = \operatorname{tr}(F_{r_1}F_{r_2}\cdots F_{r_n}) \tag{6}$$

is an adjoint invariant for $n \geq 2$ ($r_1 \cdots r_n$ are not necessarily distinct). The subscript r stands for a set of Lorentz indices on F, DF, etc. Consider for the moment covariantly constant F. If we think of F in a basis independent way as a Lie $SU(N)$ matrix whose entries are vectors in the representation $V_{1,1}$ (in the standard notation) of $SO(3, 1)$ – or equivalently as an antisymmetric Lorentz tensor with values in Lie $SU(N)$ – then it is clear that a monomial trace can be written as

$$T^{(n)} = \operatorname{tr}(\otimes^n F)$$

where the tensor product refers to $V_{1,1}$ and the trace is over the matrix indices. D^rF is similarly a matrix whose entries are in the vector space

$$V_r = (\otimes^r V_{1/2,1/2}) \otimes V_{1,1},$$

V_0 being identified with $V_{1,1}$. In general then $T^{(n)}$ is an invariant polynomial on K copies of Lie $SU(N)$ with values in the space $V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_n}$, r_j finite. The corresponding gauge-invariant local field is a function from $\mathbb{R}^4$ into this vector space.

It follows that the field $T^{(n)}(x)$ is decomposable with respect to the Lorentz group $SO(3, 1)$ and that its components give the hadronic fields of definite spin [17].

The above picture also shows that the invariant fields $T^{(n)}(x)$ are generalisations of the more familiar invariant polynomials that define the Chern classes (see, for example, [18]). Briefly, the latter are ordinary $2n$-form valued polynomials in the Lie $SU(N)$-valued 2-form F, multiplication of the matrix elements being the exterior product Λ of forms. The invariants are the monomial traces

$$T^{(n)}_{\Lambda} = \operatorname{tr}(\Lambda^n F).$$

For $\mathbb{R}^4$, $T^{(n)}_{\Lambda} = 0$ for $n > 2$; in fact the only non-vanishing Chern class is $T^{(2)}_{\Lambda}$, since F is traceless. In general $\{T^{(n)}_{\Lambda}\}$ is a subset of $\{T^{(n)}\}$, obtained by restricting the variables to F (no derivatives) and the tensor product to the exterior product.

We conclude this subsection by noting that the Bianchi identity

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0 \tag{7}$$

and the Ricci identity

$$D_\lambda D_\mu F_r - D_\mu D_\lambda F_r - [F_{\lambda\mu}, F_r] = 0 \tag{8}$$

must be imposed on $\{F_r\}$. They imply relations among the invariants in covariant derivatives of F, in addition to the relations mentioned earlier. We have been unable to express them in a natural and systematic form.

3.2. $SU(2)$

The following elementary argument reduces $SU(2)$ matrix invariant theory to $SO(3)$ vector invariant theory. A one-one correspondence between $X \in \mathrm{Lie}\, SU(2)$ and $x \in \mathbb{R}^3$ is given by $X = \sum_{a=1}^{3} x_a \sigma_a$; $x_a = \frac{1}{2}\operatorname{tr}(X\sigma_a)$ (σ_a = Pauli matrices). The adjoint group of $SU(2)$ being $SO(3)$, there is also a one-one correspondence between the adjoint action of $SU(2)$ on X and the linear action of $SO(3)$ on $\mathbb{R}^3: X \to gXg^{-1}$, $g \in SU(2)$, corresponds to $x \to R(g)x$, where $R(g) \in SO(3)$ has elements $R(g)_{ab} = \operatorname{tr}(g\sigma_b g^{-1}\sigma_a)$. $R(g)$ determines g only up to a sign, but this is immaterial in the adjoint action of g. It follows that:

There is a one-one correspondence between invariant polynomials in a set of K 2×2 traceless hermitian matrices under conjugation by $SU(2)$ and those in a set of K real 3-vectors under linear transformations by $SO(3)$; this correspondence extends also to relations among invariant polynomials.

