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Complex groups, quantum mechanics, and the dimension and reality of space

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Abstract. Orthogonal groups with complex parameters, and invariants of groups with nondefinite signature, are studied. Functions invariant under an orthogonal group are invariant under the complex group. Orthogonal transformations on coordinates induce unitary ones requiring relations among these groups. This gives the dimension and signature of space. Using the invariants of inhomogeneous unitary groups it is found that space has to be real. Requiring quantum mechanical statefunctions to be eigenfunctions of the inhomogeneous part of the group over space leads to them being complex.

1. Introduction

Groups, especially unitary and orthogonal groups, have played an increasingly important role in physics. Those with nondefinite signature, orthogonal groups with complex parameters (like the complex Lorentz group) and inhomogeneous groups, which have commuting generators forming a representation of the group given by the other generators, the homogeneous part, have interesting physical applications. We consider here some properties of these groups and use them to study the dimension of space [1, 2], the reason the coordinates (distances) are real numbers, and the complex nature of quantum mechanical statefunctions. These are closely tied to the dimension of space for group theoretical reasons (so the dimension and signature can be considered a consistency condition).

In the next section the complex groups are considered, and applied in Section III to the dimension, in Section IV to the reality of space, and to the complex nature of statefunctions in Section V. These indicate the restrictions the nature of transformations, so group theory, places on the nature of space and physical theories (here, quantum mechanics). One purpose of these sections is to raise questions, which are of great interest but which cannot be studied here. The results are summarized in Section VI.

II. The complex groups

Transformations on a real space form orthogonal groups. These can be extended to pseudo-orthogonal groups, with nondefinite signature, to complex orthogonal groups, with complex parameters – relevant because of invariance under orthogonal groups – and to complex pseudo-orthogonal groups; for $3+1$ space the complex Lorentz group. This group has been extensively discussed [3] but there appears to be no explicit statement of why it is relevant, what its algebra is and why and of its relationship to the unitary group it induces.

An orthogonal transformation is given by $x_i' = a_{ij}x_j$, with the a's orthogonal matrices. For a complex orthogonal transformation the parameters are complex with the a's still orthogonal. Expanding about the identity with complex parameters gives $a_{ij} = \delta_{ij} + \theta_1 R_{ij} + i\theta_2 R'_{ij}$, with generators R and R' the same (both are real and antisymmetric, so forming the orthogonal-group Lie algebra). Each generator appears twice; the group has twice the parameters as the orthogonal group. The number of generators can be taken twice that of the orthogonal group or the same with the number of parameters doubled. The number of commuting generators of the complex and of the orthogonal group are equal. These label the states and the orthogonal-group basis states are completely labeled. The complex group mixes them but does not introduce new ones so there are no further labels so no other commuting generators.

The complex orthogonal group is a subgroup of the special linear group and can have no more commuting generators than it. The complex Lorentz group $CO(3, 1)$ with twelve generators, a subgroup of $SL(4)$ which has three commuting generators, has two.

The complex group is relevant because a function is invariant under (is a basis vector of) a (pseudo-)orthogonal group $O(n)$ if and only if it is invariant under (is a basis vector of) $CO(n)$. Sufficiency is obvious. For necessity only the algebra is needed. But the algebras are the same. Thus if a function is invariant under all transformations of an orthogonal group it is invariant under all transformations of the complex group (whether or not these are physically realizable).

It is not surprising that functions of group parameters can be continued analytically to complex values [3].

Invariance under the complex group has other consequences. For a product of a vector on which $O(n)$ acts (like momentum) with an object like spin acted on by $SU(k)$ to be invariant, $O(n)$ must be homomorphic to an $SU(k)$ subgroup. So a unitary transformation is needed for each one of $CO(n)$ such that the product is invariant under the simultaneous application of both. The object on which $SU(k)$ acts must allow these transformations, part giving a group homomorphic to $O(n)$ the other part similarly homomorphic so giving the same unitary group – $CO(n)$ is homomorphic to a product of unitary groups on the object.

