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Orthomodularity, compatibility and commutativity in physical theories

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Abstract. A physical interpretation of the orthomodularity in a lattice-theoretical axiomatics is given in terms of the compatibility relations defined in the paper. A connection between compatibility and commutativity has also been established.

1. Introduction

This paper intends to discuss one of the fundamental axioms of the theory of physical systems, based on the mathematical structure of orthomodular lattices. We have in view the axiom of orthomodularity, which reflects the essential property of orthomodularity, which an ortholattice must possess in order to be a possible physical theory.

Let us begin with the definition of an ortholattice.

Definition 1.

a) Given $(T, \leqslant)$ a partially ordered set (" $\leqslant$ " is an order relation on T), such that there exist the lowest and the greatest elements of T, denoted respectively by 0 and 1; a mapping $T \ni a \mapsto a^{\perp} \in T$ is called on orthocomplementation on T if it has the following properties:

(i) $(a^{\perp})^{\perp} = a$ for all $a \in L$;
(ii) $\sup\{a, a^{\perp}\} = 1$, $\inf\{a, a^{\perp}\} = 0$;
(iii) $a \leqslant b \Rightarrow b^{\perp} \leqslant a^{\perp}$.

b) A triple $(T, \leqslant, \perp)$, where $(T, \leqslant)$ is a partially ordered set and "$\perp$" an orthocomplementation on T is called an orthoposet.

c) An orthoposet $(L, \leqslant, \perp)$ having the property $a, b \in L \Rightarrow a \vee b \equiv \sup\{a, b\} \in L$ is called an ortholattice.

We will not discuss here whether a physical theory may be indeed described by ortholattice. This problem has been discussed in many papers and became a common subject of the more or less standard works refering to the foundations of quantum mechanics or of physical theories [1–4]. The aim of this article is to give a physical interpretation/justification of the fact that any ortholattice describing a physical theory must be orthomodular. In order to avoid any confusion, we will explain what we mean by physical interpretation of a mathematical object.

We admit that a mathematical object (which usually has not a direct physical interpretation) *is physically significant if another mathematical object may be defined which has a satisfactory physical interpretation and whose existence in a theory implies the existence of the first object.* The object which must be interpreted here is the orthomodularity of an ortholattice L (which can be defined as a relation on L). We will show that, if L is considered as being a physical theory, then we may assume that a relation called compatibility and having a good enough physical significance (see Paragraph 2) may be defined—and therefore exists—on L. Then it will be proved that the existence of a compatibility on L implies the orthomodularity of L (see Proposition from Paragraph 2).

We want to expose now the empirical basis for defining a physically justified notion of compatibility. To this purpose, the usual notion of "yes-no" experiment (or question) will be slightly changed in a sense which will become clear in what follows. For the sake of convenience let us consider one of the standard definitions of a question: ⟨⟨We shall call a *question* any experiment leading to an alternative whose terms are "yes" and "no"⟩⟩ [3]. We will prefer to use the term *test* instead of question.

The comment of the above "definition" begins with the obvious fact that the performance of a fundamental experiment—a "yes-no" experiment—assumes implicitly that a statement which must be verified had been formulated, like "the absolute temperature takes a value in the interval [0, 1]", or "there exists an electron in the domain D of the physical space", etc. Then it is also clear that a statement may be verified if and only if we are able to find an experimental procedure which permits to decide if the considered statement is true or not, an assertion which is also implicitly contained in the above cited "definition". Consequently, we will prefer to "define" a test as an object consisting of two parts: a propositional part, which is the statement which must be verified, and the experimental procedure which decides effectively whether the statement in discussion is true or not. Further, it would be perhaps more rigorous to consider that the propositional part of a test is a set of logically equivalent statements. Concerning the second "component" of a test, we have to notice that any experiment which "measures" a proposition/statement assumes implicitly the existence of a contact of an apparatus with the system. Of course, in any theory it is assumed that such a contact is ideal, or "free of errors". Even if the ideality of a contact is difficult to define, it is intuitively acceptable, so that assumptions concerning ideality of measurements are always more or less explicitly present in theoretical works. It is also supposed that a contact may measure a proposition in any of the states which are considered interesting in a given theory. This is also an idealization which expresses the "independence" of a measurement on some theoretically nonessential experimental details. Such ideal contacts will be simply called contacts in our paper. We might assume that there exists a unique contact for measuring a given measurable proposition, but we believe that such a hypothesis is not reasonable from the empirical point of view. Indeed, we know that several essentially different concrete procedures may exist to measure a given physical quantity and, in our opinion, this empirical observation requires that, in general, more than one contact may measure a given proposition.

