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A Solvable Model of Two-Channel Scattering

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Abstract

A simple solvable model of two-channel inelastic scattering is constructed. The contact-type interaction is realized mathematically by a self-adjoint extension; if the interchannel interaction is switched off it reduces to one point interaction in each channel. The scattering matrix is calculated and the bound states and resonances are found in the weak-coupling case. The decay of the resonance state is also considered ; it is shown that the asymptotic behaviour of the non-exponential corrections to the decay law depends substantially on the spectrum of the unperturbed system.

1 Description of the model

Multichannel scattering represents one of the most important processes, especially in nuclear physics (for a general overview see [33]). The formalism to describe concrete physical situations, however, is usually not quite simple and some effort is needed to see interesting physical effects like resonances below the threshold of the excited channels. This might be a motivation to study solvable models in which these effects are manifested in a transparent way.

The aim of the present paper is to construct and analyze such a simple model. It has not only the pedagogical value since it is expected to represent a low-energy limit to the two-channel scattering with a spherically symmetric matrix potential, however, we postpone the discussion of this aspect to another publication. Moreover, the model constructed below represents the simplest and generic example in a class of models describing N-channel contact-interaction inelastic scattering.

The method of constructing the interaction is based on the concept of point interaction defined by means of self-adjoint extensions. This idea goes back to Berezin and Faddeev [8] and in the last decade it was studied systematically — see, *e.g.*, [2-4] and the monograph [1] — and generalized by adding an internal structure [34], for interactions supported by surfaces [6, 11, 37, 38], curves and graphs [9, 10, 20, 23, 25], more complicated objects [16-19], for the Dirac operators [10, 12, 13, 26] *etc.* ; many recent results are summarized in [21, 22].

To begin with, suppose we have a pair of non-relativistic particles which can exist in two states ; for definiteness one can imagine a neutron plus a nucleus having two internal states. After separating the centre-of-mass motion, the state Hilbert space is therefore $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $\mathcal{G}_j := L^2(\mathbf{R}^3)$. In view of the Bargmann superselection rule [7, 30] the reduced masses in the two channels are the same ; for simplicity we put $\mu_1 = \mu_2 = \frac{1}{2}$. Our basic assumption is that *the full interaction between the particles*, including the part which is responsible for the transitions between the two channels, *has a contact nature*, i.e., it is supported by the origin of coordinates in the relative-coordinate space only.

Following the standard contact-interaction ideology, we shall construct the class of admissible Hamiltonians from the pair of non-selfadjoint operators

$$H_{0,1} := -\Delta \quad \text{with} \quad D(H_{0,1}) = C_0^\infty(\mathbf{R}^3 \setminus \{0\}),$$

$$H_{0,2} := -\Delta + E \quad \text{with} \quad D(H_{0,2}) = C_0^\infty(\mathbf{R}^3 \setminus \{0\}).$$

Here E is a positive number which represents the *threshold energy of the inelastic channel.* In view of our basic assumption, the pair of particles whose relative-coordinate wavefunction ψ is separated from the origin does not feel the interaction, which means that $H_0 := H_{0,1} \oplus H_{0,2}$ acts on such ψ as the Hamiltonian should act. To specify the dynamics, one has to say what happens when the particles hit each other; it is clear from what we have said that *the Hamiltonian should be chosen among the self-adjoint extensions of* H_0.

These extensions can be easily constructed since the deficiency indices of H_0 are $(2,2)$. It is seen from the partial-wave decomposition ; since this part of the argument is standard [1] we skip the details. Due to the presence of the centrifugal barrier, the component of $H_{0,j}$ in the l-th partial wave is e.s.a. if $l \geq 1$ so *only the s-wave parts can be coupled.* In this way the problem is substantially simplified : passing to the reduced radial wave functions

$f(r) := r\psi(r)$, we take

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \quad \text{with} \quad \mathcal{H}_j := L^2(\mathbf{R}_+) \tag{1}$$

as the state space of the problem, and the "starting operator" will be $h_0 := h_{0,1} \oplus h_{0,2}$, where

$$\begin{aligned} h_{0,1} &:= -\frac{d^2}{dr^2} \quad \text{with} \quad D(h_{0,1}) = C_0^\infty(0,\infty)\,, \\ h_{0,2} &:= -\frac{d^2}{dr^2} + E \quad \text{with} \quad D(h_{0,2}) = C_0^\infty(0,\infty)\,. \end{aligned} \tag{2}$$

The operator h_0 has again deficiency indices $(2,2)$, and therefore a four-parameter family of self-adjoint extensions. The most convenient way to characterize them is through suitable boundary conditions :

1.1 Proposition. To any $u \equiv \{a,b,c\}$ with $a,b \in \mathbf{R}$ and $c \in \mathbf{C}$ denote by h_u the operator defined by the same differential symbol as h_0 with the domain $D(h_u) \subset D(h_0^*)$ specified by the boundary conditions

$$\begin{aligned} f_1'(0) &= af_1(0) + cf_2(0)\,, \\ f_2'(0) &= \bar{c}f_1(0) + bf_2(0)\,. \end{aligned} \tag{3}$$

Then h_u is a self-adjoint extension of h_0.