Next, multinomial invariants of a unitary or orthogonal group go into corresponding invariants for a group with a different signature. To change the signature group operators are multiplied by i for each index greater than g. This gives compact subgroups on vectors with indices less than or equal to g, and with indices greater than g. The groups leave $\sum z_j z_j^*$ or $\sum x_j x_j$ invariant; the sign for the first g terms is positive, for the others negative.

The commutator containing an invariant is a sum of subsums each consisting of terms all with the same set of indices, and with the same pairing of indices on the E's (different operator orderings differ by lower order terms), and since terms with different indices are independent, each is zero. When the signature is changed all terms in a subsum are multiplied by a factor which depends on the indices; these are the same for all terms in the subsum so the factor is. So the subsum, thus the sum – the commutator – is unaffected and remains zero. The invariant goes into an invariant.

Thus the numbers of invariants from the enveloping algebra are the same for all signatures. However the total number of invariants need not be. An example is the sign of the time for the Lorentz group. But this is not given by a multinomial in the generators.

If a group is a complex extension of a direct sum of simple groups (the Lorentz group $O(3, 1)$ is a complex extension of $O(4)$ whose algebra is equivalent to that of $O(3) \times O(3)$) then its polynomial invariants are obtained from the semi-simple group and both groups have the same number. The generators of one group are obtained from those of the other by multiplying by i's and by taking linear combinations. The first does not change the invariants, the second mixes them but does not change their number.

Multinomial invariants of a group are also invariants of the complex group. Invariants commute with all generators and the generators of the two groups are the same.

III. The dimension of space and complex orthogonal groups

Equations of motion are form-invariant under the orthogonal group. Thus they must be so under the complex extension. To see the implications of this we study, as an example, the Dirac equation in n dimensional space (any metric signature) $-\gamma_\mu p_\mu \psi(p) + m\psi(p) = 0$. Coordinates transform as $x'_\mu = \Lambda_{\mu\nu} x_\nu$; with $\Lambda = I + \epsilon\lambda$, $x'_\mu = x_\mu + \epsilon\lambda_{\mu\nu} x_\nu$, and $[\lambda_{\mu\nu}, \lambda_{\rho\sigma}] \equiv \omega^{\chi\xi}_{\mu\nu,\rho\sigma}\lambda_{\chi\xi}$. The x's are real, so the Λ's represent the complex orthogonal group $CO(n)$. The ψ is transformed by $S(I + \epsilon\lambda) = I + \epsilon T$; $[T_{\mu\nu}, T_{\rho\sigma}] \equiv \Omega^{\chi\xi}_{\mu\nu,\rho\sigma} T_{\chi\xi}$. The T's form a unitary group algebra $[T_{\mu\nu}, T_{\rho\sigma}] = \delta_{\mu r}] = \delta_{\mu\rho} T_{\nu\sigma} - \delta_{\nu\sigma} T_{\mu\rho}$. For the γ's, which are taken as not depending on the coordinates, $S(\Lambda)^{-1}\gamma_\rho S(\Lambda) = \Lambda_{\rho\nu}\gamma_\nu$. So $[\gamma_\mu, T_{\alpha\beta}] = \lambda_{\alpha\beta},_{\mu\nu}\gamma_\nu$. Then $i\gamma'_\mu p'_\mu S\psi(p') + mS\psi(p') = 0$, so $S\psi$ is a solution.

Coordinates and solutions are different so the groups, thus ω and Ω, are. They must be homomorphic: $S(\Lambda)^{-1}\gamma_\nu S(\Lambda) = \Lambda_{\nu\mu}\gamma_\mu$ (which cannot hold unless there is a unique S, up to terms commuting with the γ's, for each Λ) are satisfied by Λ_1, Λ_2 and Λ, with $S(\Lambda) = S(\Lambda_1)S(\Lambda_2)$. Then $S(\Lambda_2)^{-1}S(\Lambda_1)^{-1}\gamma_\nu S(\Lambda_1)S(\Lambda_2) = \Lambda_{\nu\rho}\gamma_\rho = \Lambda_{1\nu\mu}S(\Lambda_2)^{-1}\gamma_\mu S(\Lambda_2) = \Lambda_{1\nu\mu}\Lambda_{2\mu\rho}\gamma_\rho$, so (the γ's are independent since the p's are) $\Lambda_{\nu\rho} = \Lambda_{1\nu\mu}\Lambda_{2\mu\rho}$; Λ's and S's obey the same product relations. Their groups are isomorphic, up to multiplication of S by operators (like isospin) which commute with the γ's (groups which commute with γ and S are divided out; by unitary group we mean the factor group).