Examples of this correspondence are

$$\operatorname{tr}(XY) \to \langle x, y\rangle \text{ (scalar product } x_a y_a), \tag{9}$$

$$\operatorname{tr}(XYZ) \to [x, y, z] \text{ (determinant } \varepsilon_{abc} x_a y_b z_c). \tag{10}$$

The two fundamental theorems of vector invariant theory are [10]:

I. An irreducible set of $SO(3)$-invariants in $\{x^{(r)} \in \mathbb{R}^3, r = 1, \ldots, K\}$ consists of

$$\text{i) } \langle x^{(r_1)}, x^{(r_2)}\rangle \quad \text{for all } r_1, r_2; \tag{11}$$

$$\text{ii) } [x^{(r_1)}, x^{(r_2)}, x^{(r_3)}] \quad \text{for all } r_1, r_2, r_3. \tag{12}$$

II. An irreducible set of relations among the irreducible invariants consists of

$$\text{i) } \sum_{\pi} \sigma(\pi)[x^{(r_1)}, x^{(r_2)}, x^{(r_3)}]\langle x^{(r_4)}, x^{(s)}\rangle = 0; \tag{13}$$

$$\text{ii) } [x^{(r_1)}, x^{(r_2)}, x^{(r_3)}][x^{(s_1)}, x^{(s_2)}, x^{(s_3)}] = \det[\langle x^{(r_i)}, x^{(s_j)}\rangle]; \tag{14}$$

where the sum in i) is over cyclic permutations π of $(1, 2, 3, 4)$ of signature $\sigma(\pi)$, keeping $x^{(s)}$ fixed, and the determinant is of the matrix of scalar products as indicated.

These statements translate immediately, thanks to the correspondence (9), (10), into:

First fundamental theorem ($SU(2)$): An irreducible set of adjoint-invariants in $\{F_r \in \mathrm{Lie}\, SU(2), r = 1, \ldots, K\}$ consists of all monomial traces [equation (6)] of degree 2 and 3.

Second fundamental theorem ($SU(2)$): An irreducible set of relations among the irreducible invariants $T^{(2)}$ and $T^{(3)}$ consists of

$$\text{i) } \sum_{\pi} \sigma(\pi) T^{(3)}_{r_1 r_2 r_3} T^{(2)}_{r_4 s} = 0; \tag{15}$$

ii) $T^{(3)}_{r_1r_2r_3}T^{(3)}_{s_1s_2s_3} - \det[T^{(2)}_{r_is_j}] = 0.$ (16)

In practice, it is easy to express monomial traces of degree ≥ 4 as polynomials in quadratic and cubic monomial traces, e.g.,

$$T^{(4)}_{r_1r_2r_3r_4} = T^{(2)}_{r_1r_2}T^{(2)}_{r_3r_4} - T^{(2)}_{r_1r_3}T^{(2)}_{r_2r_4} + T^{(2)}_{r_1r_4}T^{(2)}_{r_2r_3},$$

etc.

These two theorems characterise completely the constraint-free configuration space C – the first theorem determines C_{irr} and the second the subset of C_{irr} to be identified as C. In abstract terms, $T^{(2)}$ and $T^{(3)}$ are vectors in some finite dimensional vector spaces $V^{(2)}$ and $V^{(3)}$ carrying definite (reducible) representations of $SO(3, 1)$. The space C_{irr} is therefore the space of functions from $\mathbb{R}^4$ to $V^{(2)} \oplus V^{(3)}$, i.e., some finite-dimensional representation space of $SO(3, 1)$. The constraints (15) and (16) pick out a set of vectors of $V^{(2)} \oplus V^{(3)}$ as the real roots of a finite set of polynomial equations, i.e., an algebraic variety V_Q embedded in $V^{(2)} \oplus V^{(3)}$, and C is the space of functions from $\mathbb{R}^4$ into V_Q. The constraints are, furthermore, Lorentz-invariant in the sense that $SO(3, 1)$ operates on the constraint polynomials $\{Q\}$. This gives an action of the Lorentz group on the space C, which is non-linear.

Turning this implicit (though complete) picture into an explicit list of irreducible hadronic fields is difficult without assumptions about the order of derivatives of F that one wishes to work with. For the simplest choice $\{F_r\} = \{F_{\mu\nu}\}$, the spaces $V^{(2)}$ and $V^{(3)}$ are $\otimes^2 V_{1,1}$ and $\otimes^3 V_{1,1}$ and it is straightforward to list all the Lorentz-indecomposable hadrons (fields of definite spin) as well as the constraints among them by standard techniques of reduction. The details and the results have been described elsewhere [17].