Proof : An integration by parts in combination with the standard limiting argument yield

$$(h_0^* f, g) = (\bar{f}_1' g_1 + \bar{f}_2' g_2 - \bar{f}_1 g_1' - \bar{f}_2 g_2')(0) - \int_0^\infty (\bar{f}_1 g_1'' + \bar{f}_2 g_2'' - E\bar{f}_2 g_2)(r)\, dr$$

for any $f,g \in D(h_0^*)$; it is simple to check that the corresponding boundary form vanishes under the boundary conditions (3). QED

In addition to the extensions given by the theorem there are some exceptional ones (for instance, those obtained formally by inserting $a = \infty$ or $b = \infty$ into (3)). They can be described by the standard methods (*cf.*[35], Sec.X.1; [14], Sec.XII.4) but for our model purposes it is sufficient to know the "most part" of admissible Hamiltonians.

It is obvious that c is the complex coupling constant between the channels ; if $c = 0$ we have $h_u = h_a \oplus h_b$ where the last named operators correspond to the s-wave parts of the point-interaction Hamiltonians H_α and H_β in the two channels (*cf.* [1], Chap.I.1) with the interaction strengths $\alpha := a/4\pi$ and $\beta := b/4\pi$, respectively. Hence the spectrum of h_u in the case of decoupled channels is easily obtained from the spectra of H_α and H_β ; in Section 3 below we shall discuss $\sigma(h_u)$ in the interacting case.

Let us finally remark that the above specified family of Hamiltonians is restricted further if we demand the interaction to be *time-reversal invariant.* To meet this requirement the domain of the corresponding self-adjoint extension must be invariant with respect to the complex conjugation, *i.e.*, the same must be true for the boundary conditions (3); this is true if *the coupling constant c is real-valued.* In fact one can check easily that h_u and $h_{u'}$ with $a' = a$, $b' = b$ and $c' = ce^{i\alpha}$ are unitarily equivalent by $e^{i\alpha/2} \oplus e^{-i\alpha/2}$.

2 The resolvent

To find the spectrum of h_u one has to determine first its resolvent. The form of our Hamiltonian allows us to do it explicitly.

2.1 Proposition. For any $z \in \varrho(h_u)$, denote $k := \sqrt{z}$. The resolvent $(h_u - z)^{-1}$ is integral operator on $\mathcal{H}$ with the kernel

$$G_u(r,r';k) = \begin{pmatrix} \frac{e^{ik|r+r'|}-e^{ik|r-r'|}}{2ik} & 0 \\ 0 & \frac{e^{i\kappa|r+r'|}-e^{i\kappa|r-r'|}}{2i\kappa} \end{pmatrix} + D^{-1}\begin{pmatrix} (b-i\kappa)\, e^{ik(r+r')} & -c\, e^{i(kr+\kappa r')} \\ -\bar{c}\, e^{i(\kappa r+kr')} & (a-ik)\, e^{i\kappa(r+r')} \end{pmatrix}, \tag{4}$$

where $D := (a-ik)(b-i\kappa) - |c|^2$ and $\kappa := \sqrt{k^2 - E}$.

Proof: For $u_0 \equiv \{a, b, 0\}$ the operator h_{u_0} decouples into an orthogonal sum so its resolvent kernel expresses through the known resolvent kernels of the point-interaction Hamiltonians h_a and h_b as

$$G_{u_0}(r,r';k) = \begin{pmatrix} g_a(r,r';k) & 0 \\ 0 & g_b(r,r';\kappa) \end{pmatrix},$$

where

$$g_a(r,r';k) = \frac{e^{ik|r+r'|} - e^{ik|r-r'|}}{2ik} + \frac{e^{ikr}e^{ikr'}}{a-ik}$$

and g_b is obtained by replacing a, k with b, κ, respectively. For the general case, the resolvent is given by Krein formula (*cf.* [1], Appendix A) : its kernel equals

$$G_u(r,r';k) = G_{u_0}(r,r';k) + \sum_{j,l=1}^{2} \lambda_{jl}(k^2)\, F_j^k(r) F_l^k(r')^t, \tag{5}$$

where t means transposition and the 2×1 columns F_j^k can be chosen as

$$F_1^k(r) := \begin{pmatrix} g_a(r,0;k) \\ 0 \end{pmatrix}, \quad F_2^k(r) := \begin{pmatrix} 0 \\ g_b(r,0;\kappa) \end{pmatrix}.$$