The Jacobi identity $[[\gamma, T], T] + \text{cycl} = 0$ gives $\Sigma\, (\omega^{\chi\xi}_{\mu\nu,\rho\sigma} - \Omega^{\chi\xi}_{\mu\nu,\rho\sigma})\lambda_{\chi\xi} = 0$. This is not satisfied unless $\omega = \Omega$, that is the algebras are isomorphic.

The solutions are the states of a representation of the largest invariance group of the equation. What are its linear transformations (translations are not relevant)? The p's can be replaced by sums (of the orthogonal group – the group parameters can be complex – but not the unitary group else the phase of a p, which we wish hermitian in any case, could be varied so that of a γ would have to be for invariance which is impossible with $\gamma^2 = 1$). The statefunction can be replaced by another (of the transformed p's), the γ's by a sum (the realization of the γ matrices can be changed – a similarity transformation giving the same solutions) and a sum of solutions for the same p's (including signs) with complex coefficients is a solution. There are no further degrees of freedom so no further transformations.

In $3 + 1$ dimensional space the solution of the Dirac equation is a bispinor with two halves mixed by transformations induced by rotations but not the two components of each. In n-space the k-component solution again consists of two parts mixed by rotations with the components of each not so affected. We use for the γ's the realization of Boerner[4] to show that the solution is $\psi = (u_1, u_2)$ where the components of u_1 are arbitrary and determine those of u_2. It can be seen by iteration that Boerner's ρ's can be written schematically as $\begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix}$ for ρ_1, and $\begin{pmatrix} \Omega_x & 0 \\ 0 & -\Omega_x \end{pmatrix}$ for the others, and σ's as $\begin{pmatrix} 0 & iU \\ -iU & 0 \end{pmatrix}$ for σ_1, and $\begin{pmatrix} \Omega_y & 0 \\ 0 & -\Omega_y \end{pmatrix}$ for the others, where U is a unit matrix and the Ω's, which have unit squares, are functions of $\rho(=\sigma_x)$, $\sigma(=\sigma_y)$ and the unit matrix; thus the squares of the ρ's and σ's are unit matrices. For an odd dimensional space $\tau_0 = \begin{pmatrix} \Omega_z & 0 \\ 0 & -\Omega_z \end{pmatrix}$, with Ω a function of σ_z's. The Ω's, ρ's and σ's anticommute.

These forms are true for two and three dimensions ($\nu = 1$). Suppose they are true for some ν. For $\nu + 1$ all the ρ's and σ's are multiplied by ϵ_2 so $\begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix}$, $\begin{pmatrix} \Omega_x & 0 \\ 0 & -\Omega_x \end{pmatrix}$, are replaced by

$$\begin{pmatrix} 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \\ U & 0 & 0 & 0 \\ 0 & U & 0 & 0 \end{pmatrix}$$

and similarly for the Ω's. These are of the same form giving the result by induction. The argument for τ is the same. There is one additional ρ and σ when ν is increased by 1. Here ρ, and similarly σ, is multiplied into a product of τ's symmetrical in their upper and lower halves, up to a sign (giving τ_0), because this is true of the first τ. Each 1 is replaced by ρ giving the result.