3.3. $SU(3)$

The reduction of the matrix invariant theory to vector invariant theory exploited in 3.2 is special to $SU(2)$. For the corresponding $SU(3)$ problem, we have to appeal to recent results, less complete than for vectors, concerned directly with matrix invariants [11, 19]. The strongest available forms for the fundamental theorems are:

First fundamental theorem ($SU(3)$). A basic set of adjoint-invariants in $\{F_r \in \text{Lie}\, SU(3),\ r = 1, \ldots, K\}$ is contained in the set of monomial traces of degree ≤ 6.

Second fundamental theorem. Every relation among invariant polynomials follows from the Hamilton–Cayley identity applied to the monomials of degree ≤ 6.

The term basic in the first theorem (as distinct from 'irreducible') means that the monomial traces $\{T^{(n)},\ n = 2, \ldots, 6\}$ contain a basis for the algebra of all invariant polynomials. We do not know that it is the smallest basic set – in fact, it can be shown that the symmetric tensors contained in $T^{(4)}$, $T^{(5)}$ and $T^{(6)}$ are generated by symmetric monomial traces of degrees 2 and 3 – and a complete list of irreducible invariants is unavailable. The important feature of the first theorem for us is that the degree of the monomials in the basic set is bounded above by a number independent of the number of variables. Together with the boundedness of the order of derivatives, the theorem implies that the spin of the elementary glueball fields is bounded above.

The second theorem is even less explicit and it is impracticable to list the irreducible relations. But as in the case of $SU(2)$, the constraints form a finite set of polynomial equations in the components of vectors in a finite dimensional space, invariant under the action of $SO(3, 1)$. Therefore, the constraint-free configuration space is again the space of functions from space–time into an algebraic variety on which the Lorentz group acts nonlinearly.

The complete set of irreducible hadronic fields of degree up to 3 in $\{F_{\mu\nu}\}$ will be found in [17].

4. Mesons and baryons for $SU(N)$

4.1. *General*

The theory of (complex) vector invariants is an obvious generalisation of the real case used in the previous section for the study of $SU(2)$ matrix invariants. A general reference for all theorems quoted without proof in the present section is Weyl [10].

The variables are a finite set of L vectors $\{\psi_s = \psi_{\alpha f}, D_\lambda \psi_{\alpha f}, \ldots\}$ in $\mathbb{C}^N \equiv V_{\mathrm{col}}$ (the general theorems can all be stated for arbitrary N) and an equal number of covectors $\{\psi_s^*\}$ (vectors of the dual space V_{col}^*); α is a Dirac index and f a finite flavour index. Thus, for a fixed basis in V_{col}, each colour component of $\psi_{\alpha f}$ is a vector in $V_{\mathrm{Dirac}} \otimes V_{fl}$ and that of $D^n \psi_{\alpha f}$ a vector in $W_n = (\otimes^n V_{1/2,1/2}) \otimes V_{\mathrm{Dirac}} \otimes V_{fl}$. The corresponding antiquark has components in W_n^*.

The mesons $M_{s's} = \langle \psi_{s'}^*, \psi_s \rangle$, the baryons $B_{s_1 \cdots s_N} = [\psi_{s_1}, \ldots, \psi_{s_N}]$ and the antibaryons $B_{s_1 \cdots s_N}^*$ are obviously invariants. They take values in $W_M = W_n^* \otimes W_n$, $W_B = \Lambda^N W_n$ and W_B^* respectively for some W_n, dim W_n being determined by L.

4.2. *Irreducible invariants and irreducible relations*

The basic theorems are:

First fundamental theorem: An irreducible set of invariants consists of

$$M_{s's} = \langle \psi_{s'}^*, \psi_s \rangle, \tag{17}$$

$$B_{s_1 \cdots s_N} = [\psi_{s_1}, \ldots, \psi_{s_N}], \tag{18}$$

and

$$B_{s_1 \cdots s_N}^* = [\psi_{s_1}^*, \ldots, \psi_{s_N}^*]. \tag{19}$$

Second fundamental theorem: An irreducible set of relations among the irreducible invariants consists of

