Dirac operator and Chern-Simon action

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Dirac operator and Chern - Simons action [1]

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Abstract. The effective action generated by fermions exposed to a nonabelian gauge field in three-dimensional euclidean space is examined in detail. It is shown that the symmetries of the Dirac operator allow one to specify its determinant in a unique manner in terms of the corresponding heat kernel.

The present is meant as a small present to Gérard Wanders, on the occasion of his 60th birthday. It concerns a problem which occurs in the neighbourhood of the field which he has explored in recent years [1], demonstrating that pair creation of massless fermions by an external gauge field in two-dimensional Minkowski space leads to remarkable phenomena which do not manifest themselves in the Euclidean effective action. The setting in which the following considerations take place also concerns fermions in an external gauge field, but differs from the framework studied by Wanders and his collaborators in three respects: I consider a three-dimensional space, restrict the discussion to the Euclidean situation and allow the fermions to be massive.

1. The effective action which results if the fermions are integrated out is given by the logarithm of the determinant associated with the operator

$$D = \not{D} + m \; ; \qquad \not{D} = \gamma_\mu \{\partial_\mu + iA_\mu(x)\} \tag{1}$$

Some properties of this effective action and its relation to the Chern-Simons action are discussed in the literature [2]. I wish to present a more complete analysis, emphasizing mathematical aspects.

In three dimensions, there are two inequivalent irreducible representations of the γ-matrices. For definiteness, I choose $\gamma_\mu = \sigma_\mu$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The fermions transform according to a unitary representation of the gauge group and the gauge field is represented by a hermitean matrix $A_\mu(x)$ acting on the fermion flavour. In the following, neither the structure of the gauge group nor the properties of the representation play an important role.

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Formally, the determinant of the Dirac operator is the product of its eigenvalues. Since $\not{D}$ is antihermitean, its eigenvalues are imaginary,

$$\not{D}\, u_n(x) = i\lambda_n u_n(x) \tag{2}$$

such that the determinant takes the form

$$\det D = \prod_n (i\lambda_n + m) = \mid \det D \mid e^{i\phi} \tag{3}$$

The modulus $\mid \det D \mid$ is even under $m \to -m$,

$$\ln \mid \det D \mid = \frac{1}{2} \sum_n \ln(\lambda_n^2 + m^2) \tag{4}$$

while the phase

$$\phi = \sum_n \arctan(\frac{\lambda_n}{m}) \tag{5}$$

is odd. Needless to say that the formal sums occurring here do not make sense as they stand - before discussing the properties of the determinant, we need to specify it in a less formal manner.

2. As a first step, consider the expansion of the effective action in powers of the gauge field. Setting $D = D_0 + i\not{A}$ where D_0 is the free Dirac operator, we obtain

$$\ln \det D = \ln \det D_0 + i\mathrm{Tr}(S\not{A}) + \frac{1}{2}\mathrm{Tr}(S\not{A}S\not{A}) - \frac{i}{3}\mathrm{Tr}(S\not{A}S\not{A}S\not{A}) + \ldots \tag{6}$$

where $S = (D_0)^{-1}$ is the free Euclidean propagator. Let us dispose of infrared problems by taking $m \neq 0$ and requiring the field $A_\mu(x)$ to vanish outside a compact region of space. The problem with the formal sums given above stems from large eigenvalues of the Dirac operator. In the propagator, these eigenvalues manifest themselves in the fact that $S(x-y)$ blows up when x tends to y. The short-distance singularities can be regularized, e.g. by using

$$S_\alpha(x-y) = \frac{1}{(2\pi)^3} \int d^3p \frac{(-i\not{p} + m)}{(m^2 + \not{p}^2)^\alpha} e^{ipx} \tag{7}$$

where α plays the role of a cutoff - we are interested in the limit $\alpha \to 1$.