With these γ's the Dirac equation for a space of dimension $n = 2\nu$ (or

$n = 2\nu + 1$ and either representation of the γ's) gives (the ρ_ν term not present for even dimensions)

$$p_1 u_2 + ip_2 u_2 + \left[\sum_{\text{odd } j=3} (p_j \Omega_{xj} + p_{j+1}\Omega_{yj}) + \tau_\nu p_\nu - m\right] u_1 = 0,$$

and

$$p_1 u_1 - ip_2 u_1 - \left[\sum_{\text{odd } j=3} (p_j \Omega_{xj} + p_{j+1}\Omega_{yj}) + \tau_\nu p_\nu + m\right] u_2 = 0.$$

The determinant has to be zero; it is $-\sum p_j^2 + m^2$, which is zero because the p's form an invariant $\sum p_j^2 = m^2$ (the sum can have negative signs). Thus the equations give u_2 in terms of u_1 (as in $3 + 1$ space). With no other conditions on the u's, u_1 is arbitrary. There are $k/2$ independent solutions for each set of p's, where ψ has k components so u_1 has $k/2$.

Coordinates transform under $CO(n)$, solutions under $U(k)$ and also under the product of $U(k/2)$ acting on spinor u_1, and the unitary group (if any) homomorphic to $CO(n)$. The invariance group is (at most) $U(k/2) \times CO(n)$. The order of unitary algebra $A(\nu)$ is $(\nu + 1)^2 - 1$, of orthogonal algebra $B(\nu)$ is $\nu(2\nu + 1)$, and of $D(\nu)$, $\nu(2\nu - 1)$. The number of parameters in $CO(n)$ is $2\nu(2\nu \pm 1)$, and for a spinor[4] $k = 2^\nu$. The number of parameters of $U(k/2)$ is $2^{(2\nu-2)}$, for $U(k)$ $2^{2\nu}$, so of the invariance group of the equation (the sum of the numbers for $CO(n)$ and $U(k/2)$) $2^{(2\nu-2)} + 4\nu^2 \pm 2\nu$.

The numbers of parameters of the groups for the equation and the solution are equal (only) for $v = 2$, and an even-dimensional space so $n = 4$. Also the numbers of commuting generators have to be equal giving $2 + k/2 = k$. This is satisfied only in four-space. Thus the Dirac equation is, and can be, form-invariant under orthogonal transformations because these counting conditions are satisfied. (If the solution consists of j parts instead of 2 then this gives $j = 2$).

The equation is not invariant under $O(n)$ only; we cannot divide the number of parameters by 2. There are transformations on the statefunction which correspond to orthogonal transformations with imaginary parameters because invariance under the orthogonal group gives it for the complex orthogonal group. With $O(n)$ instead of $CO(n)$ $\nu = 1$, $n = 3$ and $k = 2$. But the components of a two-component spinor are mixed by a rotation, there are no further degrees of freedom, the $SU(k/2)$ term is absent, so no values of n, k and ν which satisfy. There is no consistent solution for 3-space, or using $O(n)$.

A four dimensional space is a direct sum of two three-dimensional ones (the algebras of $O(4)$ and $O(3) \times O(3)$ are isomorphic) so does not satisfy. If it were invariant under (the algebra of) $O(4)$ it would have to be invariant under two different sets of $O(3)$ transformations separately and it is not. The equation is consistent in only $3 + 1$ space giving the dimension and signature. Thus requiring the Dirac equation to be invariant under orthogonal transformations does make it invariant under complex orthogonal transformations, and fixes the dimension of space as $3 + 1$.

A transformation of the statefunction which mixes components connected by off-diagonal elements of the γ's (isospin states for example are not so connected) acts on the γ's. For a scalar product of a γ and a p to be invariant requires a coordinate transformation and that the two groups be homomorphic (the transformation can be multiplied by a term commuting with the γ's so isomorphism is too strong). If there is no homomorphism then interactions, which act on statefunctions, change scalar products.

Interactions (nonlinear terms) act as $U(k)$ transformations on statefunctions. If the required homomorphism did not hold they would act on solutions to the linear equation (say, in-states) to give functions (out-states) which are not solutions (and not expandable in terms of solutions – the space given by interactions is larger than that of the solutions of the free equation). An interaction containing γ's (such as the EM interaction) which has a sum on indices, as is necessary to even look invariant, contains γ_1. This mixes u_1 and u_2. The resulting function can have any ratio of the upper and lower components and so not be a solution. Specific examples showing the phase (introduced by a complex transformation) explicitly have been considered (for nonrelativistic cases) previously [5].