$$\sum_{\pi_s} \sigma(\pi_s) B_{s_1 \cdots s_N} M_{t s_{N+1}} = 0 \tag{20}$$

$$\sum_{\pi_s} \sigma(\pi_s) B^*_{s_1 \cdots s_N} M_{s_{N+1} t} = 0 \tag{21}$$

$$\sum_{\pi_s} \sigma(\pi_s) B_{s_1 \cdots s_N} B_{s_{N+1} t_1 \cdots t_{N-1}} = 0 \tag{22}$$

$$\sum_{\pi_s} \sigma(\pi_s) B^*_{s_1 \cdots s_N} B^*_{s_{N+1} t_1 \cdots t_{N-1}} = 0 \tag{23}$$

and

$$B^*_{s_1 \cdots s_N} B_{t_1 \cdots t_N} - \det M^{(s,t)} = 0, \tag{24}$$

where the sum in (20)–(23) is over the $N+1$ terms arising from the cyclic permutations π_s of the subscripts 1 to $N+1$ on s, of signature $\sigma(\pi_s)$, and $M^{(s,t)}$ is the matrix with elements $M^{(s,t)}_{ij} = M_{s_i t_j}$.

The first fundamental theorem is evidently a special case of the more general mixed invariant theorem of Procesi [11]. There is, however, no statement of the second fundamental theorem for mixed invariants even remotely approaching the explicitness of equations (20) – (24). This is one of the advantages of the reduction to pure vector invariants made possible by the Ricci identity.

The relations are easy to establish. The left sides of equations (20)–(23) are all antisymmetric with respect to transpositions of any pair of indices 1 to $N+1$ and the relations simply state the fact that there are no alternating multilinear forms in more than N vectors in N dimensional space. As for equation (24), the (i, j) element of $M^{(s,t)}$ is

$$M^{(s,t)}_{ij} = \sum_{a=1}^{N} \psi^{*a}_{s_i} \psi^a_{t_j} = \sum_a A^{*(s)}_{ia} A^{(t)}_{aj} = (A^{*(s)} A^{(t)})_{ij},$$

where $A^{(s)}$ is the $N \times N$ matrix whose determinant is the corresponding baryon invariant and A^* its adjoint. Taking the determinant, equation (24) follows. What is non-trivial is to prove that these relations generate algebraically all polynomial relations, just as the difficult part of the first theorem is to show that B^*, B and M generate all polynomial invariants.

All these relations reflect the dimension of the underlying vector space; they are the reexpression, through the invariants, of the fact that every set of $N+1$ vectors in $\mathbb{C}^N$ is linearly dependent (the relations in the corresponding matrix case are the statement that every $N \times N$ matrix obeys a polynomial (Hamilton–Cayley) identity of degree N).

The two fundamental theorems allow us to characterise the true configuration space as follows. The irreducible invariants M, B and B^* take values in the linear vector space $W = W_M \oplus W_B \oplus W^*_B$ and the relations define, *a priori,* an algebraic variety W_Q embedded in W. On W, but not on W_Q, the relativity group $SL(2, C)$ and the flavour group $U(F)$ operate by linear representations (the relations mix up different spins and flavours).

The $SL(2, C) \times U(F)$ decomposition of M, B and B^* is in principle easy. In the simplest example of a covariantly constant ψ of one flavour with $SU(3)$ as the structure group, the meson invariants belong to $\otimes^2 V_{\mathrm{Dir}}$ and the baryons to $\Lambda^3 V_{\mathrm{Dir}}$, of dimension 16 and 4 respectively. While the meson spin spectrum is the correct one, the baryons have the wrong spin (in the many flavour situation, the corresponding statement is that baryons, belonging to $\otimes^3_{\mathrm{sym}} V_{fl}$, e.g., *uuu, ddd,* etc., have spin $\frac{1}{2}$ while the mesons, belonging to $V^*_{fl} \otimes V_{fl}$, are once again alright). The well-known remedy is for classical spinor fields to take values in an exterior algebra, a prescription forced by the independent need to satisfy the demands of Fermi statistics in a functional integral calculation of spinor processes.

4.3. *Bosonisation*

The relations (20)–(24) can also be converted to a covariant form, i.e., written as relations for appropriate indecomposable $SL(2, C) \times U(F)$ tensors. This procedure, however, makes them more complicated in appearance, especially equation (24), because the linear operations involved do not commute with taking the determinant. In any case, the component form is the more transparent and useful one for our present purposes.