The first term in the expansion (6) is an irrelevant constant which can be dropped, setting $\det D_0 = 1$. Since the second term vanishes upon taking the trace over the γ-matrices, the expansion starts with the contribution quadratic in the gauge field

$$\ln \det D = \frac{1}{2} \int dx dy \,\mathrm{tr}_f\{A_\mu(x) A_\nu(y)\} \Pi_{\mu\nu}(x-y) + O(A^3) \tag{8}$$

Here, the symbol f indicates that the trace only extends over the flavour quantum numbers. The kernel is given by the standard loop integral associated with vacuum polarization,

$$\tilde{\Pi}_{\mu\nu}(q) = \frac{1}{(2\pi)^3} \int d^3p \,\mathrm{tr}_s \frac{\{\gamma_\mu(-i\not{p} + m)\gamma_\nu(-i\not{p} + i\not{q} + m)\}}{(m^2 + p^2)^\alpha (m^2 + (p-q)^2)^\alpha} \tag{9}$$

where the trace now only extends over the spin indices. There is an important difference, however, to the familiar situation in $d = 4$: in three dimensions, the trace of a product of three γ-matrices does not vanish and $\Pi_{\mu\nu}$ therefore contains a contribution which is odd under $m \to -m$. Accordingly, the phase of the determinant is different from zero. Note that this contribution to the loop integral (9) is convergent - the regularization is needed only for the piece which is even under $m \to -m$. The integral is readily worked out with the representation

$$\frac{1}{N_1^\alpha N_2^\alpha} = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \int_0^1 d\xi [\xi(1-\xi)]^{\alpha-1} \{(1-\xi)N_1 + \xi N_2\}^{-2\alpha} \tag{10}$$

The result is analytic in α except for poles occurring at $\alpha = 5/4, 3/4, \ldots$. Since there is no singularity at $\alpha = 1$, the cutoff can be removed without further ado. In fact, in $d = 3$, one does not need to add any counter terms to keep one-loop integrals finite. (In the regularization (7), any one-loop graph can be reduced to the standard integral $\int d^3p(M^2 + p^2)^{-n}$ which is analytic in n except for poles occurring at half-integer values. Since the n-values of interest are integer, the cutoff can always be removed without encountering divergent expressions). In the case of the vacuum polarization, the result reads

$$\tilde{\Pi}_{\mu\nu}(q) = \frac{1}{4\pi} \int_0^1 d\xi \frac{\{2\xi(1-\xi)(q_\mu q_\nu - \delta_{\mu\nu} q^2) - m\epsilon_{\mu\nu\alpha} q_\alpha\}}{\{m^2 + \xi(1-\xi)q^2\}^{1/2}} \tag{11}$$

In accordance with gauge invariance, the polarization tensor is transverse.

The higher order terms occurring in the expansion (6) can be analyzed in the same manner. Power counting shows that the loop integrals associated with four or more external legs are convergent as they stand. The triangle graph formally diverges logarithmically; the integral over the direction of the internal momentum however eliminates this divergence, such that this graph is actually finite, too. With the regularization (7), the expansion of the determinant in powers of the gauge field thus leads to an unambiguous result. In particular, this expansion specifies the phase of the determinant in terms of a sequence of ordinary, convergent loop integrals.

3. Next consider the limiting case of very heavy fermions. The representation (11) shows that in the limit $m \to \infty$, the vacuum polarization tensor does not tend to zero, but reduces to a term linear in the momentum $\propto \epsilon_{\mu\nu\alpha} q_\alpha$. In the case of the triangle graph, the limit $m \to \infty$ is a momentum independent constant while diagrams with four or more external legs tend to zero. In the heavy mass limit, the determinant therefore reduces to

$$\ln \det D = -i \frac{m}{|\, m \,|} I + 0(m^{-1}) \tag{12}$$

where I is the Chern-Simons action

$$I = \frac{1}{8\pi} \int d^3x \epsilon_{\alpha\beta\gamma} \operatorname{tr}_f \{A_\alpha \partial_\beta A_\gamma + \frac{2i}{3} A_\alpha A_\beta A_\gamma\} \tag{13}$$

Note that the result is purely imaginary - in the modulus of the determinant, heavy quarks decouple from the gauge field, $|\det D|$ tending to one as $m \to \infty$, but in the phase of the determinant, they don't.

This finding is strange, not only because it contradicts the intuitive expectation that the occurrence of very heavy fermions should not affect the effective action of the gauge field, but also because the Chern-Simons action is not fully gauge invariant: while I is invariant under gauge transformations which differ from the identity only in a compact region of space, it is not in general invariant under gauge transformations which are nontrivial at infinity. The effective action thus appears to break invariance under general gauge transformations.