This choice ensures that the F_j^k solve the equation $(h_0^* - k^2)F_j^k = 0$, *i.e.*, they belong to the deficiency subspaces of the operator h_0. It remains to find the functions $\lambda_{jl}(k^2)$. We use the fact that $(h_u - z)^{-1}$ maps $\mathcal{H}$ into $D(h_u)$ which means that for any $f \in \mathcal{H}$, the component functions of $(h_u - z)^{-1}f$ must fulfill the boundary conditions (3). After a short calculation this requirement yields a system of four linear equations,

$$-\lambda_{11}(k^2) = \frac{c}{b-i\kappa}\lambda_{21}(k^2), \quad -\lambda_{21}(k^2) = \bar{c}\left(1 + \frac{\lambda_{11}(k^2)}{a-ik}\right),$$

$$-\lambda_{12}(k^2) = c\left(1 + \frac{\lambda_{22}(k^2)}{b-i\kappa}\right), \quad -\lambda_{22}(k^2) = \frac{\bar{c}}{a-ik}\lambda_{12}(k^2),$$

which is readily solved ; substituting into (5), we get the result. QED

3 Spectral properties

The explicit form of the resolvent makes it possible to study its singularities. In fact, it is only necessary to find zeros of the "discriminant" in the resolvent kernel. To see this, notice that for $z \in \mathbf{C} \setminus \mathbf{R}_+$, the resolvent can be writen as $(h_u - z)^{-1} = A(z) + D^{-1}(z)B(z)$, where the two operators corresponding to the matrices in (4) are bounded so that they cannot give rise to a pole. On the other hand, one can check directly from (2,3) that h_u has no embedded eigenvalue unless $c = 0$. Of course, the resolvent has also cuts associated with the continuous spectrum which come from the square roots in definitions of k and κ, but they are independent of the interaction.

Consider first the decoupled case, $c = 0$. The discriminant then factorizes so $D = 0$ holds *iff* $k = -ia$ or $\kappa = -ib$. The spectrum is the union of spectra of the two point-interaction operators, the second one of which is shifted by the constant E. One has to distinguish several cases :

$a < 0$ ... h_u has the eigenvalue $-a^2$ corresponding to the normalized eigenfunction $f(r) = \sqrt{-2a}\,(e^{ar}, 0)$

$a \geq 0$ ... the pole corresponds now to a zero-energy resonance ($a = 0$) or to an anti-bound state (*i.e.*, a resonance on the negative imaginary axis of k — hidden deeply on the second sheet of energy)

$b < 0$ ... h_u has the eigenvalue $E - b^2$ corresponding to $\sqrt{-2b}\,(0, e^{br})$

$b \geq 0$ ... h_u has a zero-energy resonance or an anti-bound state

The continuous spectrum of the decoupled operator is again union of the continuous spectra of h_a and h_b, *i.e.*, it covers the interval $[0, \infty)$ being of multiplicity two in $[0, E)$ and of multiplicity four in $[E, \infty)$.

We shall consider all the above listed cases (though we avoid mostly discussion of the situations including the zero-energy resonances) but our main interest concerns the cases when both a, b are negative. Among them, we want to pay a particular attention to the situation when $b^2 < E$, *i.e.*, the eigenvalue of h_b is embedded into the continuous spectrum of h_a.

After this preliminary, let us turn to the interacting case, $c \neq 0$. Since the deficiency indices of h_0 have been finite, the essential spectrum is not affected by the interaction (*cf.*[39], Theorem 8.8). To find the eigenvalues (or resonances) of h_u, one has to solve the equation

$$(a - ik)\left(b - i\sqrt{k^2 - E}\right) = |c|^2. \tag{6}$$

It reduces to a quartic equation, and therefore can be solved in terms of radicals, however, the formulae expressing the roots for general a, b and c are too complicated to be of any use (each of them requires more than one page). We shall concentrate therefore our attention on the weak-coupling case in which the equation (6) can be solved perturbatively ; at the same time it represents physically the most interesting situation.

3.1 Theorem. (a) Let the point spectrum of the decoupled operator h_{u_0} be non-degenerated,

$-a^2 \neq E - b^2$, then the perturbed first-channel eigenvalue (resonance) behaves for small $|c|$ as

$$e_1(c) = -a^2 + \frac{2a|c|^2}{b+\sqrt{a^2+E}} + \frac{a^2-E-b\sqrt{a^2+E}}{\sqrt{a^2+E}\left(b+\sqrt{a^2+E}\right)^3}|c|^4 + \mathcal{O}(|c|^6)\,. \tag{7}$$

In particular, the zero-energy resonance turns into an antibound state if h_{u_0} has an isolated eigenvalue in the second channel, $b < -\sqrt{E}$, and into a bound state otherwise.
(b) Under the same assumption, let h_{u_0} have an isolated eigenvalue in the second channel, $b < -\sqrt{E}$, then the perturbation changes it into

$$e_2(c) = E - b^2 + \frac{2b|c|^2}{a+\sqrt{b^2-E}} + \frac{b^2+E-a\sqrt{b^2-E}}{\sqrt{b^2-E}\left(a+\sqrt{b^2-E}\right)^3}|c|^4 + \mathcal{O}(|c|^6)\,. \tag{8}$$