If in-states obey the free-particle equation there are outgoing one-particle states (with the same spin) which do not obey it. The equation corresponds to a Hamiltonian eigenvalue equation so the resulting states are not eigenfunctions of the (same) Hamiltonian (as the in-states – and possibly of none). This implies they do not have definite energy (nor definite momentum) and suggests that there is no group for which the statefunctions are eigenfunctions of physically meaningful representation-label operations.

This difficulty can be stated somewhat differently. A unitary transformation gives a different observer [6] who measures with respect to itself and relates coordinates by an orthogonal transformation. In these spaces there are unitary transformations, so observers, with no corresponding rotations (none can be assigned as the multiplication rules differ). For these, products differ so both the free particle equation and interactions would be different for different observers which would imply the formalism is inconsistent.

For the equation in $3+1$ space the invariance group is the complex Lorentz group (with algebra equivalent to that of $SL(2, C) \times SL(2, C)) \times SU(2)$, with algebra equivalent to that of $L(4)$, the largest group of transformations of the four-component solutions. The homomorphism between $CO(3, 1)$ and $SL(2, C) \times SL(2, C)$, following from that between $O(3, 1)$ and $SL(2, C)$, allows invariance of the nonlinear equation under orthogonal transformations because there is a transformation on the complex solutions corresponding to each orthogonal transformation of the coordinates. Also $CO(3, 1)$ is homomorphic to a unitary group, another condition which must be satisfied, and is.

The relationship between the groups is independent of the Dirac equation. A statefunction (which can describe spin or orbital angular momentum, both complex so transformed by a unitary group) is a unitary-group representation basis state. Transformations of the complex (pseudo-)orthogonal group induce unitary-group transformations on it so space must allow a homorphism between

$CO(n)$ and a unitary group which mixes blocks of components but not necessarily all. If not, another unitary group mixes blocks (of size k/j); it does not commute with $CO(n)$ otherwise it would give a direct product and we would divide the latter out and consider the factor group. This is the full set of statefunction transformations so $SU(k)$ is homomorphic to $CO(n) \times SU(k/j)$. The Cartan subalgebras of $CO(n)$ and $SU(k/j)$ commute and are distinct since $SU(k/j)$ gives linear combinations of the orthogonal-group basis vectors so supplies labels, so commuting generators, to distinguish these states.

For the Dirac bispinor, spatial transformations induce related transformations on the two spinors of each of the two independent solutions. But a sum of the solutions, with complex coefficients, is a solution and these transformations form $SU(4/2)$. So the spin direction is changed by rotating the axes, by performing successive boosts in different directions, or by taking a sum of basis vectors, that is by Lorentz or by $SU(k/j)$ transformations. These do not commute for both produce rotations.

If instead of $SU(k/j)$ there were a product of such groups not all sums of statefunctions would be statefunctions (solutions). Rather there would be sets such that sums of terms in each are acceptable, but not sums from different sets. This would mean that not all $SU(k)$ transformations were possible since the transformations not allowed are of this group. Then the statefunction would break up into a direct sum each term of which we consider separately. This (implausible) case does not affect the argument.

What are k, j and n? Equality of the numbers of parameters and commuting generators is necessary for a homomorphism. For the parameters this gives $2v(2v \pm 1) + (k/j)^2 = k^2$, and for commuting generators $v + k/j = k$. These must have simultaneous solutions for j, v, and k, and these must all be integers. (There are no solutions if the $SU(k/j)$ term is missing). They have only one integer solution, $k = 4$, $j = 2$ and $v = 2$ with a minus sign.

Statefunctions in $3 + 1$ space with more than four components are basis vectors of larger representations of the unitary group on the four-component function (which determines the number of group parameters). The other representations do not affect the argument. (However it is well-known [7] that relativistic wave equations have serious problems, except for the four-component Dirac equation.)