Equation (24) expresses every baryon-antibaryon bilinear (every spin and flavour component of all baryon currents, including those involving covariant derivatives) as a homogeneous polynomial of order N in specific mesons with specific coefficients. We shall call these the bosonisation formulae of chromodynamics. The origin of these formulae here is quite distinct from that of the bosonisation formulae which hold in all 2-dimensional field theories of fermions. In the latter, they arise from the peculiar simplicity of 2-dimensional kinematics [12] while in the present context they, as well as the other relations, are consequences of the fact that the mesons and baryons are all composites of the same elementary fields. Furthermore, these formulae involve hadrons of all spins in a fairly complicated polynomial expression and they respect the full flavour symmetry as they must.

In spite of these fundamental differences, these formulae can be used to eliminate all baryon fields from whatever (for the present, unknown) effective Lagrangian controls the dynamics, as long as L_{eff} conserves baryon number. We make this assumption. Given, then, that the dynamics is described by some (unknown) polynomial L_{eff} in the mesons alone, we now turn to a complete description of the constraint-free mesonic configuration space.

Consider the $N \times L$ matrix A of the colour components of all quarks:

$$A_{as} = \psi^a_s, \; 1 \le a \le N, \; 1 \le s \le L.$$

The (s, t) element of the $L \times L$ matrix A^*A is

$$(A^*A)_{st} = \sum_a \bar{\psi}^a_s \psi^a_t = \langle \psi^*_s, \psi_t \rangle = M_{st}.$$

The set of all irreducible meson invariants can therefore be arranged as a $L \times L$ matrix M which is hermitian positive semi-definite. For $L > N$, there is also a rank restriction on M: the eigenvalues of A^*A are $\{\lambda_1, \ldots, \lambda_N, 0, \ldots, 0\}$, $\{\lambda\}$ being the eigenvalues of the $N \times N$ matrix AA^*; therefore, generically, rank $M = N$. This condition is a set of polynomial relations among the invariants M_{st}, clearly generated by the irreducible relations

$$\det M_{(N+1)} = 0, \tag{25}$$

$M_{(N+1)}$ being any $N+1 \times N+1$ submatrix of M.

Conditions (25) are precisely the content of the second fundamental theorem for meson invariants, i.e., they exhaust the relations (20)–(24) when projected on to the mesonic invariants. First convert equation (20) into a constraint on mesons by multiplying by $B^*_{t_1 \cdots t_N}$ and using equation (24). The left side is the expansion in minors of the determinant of

$$M^{(s,t)}_{(N+1)} = \begin{bmatrix} M_{s_1 t_1} & \cdots & M_{s_1 t_{N+1}} \\ \vdots & & \\ M_{s_{N+1} t_1} & \cdots & M_{s_{N+1} t_{N+1}} \end{bmatrix}$$

with $t_{N+1} = t$. Its vanishing for every set of pairs of $(N+1)$-plets of indices is just the rank condition. Next, multiply equation (22) by $B^*_{r_1 \cdots r_N}$ and again use equation (24). We have

$$0 = \sum_{\pi_s} \sigma(\pi_s) B_{s_1 \cdots s_N} \det \begin{bmatrix} M_{r_1 s_{N+1}} & \cdots & M_{r_N s_{N+1}} \\ M_{r_1 t_1} & \cdots & M_{r_N t_1} \\ \vdots & & \\ M_{r_1 t_{N-1}} & \cdots & M_{r_N t_{N-1}} \end{bmatrix}$$

$$= \sum_{j=1}^{N} \Delta_j \sum_{\pi_s} \sigma(\pi_s) B_{s_1 \cdots s_N} M_{r_j s_{N+1}},$$

where Δ_j, a polynomial of degree $N-1$, is the cofactor of $M_{r_j s_{N+1}}$. The coefficients of Δ_j are the expressions occuring on the left side of the relations (20). It follows that this equation is an algebraic consequence of (20) which, as seen above, project onto the mesonic subspace as the rank condition (25). We have thus the

Fundamental theorem on meson invariants: The irreducible meson invariants, modulo relations, form the space $H(L, N)$ of positive semi-definite matrices of dimension L and rank N.

The effective hadronic configuration space C_M is therefore the space of functions from space-time into $H(L, N)$.