4. Indeed, the fact that the expansion of the phase in powers of the gauge field leads to well-defined kernels, given by convergent loop integrals, is misleading. Although these integrals do not indicate the occurrence of an ambiguity, there are perfectly respectable regularization schemes which lead to a different expression for the phase ϕ. As an example, consider Pauli-Villars-regularization. In this scheme, the vacuum polarization tensor is given by

$$\tilde{\Pi}_{\mu\nu}(q)_{PV} = \frac{1}{(2\pi)^3}\int d^3p \sum_k c_k \frac{\mathrm{tr}\{\gamma_\mu(-i\not{p} + M_k)\gamma_\nu(-i\not{p} + i\not{q} + M_k)\}}{(M_k^2 + p^2)(M_k^2 + (p-q)^2)} \tag{14}$$

where $c_0 = 1, M_0 = m$ is the unregularized contribution. Inspection of the integrand shows that one regulator term with $c_1 = -1$ suffices to insure convergence. Now, consider the part which is odd under $M_k \to -M_k$. Since the individual contributions with $k = 0$ and $k = 1$ are convergent, they coincide with the odd part of the expression given in eq. (11). If we send the regulator mass to infinity, a finite contribution remains from the regulator term. A similar phenomenon occurs with the triangle graph. The net result can be inferred from eq. (12): applying Pauli-Villars regularization to all of the loop integrals occurring in the expansion of the determinant in powers of A_μ, one obtains

$$\ln\det D_{PV} = \ln\det D + i\frac{M_1}{\mid M_1 \mid} I \tag{15}$$

where $\ln\det D$ stands for the quantity specified in paragraph 2. In particular, if the sign of the regulator mass coincides with the sign of m, the Pauli-Villars-determinant decouples when m is sent to infinity, $\det D_{PV} \to 1$.

5. There is no physics in regularization schemes and the mathematical structure of the ambiguities associated with the short distance singularities is well known. The vacuum polarization tensor $\Pi_{\mu\nu}(z) = \mathrm{tr}[\gamma_\mu S(z)\gamma_\nu S(-z)]$, e.g., involves a product of two distributions which are singular at $z = 0$. In this context, the external gauge field plays the role of a test function on which the distribution $\Pi_{\mu\nu}(z)$ is to be evaluated. The problem is that the integral $\int d^3z f_{\mu\nu}(z)\Pi_{\mu\nu}(z)$ is unambiguous only on test functions $f_{\mu\nu}(z)$ which vanish at $z = 0$, together with their first derivatives - the function must have a zero of sufficiently high order to cancel the singularity of $\Pi_{\mu\nu}(z)$ which is of order $\mid z \mid^{-4}$. What the regularization introduced in eq. (7) achieves is that it provides us with one specific extension of the distribution $\Pi_{\mu\nu}(z)$ to test functions which do not satisfy this condition. Pauli-Villars-regularization leads to a different such extension. The key point here is that the integral $\int d^3z f_{\mu\nu}(z)\Pi_{\mu\nu}(z)$ is unique only up to multiples of $f_{\mu\nu}(0)$ and $\partial_\lambda f_{\mu\nu}(0)$. This implies that different regularizations of the vacuum polarization diagram lead to results which differ at most by multiples of $\int d^3x A_\mu(x)A_\nu(x)$ and $\int d^3x \partial_\lambda A_\mu(x)A_\nu(x)$. Extending

this analysis to diagrams containing any number of external legs, one concludes that the determinant is unique up to

$$\ln \det D' = \ln \det D + \int d^3x P\{A(x), \partial A(x), \ldots\} \tag{16}$$

where P is a polynomial in the gauge field and its derivatives. The degree of the polynomial and the number of derivatives which occur depends on the dimension of space. In $d = 3$, P is of the general form

$$P = c^a_\mu A^a_\mu + c^{ab}_{\mu\nu} A^a_\mu A^b_\nu + c^{ab}_{\lambda\mu\nu} \partial_\lambda A^a_\mu A^b_\nu + d^{abc}_{\lambda\mu\nu} A^a_\lambda A^b_\mu A^c_\nu \tag{17}$$