Some Lorentz-group representations are infinite dimensional; this is not relevant. Representation basis states are obtained by applying boost operators to a four-component bispinor – the Lorentz algebra is isomorphic to an extension of that of $SU(2) \times SU(2)$ so the bispinor contains a spinor from each. Unitary transformations act only on the bispinor (its components are relatively complex; a magnetic field gives a phase difference between the components of a spin-1/2 particle), not on the different angular momentum states in the statefunction of an orbiting pair of particles with moving center-of-mass, for these have fixed phase since their sum gives a definite function, that transformed from rest.

The difference between this and the finite-dimensional case is that for each orientation of the axes there are different spin states, so another set of transformations on the statefunction, but here the statefunction in every frame is determined by that in the rest frame. For a fixed frame no further transformations are possible. So the transformation group on the statefunction is the orthogonal group (times that on the statefunction at rest). If there were other unitary transformations then in a given frame there would be different statefunctions all reducing to the same one in the rest system. But this cannot be since there are no variables, needed to distinguish the functions, on which these depend.

There is a further condition; orthogonal transformations of real coordinates induce unitary transformations of complex statefunctions, so the algebra of the orthogonal group over space must be isomorphic to the algebra of a unitary group (the factor group obtained by dividing out any subgroup independent of rotations).

Which orthogonal algebras satisfy? The ranks (the v's) and the orders must be equal. Thus $(v+1)^2 - 1 = v(2v \pm 1)$, so $v = 1$ (for B), and 3 (for D). The algebras of $U(1)$ and $O(2)$ are also isomorphic ($v = 0$). For $v = 1$ $SU(2)$ and $O(3)$ and also $SU(1, 1)$ and $O(2, 1)$ are homomorphic. For $v = 3$ there is a homomorphism between $SU(4)$ and $O(6)$. Besides algebras over real numbers there are ones over complex numbers. The algebra of $O(4)$ is not simple but the complex extension is the algebra of the orthogonal group (the Lorentz group); complex statefunctions transform under $SL(2, C)$. So $3+1$ dimensional space satisfies. The homomorphisms for all complex classical groups are listed by Barut and Raczka [8]; there are no others.

The number of components determined by all conditions, on the number of generators, the number of commuting generators and the number of irreducible spinor components, is integral only for a dimension of $3+1$, but for this all three conditions (accidentally?) give the same number of components. This also allows the unitary-orthogonal homomorphism. That is there are conditions on both the unitary group and its representations, and on the Clifford algebra representations. All are satisfied in a space of $3+1$ dimensions, only.

That these arguments give the dimension, and signature, of space has an implication; quantum mechanics requires relativity.

The fundamental assumptions here are that the coordinates of space are real and the quantum mechanical statefunctions complex. This leads to the question of why these hold. To study this we again use an extension of orthogonal groups, pseudo-orthogonal groups.

IV. The reality of space

The coordinates of space are real numbers. This is so obvious, so central to our thinking, that it may not occur to us to ask why. But ultimately all physics is quantum mechanical and it is based on a complex space. The distance between particles is a quantum mechanical quantity. Why is it not complex?

Is consistent physics in a complex space possible? Of course no definitive answer can be given. But there are reasons to think it is not and that space must be real. Specifically the inhomogeneous part of the group over space (the translation operators which define space) must transform as a unitary representation of the homogeneous part giving a real space.

The inhomogeneous part is a representation of the homogeneous part which is $U(n)$ or a subgroup. The only complex space we need study is the defining representation; consider the transformations on space, an m-dimensional representation space of $U(n)$ which is a $U(m)$ representation space provided all $U(m)$ transformations are possible. This would not be if restrictions were imposed limiting the transformations to a subgroup. If we require the generators be hermitian the group is the $U(m)$ subgroup $O(m)$.

It is also possible that some vectors could not be rotated into others; we could rotate around some axes but not others. In saying that the space is a representation of $U(n)$ we imply that the number of homogeneous generators is n^2, fewer than the m^2 of $U(m)$ so not all of the latter can be implemented (otherwise space would form the defining representation of $U(m)$). As we are considering complex spaces (so not being limited to the $O(m)$ subgroup) the nonimplementable transformations do not only change phases. Some mix different vectors (say, directions). In these spaces there are linear combinations of the $U(m)$ generators which form the $U(n)$ subgroup and can be implemented. The remainder cannot be.