4.4. *The topology of the mesonic space*

To what extent does the exact mesonic configuration space of classical chromodynamics resemble the configuration space of popular nonlinear models such as the $SU(n)$ σ-model [20, 21; for the principles underlying such models, see 22]? One immediate difference is that in the latter, the nonlinearity is generally restricted to the action of the flavour group; the Lorentz group acts trivially on the manifold in which the fields take values (all fields are scalar fields) and this is the only way in which this action can be made linear. In contrast, the set of irreducible meson invariants of chromodynamics necessarily includes mesons of non-zero spin, as does the set of mesons in terms of which the currents are eliminated through the bosonisation formulae.

In spite of this, C_M is, topologically, very similar to the familiar examples mentioned above. The space $H(L, N)$ is amenable to a fairly detailed topological characterisation which we carry out in the appendix. It is a manifold having the structure of a fibre bundle with the complex Grassmannian manifold $G(L, N)$ of N dimensional subspaces of $\mathbb{C}^L$ as base and the space of positive definite matrices on $\mathbb{C}^N$, $H^+(N)$, as fibres. $H^+(N)$ being contractible all its homotopy groups vanish, and it follows that $H(L, N)$ and $G(L, N)$ have identical homotopy groups: the effective mesonic theory has the same topological properties as a non-linear Grassmannian model. We note in passing that when $L = N + 1$ (for covariantly constant $SU(3)$ quarks in 1 flavour, for example) $G(L, N)$ is the complex projective space $P^N(C)$ $(= CP^N)$.

The homotopy groups $\pi_k(H(L, N)) = \pi_k(G(L, N))$ are also computed in the appendix for small k. The results are ($G(L, N)$ is always connected), for $L > 3$:

$$\pi_1(H(L, N)) = 0,\ \pi_2(H(L, N)) = \mathbb{Z},\ \pi_3(H(L, N)) = 0 \qquad \text{for all } N < L; \tag{26}$$

$$\pi_4(H(L, N)) = \mathbb{Z} \qquad \text{for } N < L - 1; \tag{27}$$

$$\pi_4(H(N + 1, N)) = \pi_4(P^N(C)) = 0. \tag{28}$$

For a sufficiently large number n of flavours, or by including sufficiently large order of derivatives, we have then that the topology of $H(L, N)$ is the same for any $N > 1$. Equivalently, for a fixed N, the topology of $H(L, N)$ is independent of the number of flavours and the order of derivatives considered. From the homotopy shifting property for spaces of functions, $\pi_k(\text{Maps}\,(S^d \rightarrow H(L, N)) = \pi_{k+d}(H(L, N))$, we conclude that the configuration spaces $\text{Maps}\,(S^d \rightarrow H(L, N))$ are connected for $d = 1$ and 3 while they have an infinite number of connected components for $d = 2$ and 4; whether the bosonic configuration space of chromodynamics can support topological excitations is determined by the dimension d of space–time and not by the number of colours. For d up to 4, finite energy solitons of topological origin are possible only for $d = 3$, while for $d = 4$, the topological excitations are instanton configurations having finite action (θ-vacua in the corresponding quantum theory). Though we do not clearly understand the physical reasons for this deviation from the $U(n)$ σ model [20, 23–25], it is possibly another indication of the need to take classical fermions as anticommuting objects.

5. Concluding remarks

5.1. *Anticommuting quarks*

In the work described above, we have met three distinct features pointing to the need to 'multiply' classical quarks anticommutatively: i) the statistics of baryons, ii) their spin spectrum and iii) the topology of the mesonic space C_M. In view of the fact that qualitatively our results, especially the existence of a bosonisation, come so close to approximate hadronic models suggested on the basis of large N QCD [26, 27], it is important to see how they are modified when quarks take values in an exterior algebra. The remarks below are only a beginning in this direction.

We assume that there is a multiplication of the components of the vectors $\{\psi_s\}$ which is anticommutative: $\psi_s^a\psi_t^b = -\psi_t^b\psi_s^a$, $\psi_s^{*a}\psi_t^{*b} = -\psi_t^{*b}\psi_s^{*a}$, $\psi_s^a\psi_t^{*b} = -\psi_t^{*b}\psi_s^a$; all polynomials in quarks are defined with respect to this multiplication. They can also be multiplied by complex numbers, and therefore the groups $SU(N)$, $SL(2, C)$ and $U(F)$ act on them. Our interest is in the (non-commutative) algebra of such polynomials, invariant with respect to the $SU(N)$ action, and in the way $SL(2, C) \times U(F)$ acts on this invariant algebra.