6. Without further ado, the determinant of the Dirac operator is unambiguous only up to a local polynomial of this structure. The ambiguity can however be reduced considerably by exploiting symmetries. The specific schemes discussed above show that there are regularizations for which the determinant is invariant under the Euclidean group of motions. Restricting the category of determinants under consideration to those which respect this symmetry, the coefficients $c^a_\mu, c^{ab}_{\mu\nu}, \ldots$ must be independent of x (translation invariance) and must be proportional to either $\delta_{\mu\nu}$ or $\epsilon_{\lambda\mu\nu}$ (rotation invariance). Furthermore, invariance under gauge transformations of compact support eliminates the photon mass term $c^{ab}_{\mu\nu} A^a_\mu A^a_\nu$ and requires the remainder to only occur in the combination which shows up in the quantity I defined in eq. (13). With these symmetry requirements, the determinant therefore becomes unique up to a multiple of the Chern-Simons-action, $\int d^3xP = cI$.

7. This naturally leads to the question of whether the remaining ambiguity can be fixed in such a way that the determinant of the Dirac operator becomes invariant also with respect to gauge transformations which are nontrivial at infinity. The above setting, where we restricted the gauge field to a compact region of space ab initio, is however not suitable to study this problem. To analyze nontrivial gauge transformations, I consider a compactified version of Euclidean space and replace R^3 by the torus T^3, identifying the points $x + a$ with x. Here, a is a vector whose components are integer multiples of the sides, L_1, L_2, L_3 of the torus, $a = (n_1L_1, n_2L_2, n_3L_3)$. Alternatively, one can take a sphere- as long as the manifold is compact and does not have boundaries, the following analysis goes through without essential modifications. In order for the gauge field to live on the torus rather than on R^3, it must not distinguish between $a+x$ and x. Periodic gauge fields, $A_\mu(x + a) = A_\mu(x)$, obviously satisfy this condition. Since I wish to allow for arbitrary gauge transformations, this category is however too restrictive. I instead require the gauge field only to be periodic up to a gauge transformation,

$$A_\mu(x + a) = U_a(x)A_\mu(x)U^+_a(x) - iU_a(x)\partial_\mu U^+_a(x) \tag{18}$$

where the unitary matrix $U_a(x)$ belongs to the fermion representation of the gauge group. Likewise, the Fermi field is subject to the boundary condition

$$\psi(x + a) = (-1)^{n_1+n_2+n_3} U_a(x)\psi(x) \tag{19}$$

(The sign appearing here is needed to insure the standard connection between the fermionic determinant and the partition function - it does not play any role in the following, because the ambiguities occurring in the notion of the determinant stem from the local properties of the differential operator and are not affected by boundary conditions).

Under the action of the gauge group, $A_\mu(x)$ and $U_a(x)$ transform according to

$$A_\mu(x)' = U(x)A_\mu(x)U^+(x) - iU(x)\partial_\mu U^+(x)$$
$$U_a(x)' = U(x+a)U_a(x)U^+(x) \tag{20}$$

The boundary condition (19) insures that, on the torus, $-i\not{D}$ is hermitean; its eigenvalues λ_n are therefore real and they are invariant under the transformation (20). If we manage to express the determinant in terms of these eigenvalues, the result will necessarily be invariant under arbitrary gauge transformations.

8. Consider first the modulus $|\det D|$. The formula (4) shows that, up to a factor of two, $\ln|\det D|$ coincides with the effective action associated with the operator $m^2 - \not{D}^2$. Since this operator is hermitean and positive, its determinant can readily be constructed with the heat kernel technique [3].

For small values of t, the trace of the heat kernel associated with $(-\not{D}^2)$ behaves like

$$\mathrm{Tr}\{\exp t\not{D}^2\} = \sum_n e^{-t\lambda_n^2} = \frac{1}{(4\pi t)^{\frac{3}{2}}}\{h_0 + th_1 + t^2h_2 + \ldots\} \tag{21}$$

where the Seeley coefficients $h_0, h_1, \ldots$ are local polynomials of the gauge field. The compactification merely restricts the domain of integration occurring in the explicit expression for these coefficients - in the trace of the heat kernel, the boundary conditions manifest themselves only through terms which are exponentially small as $t \to 0$. To see this explicitly, consider the heat kernel

$$K(x,y,t) = (x \mid \exp t\not{D}^2 \mid y) \tag{22}$$

which represents a matrix in spin and flavour space, obeying the heat equation

$$\frac{\partial}{\partial t}K = \not{D}^2 K \tag{23}$$

and the initial condition

$$K(x,y,t) \underset{(t \to 0)}{\rightarrow} \sum_a (-1)^{n_1+n_2+n_3}\delta^3(x-y+a)U_a^+(x) \tag{24}$$