Taking account of these considerations we give the following "definition" of a test: *a test e is a pair* $(P_e, \kappa(e))$, *where* P_e *is a set of logically equivalent propositions and* $\kappa(e)$ *a set of contacts for measuring any proposition from* P_e. It is important to realize that any test may be considered as in independent object, i.e. an object which may be thought of as being independent on any theory. Consequently, in our language a theory would be interpreted always as a collection of tests which may be organised at least as an ortholattice.

Reconsidering now our "definition" of a test e as a pair $(P_e, \kappa(e))$, let us remark that it is in fact a reformulation of the standard "definition" of a question, which had been the start point of our considerations. We have chosen this unusual manner of defining a test since it is very useful when the mathematical properties of a compatibility must be physically interpreted. We will "define" here the so-called empirical compatibility, which will be used in Paragraph 2 for defining the mathematical notion of compatibility on our ortholattice. *Given* $e_1 \equiv (P_{e_1}, \kappa(e_1))$, $e_2 \equiv (P_{e_2}, \kappa(e_2))$ *two tests, we say that they are empirically compatible if* $\kappa(e_1) \cap \kappa(e_2) \neq \varnothing$ (here $\varnothing$ denotes the empty set and "$\cap$" the intersection of sets). This statement affirms that there exists at least one contact which allows to measure both P_{e_1} and P_{e_2} propositions by an unique single-measurement (i.e. simultaneously). It is necessary to obtain both answers by a single-measurement, since we know that the results obtained by measuring a proposition, or a physical quantity, on different copies of a given state are generally different.

Now the notion of empirical compatibility will be used to make some remarks which will suggest finally that any set of mutually empirically compatible tests may be considered as a subset of a Boolean algebra of tests. To be more precise, let us take a set E of mutually empirically compatible tests and let us take $e \in E$ an element of E. It is easy to see that the proposition non-P_e (the negation of the proposition P_e) is measurable, since it may be measured by any contact from $\kappa(e)$. Consequently it defines a test which will be denoted by e'. It follows that the set E may be expanded to the set $E \cup E' \equiv \bar{E}(E' = \{e'; e \in E\})$ which also consists of mutually empirically compatible tests. Now let $e_1, e_2 \in E$ be two tests and consider the proposition $P_{e_1} \wedge P_{e_2}$ to be true if and only if both P_{e_1} *and* P_{e_2} are true. Similarly, a proposition $P_{e_1} \vee P_{e_2}$, which is true if and only if P_{e_1} is true *or* P_{e_2} is true, may be defined. Obviously, both $P_{e_1} \wedge P_{e_2}$ and $P_{e_2} \vee P_{e_1}$ are measurable since $\kappa(e_1) \cap \kappa(e_2) \neq \varnothing$ and any contact from this intersection is good for measuring these propositions. These observations *suggest* (but to not prove) that all propositions constructed by applying repeatedly the algebraic operations " $\wedge$ " and " $\vee$ " to the elements of the set E are also measurable. This construction leads evidently to a Boolean algebra of tests ξ, having an orthocomplementation a natural extension of the mapping $\bar{E} \ni e \mapsto e' \in \bar{E}$. Given $e_1, e_2 \in \xi$ two tests, we will write $e_1 \to e_2$ (and usually say that e_1 "implies" e_2) if the proposition "P_{e_1} is true and P_{e_2} is false" is false (note that this proposition is measurable!). It is clear that if $e_1 \to e_2$ and $e_2 \to e_1$

imply $e_1 = e_2$, then "$\rightarrow$" is an order relation on ξ. Moreover, the order relation "$\rightarrow$" is that which is strictly related to the structure of the ortholattice defined on ξ by the operations "$\wedge$", and "$\vee$" and the mapping $e \mapsto e'$ (see Definition 1). These facts will be considered in Paragraph 2 as a satisfactory direct justification of the properties (C2) and (C3) of a compatibility.