The scalar products

$$M_{s's} = \langle \psi_{s'}^*, \psi_s \rangle = \sum_{a=1}^{N} \bar{\psi}_{s'}^a \psi_s^a \tag{29}$$

and the determinants

$$B_{s_1 \cdots s_N} = [\psi_{s_1}, \ldots, \psi_{s_N}] = \sum_P \sigma(P) \psi_{s_1}^{a_{P(1)}} \cdot \ldots \cdot \psi_{s_N}^{a_{P(N)}}, \tag{30}$$

where P is a permutation of $\{1 \cdots N\}$, of signature $\sigma(P)$, are then easily seen to be irreducible invariants. Because of the non-commutativity of multiplication, it is necessary to stick to a convention in the order of factors. While for mesons reordering means at most a change of sign, $B_{r_1 \cdots r_N}$ is not the same as, for example, $\sum \sigma(P) \psi_{s_{P(1)}}^{a_1} \cdots \psi_{s_{P(N)}}^{a_N}$. In fact, as is well known, $B_{s_1 \cdots s_N}$ is symmetric in $s_1, \ldots, s_N$ and the baryon invariants take values in the symmetric tensor product $\otimes_{\text{sym}}^n W_n$ rather than in $\Lambda^N W_n$, thus restoring the correct spin and flavour spectrum. Furthermore, for N odd, B belongs to the odd part of the exterior algebra; the classical baryons anticommute.

The relations (20)–(24) have also anticommuting counterparts of which we write down only the bosonisation formulae:

$$B_{s_1 \cdots s_N}^* B_{t_1 \cdots t_N} + \text{perm}\, M^{(st)} = 0, \tag{31}$$

where perm denotes the permanent of the matrix, namely the homogeneous polynomial

$$\text{perm}\, M = \sum_P M_{1P(1)}, \ldots, M_{NP(N)}$$

obtained from the determinant by changing all $-$ signs to $+$. There is no sign or ordering ambiguity in defining the determinant or permanent of a meson matrix since the elements belong to the even part of the exterior algebra.

Thus bosonisation is certainly possible. What is not certainly known, lacking precise forms of the fundamental theorems in the anticommuting case, is whether the counterpart of the rank condition which is

$$\text{perm}\, M_{(N+1)} = 0$$

exhausts the constraints on the mesonic space. We cannot therefore make any statement about the topology of this space.

5.2. *A possible approach to quantisation*

We began this paper by noting the lack, so far, of a framework for quantising field theories with differential nonabelian gauge constraints and we have seen that when the dynamical variables are restricted to gauge-invariant polynomial fields, what results is a field system subject to algebraic constraints. It is legitimate then to ask whether setting up a framework for quantising algebraically constrained theories is likely to be a less formidable task. We conclude by offering a few very preliminary remarks on this question.

It is worth repeating that it is only by reference to the embedding linear space that nonlinear V_Q valued fields can be assigned quantum numbers like spin and flavour. An 'axiomatic' statement of the principles governing a quantum V_Q model must therefore start from the linear V model on which the constraints are independently imposed. One possible way of doing this is indicated by a heuristic Euclidean functional integral approach.

The irreducible constraints $\{Q(\{\phi\})\} = 0$ can be incorporated in Green functions by inserting a δ-function factor in the integrand for each constraint; for example, the partition function is

$$Z = \int d\{\phi\} \exp(-S(\{\phi\})) \prod_i \delta(Q_i(\{\phi\})),$$

where S is the action of the V model. The 'heat kernel' trick, i.e., the approximation $\delta(q) = (\lambda/\pi)^{1/2} \exp(-\lambda q^2)$, gives Z as the formal limit as $\{\lambda_i\} \to \infty$ of the sequence

$$Z_\lambda = \int d\{\phi\} \exp(-S(\{\phi\}) - \sum_i \lambda_i Q_i(\{\phi\})^2)$$

of partition functions for theories with the modified effective actions

$$S_\lambda = S + \sum_i \lambda_i Q_i^2.$$

Each constraint thus adds an interaction term to the action, of infinite strength (this is precisely the way linear and nonlinear σ-models are related).