Denote the kernel which corresponds to a single heat source sitting at y by $\hat{K}(x,y,t)$. It obeys the same differential equation as K, but contains only one term, $\delta^3(x-y)$, in the initial condition (24). The kernel we are interested in can be expressed in terms of $\hat{K}$ as

$$K(x,y,t) = \sum_a (-1)^{n_1+n_2+n_3} U_a^+(x)\hat{K}(x+a,y,t) \tag{25}$$

Note that $\hat{K}$ does not know about the boundary conditions imposed when restricting $\not{D}$ to the torus. For small t, the standard heat kernel expansion for $\hat{K}$ reads

$$\hat{K}(x,y,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{(x-y)^2}{4t}}\{H_0(x,y) + tH_1(x,y) + \ldots\} \tag{26}$$

The quantity $\mathrm{Tr}\exp(t\not{D}^2)$ is the integral over the torus of the matrix trace of $K(x,x,t)$. In the representation (25), the terms with $a \neq 0$ are suppressed exponentially and the Seeley coefficients of K are therefore given by

$$h_r = \int_V d^3x \,\mathrm{tr}_{s,f}\{H_r(x,x)\} \tag{27}$$

Explicitly, we have, e.g. [4]

$$h_0 = 2N_f V \ , \ h_1 = 0$$
$$h_2 = \frac{1}{3}\int_V d^3x \,\mathrm{tr}_f(F_{\mu\nu}F_{\mu\nu}) \tag{28}$$

where N_f is the dimension of the fermion representation and $V = L_1L_2L_3$ is the volume of the torus. Note that the coefficient h_1 vanishes, because a gauge invariant local polynomial of dimension two does not exist. This implies that the modulus of the determinant can be represented as a convergent integral,

$$\ln \mid \det D \mid = -\frac{1}{2}\int_0^\infty \frac{dt}{t} e^{-tm^2}\{\mathrm{Tr}\exp t\not{D}^2 - \frac{2N_f V}{(4\pi t)^{\frac{3}{2}}}\} \tag{29}$$

Indeed, gauge invariance fixes the modulus uniquely: a local polynomial of the type (17) is gauge invariant only if all of the coefficients are zero. Hence any other construction of $\mid \det D \mid$ which leads to a gauge invariant result will lead to the same result.

In the limit $m \to \infty$, the behaviour of $\mid \det D \mid$ is controlled by the expansion (21). The leading contribution stems from the Seeley coefficient h_2,

$$\ln \mid \det D \mid = -\frac{1}{48\pi \mid m \mid}\int d^3x \,\mathrm{tr}_f F_{\mu\nu}F_{\mu\nu} + 0(m^{-3}) \tag{30}$$

In the real part of the effective action, fermions thus decouple if their mass tends to infinity.

9. Let us now turn to the phase which formally represents the sum of $\arctan(\lambda/m)$ over all eigenvalues. To express this sum in terms of a heat kernel, I make use of the integral representation

$$\arctan(\frac{\lambda_n}{m}) = \lambda_n \int_0^\infty dt e^{-t\lambda_n^2} f(t) \tag{31}$$

where $f(t)$ is related to the error function,

$$f(\tau^2) = \frac{m}{\tau}\int_\tau^\infty dx e^{-x^2m^2} \tag{32}$$

The sum over the eigenvalues then takes the form

$$\sum_n \lambda_n e^{-t\lambda_n^2} = (-i)\mathrm{Tr}\{\not{D}\exp t\not{D}^2\} \tag{33}$$

For small values of t, the representation of the kernel given above shows that

$$\mathrm{Tr}\{\not{D}\exp t\not{D}^2\} = \frac{1}{(4\pi t)^{\frac{3}{2}}}\{\tilde{h}_0 + t\tilde{h}_1 + t^2\tilde{h}_2 + \ldots\}$$

$$\tilde{h}_r = \int_V d^3x \mathrm{tr}_{s,f}\{\gamma_\mu D^x_\mu H_r(x,y)\}_{x=y} \tag{34}$$