(c) Under the same assumption, suppose now that h_{u_0} has an embedded eigenvalue, $-\sqrt{E} < b < 0$, then it turns under the perturbation into a second-sheet pole with

$$\operatorname{Re} e_2(c) = E - b^2 + \frac{2ab|c|^2}{a^2-b^2+E} + \mathcal{O}(|c|^4)\,, \tag{9}$$

$$\operatorname{Im} e_2(c) = \frac{2b|c|^2\sqrt{E-b^2}}{a^2-b^2+E} + \mathcal{O}(|c|^4)\,. \tag{10}$$

(d) Finally, suppose that h_{u_0} has a degenerate eigenvalue, $b = -\sqrt{a^2+E}$. Under the perturbation it splits into

$$e_{1,2}(c) = -a^2 \mp 2\sqrt{-a}\sqrt[4]{a^2+E}\,|c| + \frac{2a^4+4a^2E+E^2}{2a(a^2+E)^{3/2}}|c|^2 + \mathcal{O}(|c|^3)\,. \tag{11}$$

Proof : One can rewrite the spectral condition as

$$(\alpha - iz)\left(\beta - i\sqrt{z^2-1}\right) = |\gamma|^2\,, \tag{12}$$

where $\alpha := a/\sqrt{E}$ and β, γ, z are similarly renormalized a, b, k. To prove the assertion (a), we use the Ansatz

$$z = -i\alpha + \sum_{n=1}^{\infty} c_n|\gamma|^{2n}\,. \tag{13}$$

Substituting it into (12) and expanding the *lhs* in powers of $|\gamma|^2$, we get an infinite set of equations,

$$\begin{aligned} -ic_1\left(\beta+\sqrt{\alpha^2+1}\right) - 1 &= 0\,,\\ -ic_2\left(\beta+\sqrt{\alpha^2+1}\right) + \frac{\alpha c_1^2}{\sqrt{\alpha^2+1}} &= 0\,,\\ -ic_3\left(\beta+\sqrt{\alpha^2+1}\right) + \frac{2\alpha c_1c_2}{\sqrt{\alpha^2+1}} + \frac{ic_1^3}{2\left(\alpha^2+1\right)^{3/2}} &= 0\,, \end{aligned} \tag{14}$$

etc., which can be solved recursively. Returning to the original quantities, we have

$$\begin{aligned} k_1 &= -ia + \frac{i|c|^2}{b+\sqrt{a^2+E}} + \frac{ia|c|^4}{\sqrt{a^2+E}\left(b+\sqrt{a^2+E}\right)^3} \\ &\quad -i\,\frac{bE+(E-4a^2)\sqrt{a^2+E}}{2\sqrt{E}(a^2+E)^{3/2}\left(b+\sqrt{a^2+E}\right)^4}\,|c|^6 + \mathcal{O}(|c|^8) \end{aligned} \tag{15}$$

so taking the square we arrive at (7). Notice that k_1 is purely imaginary ; if $a=0$ its sign for small $|c|$ is given by the denominator of the second term on the *rhs.* The assertions (b),(c) are proved in a similar manner ; we get

$$\begin{aligned} k_2 &= i\sqrt{b^2-E} - \frac{ib|c|^2}{\sqrt{b^2-E}\left(a+\sqrt{b^2-E}\right)} \\ &\quad -i\,\frac{aE+(2b^2+E)\sqrt{b^2-E}}{2(b^2-E)^{3/2}\left(a+\sqrt{b^2-E}\right)^3}\,|c|^4 + \mathcal{O}(|c|^6) \end{aligned} \tag{16}$$

leading to (8) and

$$\begin{aligned} k_2 &= \sqrt{E-b^2} + \frac{b|c|^2}{\sqrt{E-b^2}\left(a-i\sqrt{E-b^2}\right)} \\ &\quad + \frac{i(2b^2+E)\sqrt{E-b^2}-aE}{2(E-b^2)^{3/2}\left(a-i\sqrt{E-b^2}\right)^3}\,|c|^4 + \mathcal{O}(|c|^6)\,, \end{aligned} \tag{17}$$

respectively. In the last named case k_2 is no longer purely imaginary and its square equals

$$e_2(c) = E - b^2 + \frac{2b|c|^2}{a-i\sqrt{E-b^2}} + \frac{i(b^2+E)-a\sqrt{E-b^2}}{\sqrt{E-b^2}\left(a-i\sqrt{E-b^2}\right)^3}\,|c|^4 + \mathcal{O}(|c|^6)\,; \tag{18}$$

the real and imaginary parts are up to the order $\mathcal{O}(|c|^4)$ given by (9,10). The resolvent of h_u is analytic outside the real axis so the pole specified by (18) is not on the first sheet but can be reached by analytic continuation ; notice that the imaginary part (10) is negative.