As it will be seen, a compatibility is a mathematical object defined by some conditions inspired by the properties of the empirical compatibility. Since the existence of a compatibility on an ortholattice T implies the orthomodularity of T, it is important—in our opinion—to establish the connection between a compatibility given in Definition 2 and the commutativity relation (see Paragraph 2), which is commonly considered as describing the compatibility of tests. In Paragraph 3 an interesting result is obtained, which ensures us that on orthomodular atomic lattices the only compatibility which may be defined is the commutativity relation.

2. Compatibility and orthomodularity

Let $(L, \leqslant, \perp)$ be an ortholattice. We know that L is said to be orthomodular if the implication "$a \leqslant b \Rightarrow \exists c, (c, a)\perp, b = c \vee a$" is valid (here $(c, a)\perp$ means that the elements a, c are orthogonal, i.e., $a \leqslant c^{\perp}$). It is also well-known that any theory $(L, \leqslant, \perp)$ is supposed to be an orthomodular lattice. Nevertheless, the orthomodularity is considered by many authors as a purely technical condition. We will show here that it is possible to obtain a good interpretation of orthomodularity if we consider that compatibility enters from the beginning as a fundamental object in the mathematical apparatus of any theory. More precisely, if an ortholattice $(L, \leqslant, \perp)$ is a possible physical theory, then there must exist a relation C on L ($C \subseteq L \times L$) which represents the mathematical description of the empirical compatibility. In order to set up the properties of a compatibility on L, we need some notions concerning the general properties of a relation on an arbitrarily given set. Given $R \subseteq A \times A$ a relation on A, we will write $(a, b)R$ instead of $(a, b) \in R$. Given R a reflexive and symmetric relation on A, we say that $B \subseteq A$ is a R-class if it is maximal with the property $a, b \in B \Rightarrow (a, b)R$. The existence of R-classes is a simple consequence of the Zorn's lemma.

Taking into account the interpretation of the elements of L as tests and the "definition" of empirical compatibility, it is easy to see that C must be a reflexive and symmetric relation. It is also clear that the following assertions may be assumed to hold for any compatibility:

(C1) $(a, b)C \Rightarrow (a, b^{\perp})C$;

(C2) *if* $A \subseteq L$ *is a* C*-class and* $a_i \in A$, $i \in I$, *then* $\bigwedge_{i \in I} a_i \in A$

On any ortholattice L we may define a relation K, called commutativity and defined as follows: $(a, b)K$ if $a = (a \wedge b) \vee (a \wedge b^{\perp})$. Let us consider now $A \subseteq L$ a C-class. By using (C1) it is easy to prove the implication $a \in A \Rightarrow a^{\perp} \in A$. By combining this result with the property (C2), it may be proved that A is an ortholattice. Since the elements of A are mutually compatible, the C-class A may be identified with the lattice of propositional parts of its elements. But it is well-known that any ortholattice of propositions is a Boolean algebra. This reasoning justifies the following property of the compatibility relation:

(C3) *if* $A \subseteq L$ *is a* C*-class, then* $a, b \in A \Rightarrow (a, b)K$.

Therefore, taking account of properties (C1)–(C3), we may affirm that *any* C*-class with the order relation and the orthocomplementary intherited from L is a Boolean algebra.* In order to establish another property of a compatibility relation, we have to discuss the more subtle problem dealing with the possibilities to decide by measurements whether $a \leqslant b$ or not. At first sight, it seems that a and b must be compatible, since the order relation "$\leqslant$" is related to the experimental implication "$\rightarrow$". In fact, there exists a possibility to establish whether $a \leqslant b$ is true or not when a weaker condition than the compatibility of a and b is fulfilled. Indeed, let us assume that, given a and b two tests, there exists a family of tests $\{a_i\}_{1 \leqslant i \leqslant n}$ such that $a_1 = a$, $a_n = b$, $(a_i, a_{i+1})C$ and $a_i \rightarrow a_{i+1}$ for all i, $1 \leqslant i \leqslant n-1$. Obviously, in these conditions we may write $a \leqslant b$. Translating this observation into the mathematical language of ortholattice, we get a new property of the compatibility C:

(C4) *if* $a, b \in L$, $a \leqslant b$, *then there exists a family* $\{a_i\}_{1 \leqslant i \leqslant n}$, $a_1 = a$, $a_n = b$, $(a_i, a_{i+1})C$ *and* $a_i \leqslant a_{i+1}$ *for all* i, $1 \leqslant i \leqslant n-1$.