We have thus a sequence of linear theories, in all of which the fields have values in the same linear space V and the same form of interaction, differing only in a set of coupling constants $\{\lambda\}$. For finite $\{\lambda\}$ these theories are entirely conventional and it is reasonable to expect that standard axioms will be valid for them. In particular, the 'axiomatic' results such as the CPT and the spin-statistics theorems (see for example [28]) as well as the existence of asymptotic fields and the scattering operator [29] would follow. We recall that boundedness of the spins of $\{\phi\}$ is a necessary condition for this [27]; it is guaranteed by the boundedness of the order of covariant derivatives (locality) and of the degree of irreducible invariant polynomials (the theorem of Procesi [11]).

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Appendix: the topology of $H(L, N)$

We describe here the topological properties of the space $H(L, N)$ of positive semidefinite complex matrices of dimension L and rank N. We have learned this material from M. S. Narasimhan and thank him for permission to present it here.

It is a reasonable expectation that the space $H(L, N)$ is related to the complex Grassmannian manifold $G(L, N)$, the space of N-dimensional subspaces of $\mathbb{C}^L$. Specifically, let W be a point of $G(L, N)$ and let p be the orthogonal projection onto W. Then $p = p^2$ is hermitian positive semidefinite of rank $N : p \in H(L, N)$. Conversely, for any hermitian p of rank N satisfying $p^2 = p$, the image of p is an N-dimensional subspace W on which p is the orthogonal projection. Thus $G(L, N)$ is the subset of $H(L, N)$ consisting of operators which are also projections.

More generally, associate to each $m \in H(L, N)$ the pair (W, m_W), where W is the N-dimensional subspace of $\mathbb{C}^L$ which is the range of m (the orthogonal complement W is the kernel of m) and m_W is the restriction of m to W. Since m takes the whole of $\mathbb{C}^L$ into W linearly, m_W is a linear operator on W; it is in fact onto (rank $m = N$) and since to begin with m is semidefinite, m_W is a positive definite operator on W. Conversely, given any N-dimensional subspace W and a positive definite operator m_W on it, the extension m defined by $mW = m_W W$, $mW^\perp = 0$, is a semidefinite operator of rank N on $\mathbb{C}^L$. It is easy to see that the association $m \leftrightarrow (W, m_W)$ is one-one, i.e., $H(L, N)$ can be identified with the space of pairs $\{(W, m_W)\}$, $W \in G(L, N)$, $m_W \in H^+(W)$, the space of positive definite operators on W. We have thus a fibration $H(L, N) \to G(L, N)$ with $H^+(W)$ as the fibre over W.

$H^+(W)$ is contractible; in particular, all its homotopy groups vanish. Hence the total space $H(L, N)$ and the base $G(L, N)$ have identical homotopy groups: $\pi_k(H(L, N)) = \pi_k(G(L, N))$ for all k. This follows from the homotopy exact sequence

$$\cdots \to \pi_k(H^+(N)) \to \pi_k(H(L, N)) \to \pi_k(G(L, N)) \to \pi_{k-1}(H^+(N) \to \cdots$$

To compute the homotopy groups of $G(L, N)$, we identify it with $U(L)/U(N) \times U(L-N)$. We then have, for

$$\pi_k \equiv \pi_k(U(N) \times U(L-N)) = \pi_k(U(N)) \times \pi_k(U(L-N)),$$

$$\left.\begin{array}{l}\pi_0 = 0 \\ \pi_1 = \mathbb{Z} \times \mathbb{Z} \\ \pi_2 = 0\end{array}\right\} \quad \text{for all } L \text{ and } N \ (N < L);$$

$$\begin{aligned}\pi_3 &= \mathbb{Z} \text{ for } N > 1,\ L = N+1; \\ &= \mathbb{Z} \times \mathbb{Z} \text{ for } 1 < N < L-1; \\ \pi_4 &= \mathbb{Z}_2 \text{ for } N = 2,\ L = 3 \text{ or } L > 5; \text{ and } N \geq 3,\ L = N+2 \\ &= \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ for } N = 2,\ L = 4; \\ &= 0 \text{ for } N \geq 3,\ L = N+1 \text{ or } L \geq N+3.\end{aligned}$$

It is now a matter of matching images and kernels in the homotopy exact sequence for the fibration $U(L) \to G(L, N)$ to derive the results given in the text, equations (26)–(28).

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