Finally, in the case (d) the perturbation series (7,8) lose meaning. We replace therefore the Ansatz (13) by

$$z = -i\alpha + i\sum_{k=1}^{\infty} b_k|\gamma|^k\,; \tag{19}$$

substitution into (12) yields a system of equations which can be again solved recursively, however, for the odd coefficients we get now two solutions differing in sign,

$$k_{1,2} = -ia \pm i\,\frac{\sqrt[4]{a^2+E}}{\sqrt{-a}}\,|c| - \frac{iE^2|c|^2}{4a^2(a^2+E)^{3/2}} \pm i\,\frac{\sqrt{-a}E(5E-8a^2)}{32a^4(a^2+E)^{5/4}}\,|c|^3 + \mathcal{O}(|c|^4) \tag{20}$$

For both signs and $|c|$ small enough, this value is negative imaginary, and its square yields (11). QED

Due to (a),(b) the interaction always *repels* the eigenvalues ; in the case (d) it removes the degeneracy. The appearance of the powers of $|c|$ instead of $|c|^2$ here is quite natural, because the original eigenvalue was twice degenerated so the expansion (11) is nothing else than a few first terms of the corresponding Puiseaux series – *cf.*[28], Chap.II.

3.2 Remark. We have mentioned that the continuous spectrum of h_u is the same as in the decoupled case. One can check, moreover, that it is again absolutely continuous. It is a simple exercise to express $(f,(h_u-z)^{-1}f)$ from (4) and to check, *e.g.*, that for $f_j \in C_0^\infty(0,\infty)$ it has a finite limit as z approaches $\mathbf{R}_+$ from above, except possibly at $z = 0, E$ and the embedded eigenvalue $E-b^2$ (for $c=0$), and that the limit is a continuous function of z. For any interval $I \subset (0,\infty)$ not containing the above named points, $E_{h_u}(I)f \in \mathcal{H}_{ac}$ by Theorem XIII.9 of [35], and by density argument, $I \subset \sigma_{ac}(h_u)$ so h_u has no singularly continuous spectrum.

4 The scattering matrix

Consider now the scattering problem in our model. Since the interaction represents a finite-rank perturbation to the free resolvent, existence and completeness of the wave operators follow from Kato-Birman theory – *cf.* [35], Theorem XI.9. Furthermore, Remark 3.2 shows that our scattering problem is also asymptotically complete.

More interesting is, however, the explicit form of the S-matrix. One can find it easily in the time-independent framework : it is sufficient to take the wave function

$$f(r) = \begin{pmatrix} e^{-ikr} - A\,e^{ikr} \\ B\,e^{i\kappa r} \end{pmatrix}. \tag{21}$$

which is *locally* square integrable. Substituting it to the boundary conditions (3), we get a system of linear equations which is solved by

$$A \equiv S_0(k) = \frac{(a+ik)(b-i\kappa)-|c|^2}{D}, \tag{22}$$

where D is the same as in Proposition 2.1, and

$$B = \frac{2ikc}{D} \tag{23}$$

Consider first the case when $k^2 \le E$, *i.e., the inelastic channel is closed.* A simple calculation shows then that $|A| = 1$, and since the reflection amplitude A is (with the choice of sign we made in (21)) just the s-wave part of the first-channel scattering matrix (or more exactly, its on-shell component – *cf.*[5], Section 5-7), we have checked that the *elastic scattering is unitary for* $k^2 \le E$. Furthermore, one has $S_0(k) = e^{2i\delta_0(k)}$ so another simple calculation yields the phase shift

$$\delta_0(k) = \arctan\frac{k(b-i\kappa)}{a(b-i\kappa)-|c|^2} \pmod{\pi}. \tag{24}$$

Notice that if the channels are decoupled, the above expression reduces to the known formula for the point-interaction phase shift in the first channel – cf.[1], p.38.

The most interesting situation is when h_{u_0} has an embedded eigenvalue which turns under the interaction into a resonant state whose lifetime is

$$T(c) = -\frac{a^2-b^2+E}{4b|c|^2\sqrt{E-b^2}}\,(1+\mathcal{O}(|c|^2)) \tag{25}$$

by Theorem 3.1c (in the next section, we shall analyze the decay problem in detail). At the same time, the phase shift should exhibit a peculiar sheer change. Since κ is purely imaginary in this case, we may rewrite the expression (24) as

$$\delta_0(k) = \arctan\frac{k(b+\sqrt{E-k^2})}{a(b+\sqrt{E-k^2})-|c|^2}\,. \tag{26}$$