We consider that the properties (C1) – (C4) define completely a compatibility relation, so that we will give the following definition:

Definition 2. Let $(L, \leqslant, \perp)$ be an ortholattice. A reflexive and symmetric relation $C \subseteq L \times L$ is said to be a compatibility on L if it has the properties (C1)–(C4).

The previous considerations lead us to the idea that any physical theory must be an ortholattice having a compatibility relation defined on it. If this assumption is admitted, then it follows from the following simple proposition that any theory is an orthomodular lattice.

Proposition: *Let* $(L, \leqslant, \perp)$ *be an ortholattice such that there exists a compatibility* $C \subseteq L \times L$. *Then* L *is an orthomodular lattice.*

Proof. Let us take $a, b \in L$, $a \leqslant b$. There exists a finite family $(a_i)_{1 \leqslant i \leqslant n}$, $a_i \in L$ such that $a = a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = b$ and $(a_i, a_{i+1})C$ for all i, $1 \leqslant i \leqslant n-1$. Since a_i, a_{i+1} are elements of a C-class and any C-class is an orthomodular sublattice of L, there exists $c_i \in L$ such that $a_{i+1} = c_i \vee a_i$, $(c_i, a_i)\perp$. The element $c = \bigvee_{i=1}^{n-1} c_i$ has the properties $(a, c)\perp$ and $b = a \vee c$. Indeed, we have $a \leqslant a_i$ for all i, $2 \leqslant i \leqslant n-1$. Since $a_i \leqslant c_i^{\perp}$, it results $a \leqslant c_1^{\perp} \wedge c_2^{\perp} \wedge \cdots \wedge c_{n-1}^{\perp} = (c_1 \vee c_2 \vee \cdots \vee c_{n-1})^{\perp} = c^{\perp}$. It remains to see that

$$b = a_{n-1} \vee c_{n-1} = a_{n-2} \vee c_{n-2} \vee c_{n-1} = \cdots =$$
$$= a \vee c_1 \vee c_2 \vee \cdots \vee c_{n-1} = a \vee c, \qquad \text{Q.E.D.}$$

This proposition shows that orthomodularity of an ortholattice/physical theory L is a consequence of the existence of a compatibility relation on L. Now it is important to find what is the connection between a compatibility C and the commutativity relation K, which is commonly considered as describing the empirical compatibility. This will be done in the next paragraph.

3. Compatibility and commutativity

In the previous paragraph the commutativity relation K on an arbitrarily given ortholattice has been defined. This relation is very important since it may be proved that *$K \subseteq L \times L$ is symmetric if and only if L is an orthomodular lattice* [5]. Consequently, on any orthomodular lattice there exists at least one compatibility, since, by using the previous statement, it is easy to show that K is a compatibility provided L is orthomodular.

Now let us consider that L is orthomodular and let C be a compatibility on L. Then we may prove without difficulty that $C \subseteq K$. It is more interesting when the inclusion $K \subseteq C$ is also true. We will see below this is true when L is atomic. The proof of this statement is based on the following lemma:

Lemma. *Let $(L, \leqslant, \perp)$ be an ortholattice such that there exists $R \subseteq L \times L$ a relation having the following properties*:

(i) *R is reflexive and symmetric*;
(ii) *any R-class is a Boolean orthosublattice of L*;
(iii) *$a \leqslant b \Rightarrow (a, b)R$. Then L is orthmodular and $R = K$.*

Proof. We will prove first that $a \leqslant b \Rightarrow (a, b)R$ if and only if $(a, b)\perp \Rightarrow (a, b)R$. Suppose that the implication $a \leqslant b \Rightarrow (a, b)R$ is true. Then, $(a, b)\perp \Rightarrow a \leqslant b^{\perp} \Rightarrow (a, b^{\perp})R \overset{(ii)}{\Rightarrow} (a, b)R$. Conversely, if the implication $(a, b)\perp \Rightarrow (a, b)R$ is true, then we may write $a \leqslant b \Rightarrow a \leqslant (b^{\perp}) \Rightarrow (a, b^{\perp})\perp \Rightarrow (a, b^{\perp})R \Rightarrow (a, b)R$. Since R has the properties (i)–(iii), it is a compatibility on L, so that L is orthomodular and $R \subseteq K$. Let us consider now $a, b \in L$ such that $(a, b)K$. Then $b = (a \wedge b) \vee (a^{\perp} \wedge b)$. Since $(a \wedge b, a^{\perp} \wedge b)R$, $(a, a \wedge b)R$ and $(a, a^{\perp} \wedge b)R$, there exists a R-class which includes the set $\{a, a \wedge b, a^{\perp} \wedge b\}$ and we get immediately that $(a, b)R$, Q.E.D.