Not being able to perform all $SU(m)$ transformations is analogous in ordinary space to not being able to rotate in the xy plane (at least) without simultaneously rotating in the zt plane. It would be impossible to perform rotations from some fixed axes without also causing the object to move. There would be no observers at rest in these frames and no physical meaning of, or consistent way to measure with respect to, these axes. Not all mathematically definable coordinate systems would allow observers so either there could be no particles whose spin and velocity would point in any direction except for some fixed one, or if there were such particles there would be no way of measuring with respect to them. Consistent physics in such spaces is implausible and we assume they are not possible.

Thus we consider only complex spaces given by the defining representation of the group over it. These have only one invariant [9]; it contains both E's and p's. They have none containing only momenta so have no invariant distance, mass, scalar products or angles.

This does not mean that if we rotate the distance between two objects changes. If distance if taken as $z_1^2 + z_2^2$, then E_2^1 gives $2z_1z_2$, a different function. Thus what to one observer is a distance, to one rotated (even slightly) is some number with no meaning at all.

In ordinary space there are many ways of forming invariants and (invariant) scalar products. But here there is only one invariant. Invariant Lagrangians, Hamiltonians and interactions are impossible, implying different forms, different dependence on variables, in different frames. Interactions give creation and annihilation of particles so the number of particles would differ for rotated observers. Particles, having different interactions, would behave differently for different observers. Consistent physics in a complex space seems unlikely.

This is not surprising. Space is defined by the translation operators which must be real for invariance. There are no invariant complex numbers; their phases can be changed. The unitary group is that which keeps real numbers, the sum of absolute squares, invariant. There is no Lie group keeping complex numbers invariant.

Thus only real spaces can have invariants consisting only of momenta or coordinates. At best it is difficult to imagine a consistent, meaningful Universe in which the concept of distance has no meaning. Space must be real to give an invariant distance (among other such requirements).

A space which is a sum of a representation and its complex conjugate (determined by the $U(n)$ operator) has an invariant distance but there is no way of rotating from a (basis) vector to its complex conjugate (unless we consider space as a defining representation of $SU(2n)$). The representation is reducible so this space is a direct sum of two spaces. It is not isotropic. Distance (squared) would be defined as the coordinate times the coordinate in a different direction, the magnitude (squared) of a vector, the vector times a different vector. Such a space is not plausible either.

However the Hamiltonian (the translation operator conjugate to the proper time) should be invariant (for consistency) so it depends on only $z * z$. This is real; if space were complex the invariants would be absolute squares and products. So momentum operators, which define space, would be real so give a real space. This reducible representation still leads to a real space.

Complex spaces are ruled out. The representation (of whatever the homogeneous group is) formed by the inhomogeneous part has hermitian generators. Space is a representation space of the orthogonal group (the homogeneous part) over the inhomogeneous part. Then the previous arguments give the dimension so the underlying group. There is only one inhomogeneous part, so one homogeneous part, which satisfies. The group over the space is determined by the inhomogeneous part, which defines it, and this part is determined by consistency.

This gives another way of finding the dimension. Space is required to be isotropic (enough so there can be observers in all, or many, coordinate systems) and that it be real (so it has reasonable scalar products).

By the isotropy requirement the dimension of space must equal n (it could not be less than n) so it is either the $SU(n)$ defining representation or if required to be real the defining representation of $O(n)$.

The defining representation is real only if it is an orthogonal group representation. There is no orthogonal group which allows this. But the defining representation of $SL(2, C)$ is four-dimensional and it is homomorphic to $O(3, 1)$ so a $3 + 1$ dimensional space satisfies; it is isotropic and real.

The representation of $SL(2, C)$, a complex extension of a direct product of simple groups $SU(2) \times SU(2)$, is reducible being a sum of two two-dimensional spinors, in the usual notation $(\frac{1}{2}, 0) + (0, \frac{1}{2})$, and this is four dimensional. The defining representation is not $(\frac{1}{2}, 0)$; the group has six parameters and this allows only a three-parameter transformation group. Space forms the vector representation, $(\frac{1}{2}, \frac{1}{2})$, which is also four-dimensional (essentially our Universe exists because both $2+2$ and 2×2 equal 4). There are two different representations having the same dimension.