If there is a bound state in the first channel, $a<0$, we have $\delta_0(k)<0$ in the interval $(0,\sqrt{E-b^2})$ with zero value at its endpoints. Furthermore, $\delta_0(k)$ reaches the value $\frac{\pi}{2}$ at $k_0 := \sqrt{E-(|c|^2a^{-1}-b)^2}$ provided $|c|^2<ab$; in the opposite case the argument of the arctan in (26) which we denote as $f(k)$ has no singularity at all in $(0,\sqrt{E})$. For a,b fixed and $|c|$ small enough, the phase-shift plot is composed of the background $\arctan(k/a)$ and the jump to the next branch of this function around the singularity. The energies at which $\delta_0(k)=\pm\frac{\pi}{4}$, *e.g.*, are

$$\lambda_\pm = E-b^2+\frac{2b|c|^2}{a^2-b^2+E}(a\pm\sqrt{E-b^2})+\mathcal{O}(|c|^4)$$

which corresponds to (9),(10).

On the other hand, the jump may be substantially deformed, even for small $|c|$, if $|a|$ is small enough. As an example, take an arbitrarily small $|c|$ and $a:=|c|^2/b$. If $E-b^2\ll E$, $\delta_0(\cdot)$ increases steadily from small negative values at small k to

$$\delta_0(k) = \frac{\pi}{2}-\frac{a\sqrt{E-k^2}}{bk}+\mathcal{O}(|c|^4)$$

around $k=\sqrt{E}$.

One can also see the resonance behaviour in the cross section. Since the scattering is obviously isotropic, we have $\sigma_{tot}=\sigma_{l=0}=\frac{4\pi}{k^2}\sin^2\delta_0$. In the embedded-eigenvalue case therefore the relation (26) yields for $k^2\le E$ the following expression

$$\sigma_{tot} = \frac{4\pi\left(b+\sqrt{E-k^2}\right)^2}{\left[a\left(b+\sqrt{E-k^2}\right)-|c|^2\right]^2+k^2\left(b+\sqrt{E-k^2}\right)^2}\,. \tag{27}$$

For a,b fixed and $|c|$ small enough, the cross section varies rapidly (has a dip followed by a peak) in the region of the width $\approx T(c)^{-1}$ around the value $k=k_0$. In a similar way, one can calculate other scattering quantities in the considered situation, as the time delay given by (22) and the Eisenbud-Wigner formula (*cf.*[5], Section 7-2 ; [32]) *etc.*

Finally, let us turn to the case $k^2 > E$. The reflection and transmission amplitudes then satisfy

$$|A|^2 + \frac{\kappa}{k}|B|^2 = 1 . \tag{28}$$

This relation expresses conservation of probability flow : it is easy to see that its radial components $J_i := 2\,\mathrm{Im}\,\bar{f}_i f_i'$ are

$$J_1 = -2k\left(1 - |A|^2\right) \quad \text{and} \quad J_2 = 2\kappa|B|^2$$

in the first and second channel, respectively. The elastic scattering is now non-unitary since $|A|^2 < 1$ due to (23) and (28). The last mentioned relation can be also expressed as

$$|S_{0,1\to1}(k)|^2 + |S_{0,1\to2}(k)|^2 = 1 ,$$

it means, as a part of the full two-channel S-matrix unitarity condition.

5 Decay of the resonant state

Now we want to examine in more detail how the resonant state arising from the embedded eigenvalue of h_{u_0} looks like and how it decays. The most natural choice for the "compound nucleus" wavefunction is the eigenstate of the unperturbed Hamiltonian (*cf.* [27] ; we shall return to this problem at the end of the section),

$$f \;:\; f(r) = \sqrt{-2b}\begin{pmatrix} 0 \\ e^{br} \end{pmatrix} ; \tag{29}$$

recall that $-\sqrt{E} < b < 0$. To find how this state decays, one has to compute the reduced resolvent $(f, (h_u - k^2)^{-1} f)$ and analyze its analytic structure – *cf., e.g.,*[15], Section 3.1. Using Proposition 2.1, one gets after a simple integration

$$(f, (h_u - k^2)^{-1} f) = \frac{|c|^2 + (a - ik)(b + i\kappa)}{(b + i\kappa)^2[|c|^2 - (a - ik)(b - i\kappa)]} . \tag{30}$$

Notice first that the function on the *rhs* corresponds to a four-sheeted Riemann surface with respect to the complex energy $z = k^2$ (for brevity, we shall speak about the z-plane): it has cuts along the intervals $[0, \infty)$ and $[E, \infty)$ on the positive real axis corresponding to the continuous spectrum as indicated on Fig.1. In the momentum plane (or k-plane) we have a two-sheeted surface with the cuts along $(-\infty, -\sqrt{E}\,]$ and $[\sqrt{E}, \infty)$ coming from the square root in the definition of κ.