Theorem. *Let $(L, \leqslant, 1)$ be an atomic ortholattice and C a compatibility on L. Then $C = K$.*

Proof. It is sufficient to prove the implication $a \leqslant b \Rightarrow (a, b)C$ (see Lemma). The implication $a \lessdot b \Rightarrow (a, b)C$ (here $a \lessdot b$ means "b covers a", i.e., $a \leqslant x \leqslant b \Rightarrow x = a$ or $x = b$) is true, since $a \lessdot b$ and $(a, b)\not{C}$ (a, b are not in the relation C) implies that there exists $x \in L$, $a < c < b$, a is not covered by b, which is absurd. If p is an atom of L, $a \in L$, $(a, p)^{\perp}$, then $a \lessdot a \vee p$ and $p = (a \vee p) \wedge a^{\perp}$ Taking account of properties (C1)–(C4), we may write the following sequence of implications:

$$a \lessdot a \vee p \Rightarrow (a, a \vee p)C \Rightarrow (a^{\perp}, a \vee p)C$$

$$\Rightarrow (a^{\perp}, (a \vee p) \wedge a^{\perp})C \Rightarrow (a^{\perp}, p)C \Rightarrow (a, p)C.$$

Consider now $a, b > 0$, $(a, b)^{\perp}$. Since an atomic ortholattice is also atomistic, there exists Ω_b a set of mutually orthogonal atoms such that $b = \vee \Omega_b$. Since $(a, \alpha) \perp$ for all $\alpha \in \Omega_b$, we get $(a, \alpha)C$ for all $\alpha \in \Omega_b$. Therefore $\Omega_b \cup \{a\}$ is contained in a C-class, so that $(a, b)C$. It results that the implication $(a, b) \perp \Leftrightarrow (a, b)C$ is true, and applying Lemma, we get $K \subseteq C$. Q.E.D.

4. Comments

We could not find an example of a compatibility of an orthomodular lattice which differs from K. It is also difficult—if not impossible—to prove that the only compatibility is K. Both these problems are important from the mathematical point of view. For the development of a physical theory it is sufficient to justify that the only reasonable solution for describing mathematically the compatibility relation is to identify it with the compatibility K. Suppose we know that for any theory L there exists an atomic orthomodular lattice $\tilde{L}$ and $\varphi: L \rightarrow \tilde{L}$ a mapping such that the following assertions are satisfied:

(a) $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$;
(b) $\varphi(x^{\perp}) = \varphi(x)^{\perp}$ for all $x \in L$;
(c) $x_i \in L, i \in I$ and $\bigwedge_{i I} x_i$ exists in $\tilde{L} \Rightarrow \bigwedge_{i \in I} \varphi(x_i)$ exists in $\tilde{L}$ and $\varphi(\bigwedge_i x_i) = \bigwedge_i \varphi(x_i)$.

It will be seen in another paper that this statement is physically justified. This means that any theory may be enlarged up to an atomic theory. Since, according to Theorem, the only compatibility existing on an atomic theory is the relation K, there are no reasons to believe that the empirical compatibility might be described by a relation which differs from K.

REFERENCES

[1] G. BIRKHOFF and J. VON NEUMANN, Ann. Math., *37*, 823, 1936.
[2] C. PIRON, Helv. Phys. Acta *37*, 439 (1964).
[3] C. PIRON, *Foundations of Quantum Physics*, W. A. Benjamin, Inc., 1976, p. 19.
[4] Dirk Aerts, The One and the Many, Ph.D. Thesis, Vrije Universitait Brussel, 1980–1981.
[5] F. MAEDA and S. MAEDA, *Theory of Symmetric Lattices*, Springer Verlag Berlin, 1970, p. 165.