The first two coefficients are readily evaluated with the formulae given ref. [4]. One obtains $\tilde{h}_0 = \tilde{h}_1 = 0$, not a surprise, since gauge invariant polynomials of dimension one or three do not exist. Hence the quantity $\mathrm{Tr}(\not{D} \exp t\not{D}^2)$ tends to zero as $t \to 0$, while the function $f(t)$ is proportional to $t^{-1/2}$. The integral

$$\phi = -i \int_0^\infty dt f(t) \mathrm{Tr}\{\not{D} \exp t\not{D}^2\} \tag{35}$$

therefore converges, thus specifying the phase of the determinant in a fully gauge invariant manner. This shows that the ambiguity encountered in the restricted framework considered in the first part of this paper disappears if invariance under arbitrary gauge transformations is imposed: a gauge invariant determinant does exist and it is unique.

In the limit $m \to \infty$, the integral (35) is dominated by small values of t. The expansion (34) shows that ϕ tends to zero in proportion to m^{-2}, the coefficient being determined by the gauge invariant polynomial $\tilde{h}_2$. Decoupling therefore also occurs in the phase.

10. Finally, consider the limit $m \to 0$. The expression for the vacuum polarization tensor given in eq. (11) implies that, to order A^2 in the expansion in powers of the gauge field, the modulus of the determinant remains a nonlocal functional of the gauge field while, in the regularization scheme used there, the phase disappears (in Pauli-Villars regularization, the phase tends to $\pm I$, the sign depending on the sign of m). To see what happens in the limit $m \to +0$ with the gauge invariant determinant on the torus, I first note that the function $f(t)$ occurring in (35) approaches $\frac{1}{2}\sqrt{\pi/t}$. Expressing the trace as a sum over eigenvalues and cutting the integral off at the lower limit $t = \epsilon^2$, we obtain

$$\phi = \lim_{\epsilon \to 0} \sum_n \frac{\lambda_n}{|\lambda_n|} F(\epsilon |\lambda_n|)$$

$$F(x) = \sqrt{\pi} \int_x^\infty dy e^{-y^2} \tag{36}$$

The formula makes sense only if all of the eigenvalues are different from zero: in the massless case, zero modes require special attention. If the gauge field under consideration admits zero modes, the determinant tends to zero as $m \to 0$ and the real part of the effective action therefore diverges logarithmically.

Consider a gauge field configuration which can be reached from $A_\mu = 0$ by a sequence of infinitesimal deformations $A_\mu \to A_\mu + \delta A_\mu$ in such a way that none of the intermediate configurations contains zero modes. The deformation δA_μ generates a shift $\delta\lambda_n$ in the eigenvalues. The corresponding change in the phase is given by

$$\delta\phi = \lim_{\epsilon \to 0} (-\epsilon\sqrt{\pi}) \sum_n \delta\lambda_n e^{\epsilon^2 \lambda_n^2} \tag{37}$$

Assume first that the gauge field is periodic, $A_\mu(x+a) = A_\mu(x)$, and use a periodic interpolation between 0 and A_μ. In this case, first order perturbation theory shows that $\delta\lambda_n$ is the expectation value of $\delta\not{A}$ in the state $u_n(x)$ and the sum (37) can therefore be written as a trace involving the heat kernel

$$\delta\phi = \lim_{\epsilon \to 0} (-\epsilon\sqrt{\pi}) \mathrm{Tr}(\delta\not{A} \exp \epsilon^2 \not{D}^2) \tag{38}$$

The heat kernel expansion implies that the limit $\epsilon \to 0$ is a local polynomial determined by the coefficient

$$H_1(x,x) = \frac{i}{2}\gamma_\mu\gamma_\nu F_{\mu\nu}(x) \tag{39}$$

The polynomial coincides with the corresponding change in the value of the Chern-Simons action. We conclude that for those field configurations which are periodic and which can continously be reached from $A_\mu = 0$ without encountering zero modes, the limit $m \to 0$ of the phase is given by the Chern-Simons action,

$$\phi = I \tag{40}$$

Let us now relax the periodicity condition and consider a gauge field A_μ which differs from a periodic one, B_μ, by a gauge transformation U. The phase is gauge invariant and for periodic fields (subject to the provisos mentioned above) it coincides with the Chern-Simons action. Hence we get