The other singularities of the function under consideration are the bound-state or resonance poles we have discussed in Section 3 ; they are also indicated on Fig.1. Notice that there is one more pole here: in the k-plane it is a mirror image of the resonace pole on the second sheet as we can see easily from the spectral equation (6). The choice between them is not difficult since in the decoupled case, $\kappa = -ib$ must be positive imaginary (remember that $b < 0$) which is true for the first-sheet pole.

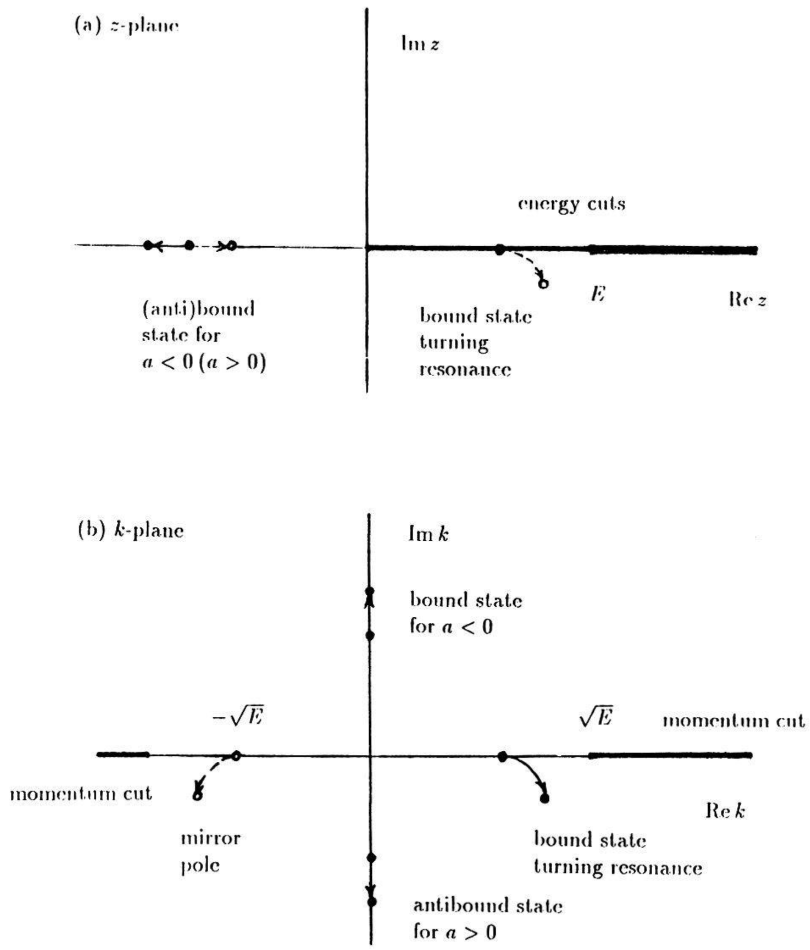

Figure 1

Analytic structure of the reduced resolvent: motion of the poles with respect to the coupling constant is indicated (for $a > 0$ the resonance pole moves down left)

Now we want to compute the reduced evolution operator of the resonant state which acts in the present case as multiplication by $v_u(t) := (f, e^{-ih_u t}f)$ on the one-dimensional subspace of $\mathcal{H}$ generated by f. To this end, one has to integrate

$$v_u(t) = \frac{1}{2\pi i} \int_\Gamma e^{-izt} (f, (h_u - z)^{-1} f)\, dz \tag{31}$$

along the curve Γ indicated on Fig.2 ; we have to consider only $t \geq 0$ since $v_u(-t) = \overline{v_u(t)}$. Alternately, one has

$$v_u(t) = \frac{1}{\pi i} \int_\Gamma e^{-ik^2 t} (f, (h_u - k^2)^{-1} f)\, k\, dk\,, \tag{32}$$