The coordinate vector belongs to a tensor representation so is real and is a representative of $O(3, 1)$. Thus a complex extension of a direct sum of unitary groups homomorphic to a pseudo-orthogonal group is required. The only space which satisfies [8] has dimension $3+1$. That is if $SU(r) \times SU(r) \sim O(2r)$ then the number of parameters must be equal so $2(r^2-1) = r(2r-1)$ giving $r=2$. Again the argument gives the signature.

Thus spacetime transforms as the defining representation of the smallest orthogonal group over it; it is real.

V. Why is the quantum mechanical statefunction complex?

Space is real but statefunctions are complex (have relatively complex components). Why? The reasons are implied by group theory.

With translation invariance a free-particle statefunction (an inhomogeneous-group representation basis vector with the inhomogeneous operators diagonal) is an eigenfunction of the translations. Its value at a point is given by the statefunction at the origin acted on by translation operators which are realized by first-order differential operators (the statefunction is a function of space). Instead of statefunctions we can consider operators (the picture is irrelevant).

Then $p_j \sim i\, d/dx_j$ with the i required else a basis function would be $\exp(px)$ which could not be a probability, would increase without limit, violate unitarity and give the position of the particle as infinity. So the statefunction is complex.

This realization for the linear parts of the translation operators and inclusion of nonlinear terms – there are interactions – gives statefunctions with relatively complex components. Even if real at a point, the translation operators contain interaction terms which perform all transformations on a statefunction giving relatively complex components at other points (the components are different functions of space – specifically the time coordinate – because in a multi-particle system each particle sees potentials which differ with direction so different components have different energy).

A particle in a magnetic field (say due to another particle so quantum mechanical) undergoes unitary group transformations. The field rotates the spin (or orbital angular momentum) and changes the phases [10] of the components differently. The velocity of the particle has no phase so is transformed by the orthogonal group. An interaction (say, due to a field with a gradient) which changes the direction of the velocity changes the phases and angular momentum direction; an orthogonal transformation induces a unitary one. If a beam of particles is split in two and different magnetic fields applied to the two beams their statefunctions have an overall phase difference and the components have different relative phases. An interference experiment measures these relative phases.

The translation operators belong to an inhomogeneous orthogonal group so cannot transform real functions into complex ones. However the eigenfunctions of the (nonlinear) translation operators are not one particle statefunctions but statefunctions of sets of interacting particles [11] – sums of products of one-particle states. The latter acted on by nonlinear operators are complex.

One-particle states are not eigenfunctions of inhomogeneous group operators and not basis functions of a representation of it, so are not solutions of the nonlinear equation of motion. They are eigenfunctions of free-particle operators, the linear parts of the translations. On a one-particle statevector interactions give terms from different representations; the space given by the interaction is larger than an irreducible representation space. In experiments we study the complex one-particle components of the statefunction of the system. Rotations affect them so we have to consider their unitary transformations; these must be homomorphic to the rotations.

VI. Conclusion

Requiring reasonable relationships between observations in different frames gives conditions on physical systems and the nature of space. This fixes its dimension and signature.

Invariance under an orthogonal group requires it under the complex orthogonal group. But these induce unitary transformations. Physics needs invariant scalar products of inhomogeneous operators – impossible in complex spaces; space is real. Using the inhomogeneous group to define space, translation invariance and interactions lead to complex statefunctions, with relatively complex components.

The analysis of transformations, and the properties of the groups thus involved, determines much about physics and about space.

Why is space real with dimension $3 + 1$, and why are quantum mechanical statefunctions complex? Because it appears that only this space and these statefunctions allow consistent implementation of transformations, thus consistent physics. It is not so much the properties of quantum mechanics and of space that determine its dimension. Rather consistency requires this set of properties. The dimension of space is thus a consistency condition. The properties of space, and of physical laws, are closely related, demonstrating again the underlying coherence of the laws governing our Universe.

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