$$\begin{aligned}\phi &= \frac{1}{8\pi}\int_V \mathrm{tr}\{BdB + \frac{2i}{3}B^3\} \\ &= \frac{1}{8\pi}\int_V \mathrm{tr}\{AdA + \frac{2i}{3}A^3 - \frac{i}{3}\omega^3\} + \frac{1}{8\pi}\int_{\partial V}\mathrm{tr}(A\omega)\end{aligned} \tag{41}$$

where I have made use of the differential one-forms $A = A_\mu dx^\mu, B = B_\mu dx^\mu$ and $\omega = iU^+\partial_\mu U dx^\mu$ to simplify the notation. Although the r.h.s. involves a volume integral over U, it actually only depends on the values of U at the boundary ∂V. The phase cannot be expressed in terms of A_μ alone, however - it also depends on the matrix $U_a(x)$ which relates the boundary values of the Fermi field on opposite sides of the surface ∂V. The result shows that for gauge fields which can continuously be deformed into $A_\mu = 0$ without encountering zero modes, the phase of the massless fermion determinant represents a gauge invariant generalization of the Chern-Simons action.

Generalizing further, let us now consider gauge fields for which the deformation into $A_\mu = 0$ necessarily gives rise to zero modes. [Constant gauge fields, $A_\mu =$ const., belong to this category if they are sufficiently strong, as can be seen by explicitly working out the corresponding eigenvalues of the Dirac operator.] Assume that only one of the eigenvalues, λ_r, goes through zero as the gauge field A_μ passes through the configuration A^0_μ. All terms in the sum (36) are then continuous there, except for the term $n = r$ whose contribution jumps from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ or vice versa. When A_μ passes through A^0_μ, the phase therefore jumps by $\pm\pi$. Denoting by N_+ the number of positive eigenvalues which become negative by the time the field reaches $A_\mu = 0$ and by N_- the number of those which do the opposite, we obtain

$$\phi = \pi(N_+ - N_-) + I \tag{42}$$

valid if the field is periodic (if this is not the case, the Chern-Simons action I must be replaced by its gauge-invariant extension, as discussed above).

Conclusion

(i) The effective action is an unambiguous notion only up to a local polynomial of the gauge field.

(ii) The polynomial can be chosen in such a way that the effective action respects all of the symmetries of the Dirac operator. Since gauge invariant local polynomials of dimension less than or equal to three do not exist, the result is unique.

(iii) If the fermion mass tends to infinity, the effective action tends to zero.

(iv) In the opposite extreme of massless fermions, the real part of the effective action remains a nonlocal functional involving arbitrarily high powers of the gauge field, but the imaginary part ϕ can be calculated in closed form. For gauge fields of compact support, ϕ coincides with the Chern-Simons action. If the gauge field instead lives on a torus, the Chern-Simons action is replaced by a gauge invariant functional involving both the gauge field and the gauge transformation which occurs in the boundary condition for the Fermi field. For sufficiently strong gauge fields, the effective action of massless fermions exhibits discontinuities related to the occurrence of zero modes.

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References

1. A.M. Din, M. Makowka and G. Wanders, *Nucl. Phys.* **B241** (1984) 613;
M. Makowka and G. Wanders, *Phys. Lett.*, **B164** (1985) 107: *Helv. Phys. Acta* **59** (1986) 1366;
D. Cangemi, M. Makowka and G. Wanders, *Phys. Lett.* **211** (1988) 107; *Comm. Math. Phys.* **121** (1989) 421; *Nucl. Phys.* **330** (1990) 205;
M. Makowka, Thèse, Université de Lausanne, 1988;
T. Gallay, Travail de diplôme, Ecole Polytechnique Fédérale de Lausanne, 1989.

2. A.N. Redlich, *Phys. Rev. Lett.* **52** (1984) 18; *Phys. Rev.* **D29** (1984) 2366;
S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975; *Ann. Phys. (N.Y.)* **140** (1982) 372;
S. Ojima, *Prog. Theor. Phys.* **81** (1989) 512.

3. J. Schwinger, *Phys. Rev.* **82** (1951) 664;
P.B. Gilkey, The index theorem and the heat equation, Publish or Perish 1974.

4. H. Leutwyler and S. Mallik, *Z. Phys. C, Particles and Fields* **33** (1986) 205, Appendix B.