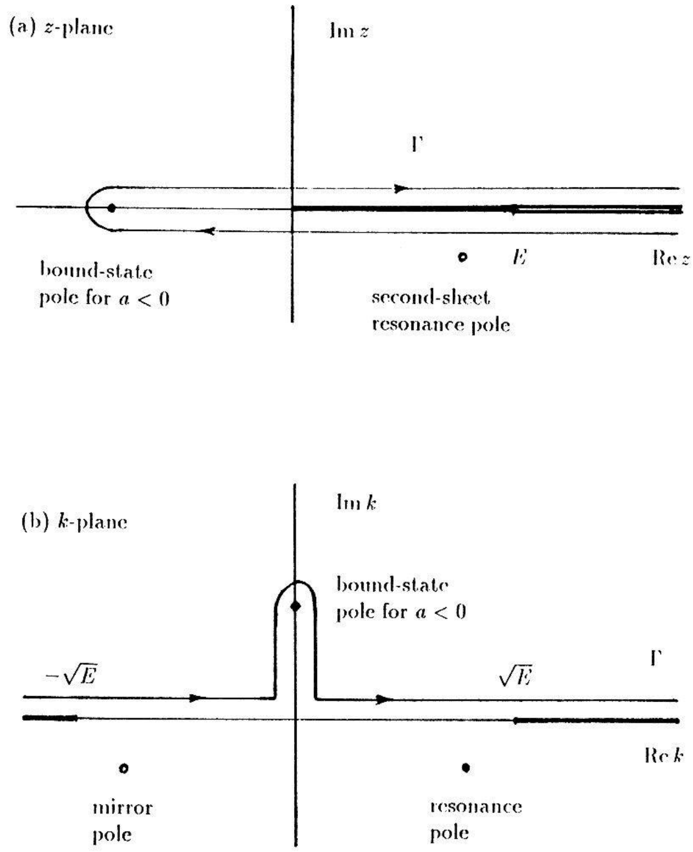

Figure 2

Integration curves for evaluation of the reduced evolution in the case there is a bound state in the first channel

where Γ is the corresponding curve in the k-plane indicated on Fig.2. If we have an antibound state in the first channel, the corresponding pole is not on the first sheet and the integration curve could be shrunk to the origin (or straightened in the k-plane). On the other hand, if the pole *is* present on the first sheet, Γ may be deformed so that we integrate over this shrunk (straightened) curve plus a small clockwise circle around the pole. The last named contribution to the integral can be calculated by the residue theorem; we get

$$\begin{aligned} v_u(t) &= -\lim_{k\to k_1}(k-k_1)\,2k_1\,\frac{|c|^2+(a-ik_1)(b+i\kappa_1)}{(b+i\kappa_1)^2\Big[|c|^2-(a-ik)(b-i\kappa)\Big]}\,e^{-ik_1^2 t} \\ &\quad+\frac{1}{\pi i}\lim_{\epsilon\to 0+}\int_{\mathbf{R}} e^{-i(k+i\epsilon)^2 t}\,\frac{|c|^2+(a-ik+\epsilon)(b+i\kappa_\epsilon)}{(b+i\kappa_\epsilon)^2\Big[|c|^2-(a-ik+\epsilon)(b-i\kappa_\epsilon)\Big]}\,(k+i\epsilon)\,dk \quad (33)\end{aligned}$$

where κ_1 corresponds to k_1 and $\kappa_\epsilon := \sqrt{(k+i\epsilon)^2-E}$. If $a>0$ we get the same expression without the first term; in the case of zero-energy resonance which we shall not discuss in detail the first term enters with the factor $\frac{1}{2}$.

By a straightforward computation using the identity $|c|^2 + (a - ik_1)(b + i\kappa_1) = 2b(a - ik_1)$ and the relation (15), we find that for $a < 0$ the first term equals

$$\frac{4ab|c|^2}{(a^2 - b^2 + E)^2}\, e^{-ik_1^2 t} \left(1 + \mathcal{O}(|c|^2)\right) .$$

In the second term of (33) the limit cannot be performed under the integral directly since the limiting function is not integrable. Fortunately, one can rewrite the integral using a simple substitution as

$$\int_0^\infty e^{-ik_\epsilon^2 t} \frac{|c|^2 + (a - ik_\epsilon)(b + i\kappa_\epsilon)}{(b + i\kappa_\epsilon)^2\left[|c|^2 - (a - ik_\epsilon)(b - i\kappa_\epsilon)\right]}\, k_\epsilon\, dk$$
$$- \int_0^\infty e^{-i\bar{k}_\epsilon^2 t} \frac{|c|^2 + (a + i\bar{k}_\epsilon)(b + i\tilde{\kappa}_\epsilon)}{(b + i\tilde{\kappa}_\epsilon)^2\left[|c|^2 - (a + i\bar{k}_\epsilon)(b - i\tilde{\kappa}_\epsilon)\right]}\, \bar{k}_\epsilon\, dk ,$$

where $k_\epsilon := k + i\epsilon$ and

$$\tilde{\kappa}_\epsilon := \begin{cases} \kappa_\epsilon & \ldots \quad 0 \le k \le \sqrt{E} \\ -\kappa_\epsilon & \ldots \quad \sqrt{E} \le k \end{cases}$$

Here we have used the fact that the limit of κ when k approaches the real axis from the upper halfplane (on the first sheet) equals $\pm\sqrt{k^2 - E}$ on the right and left cut, respectively, and $i\sqrt{E - k^2}$ in between. Writing now the expression in question as a single integral over $(0, \infty)$, the limit can be already performed giving

$$\frac{1}{\pi i} \int_0^{\sqrt{E}} \frac{e^{-ik^2 t}}{(b + i\kappa)^2} \frac{-4ik^2 b|c|^2}{|c|^4 - 2a|c|^2(b - i\kappa) + (a^2 + k^2)(b - i\kappa)^2}\, dk$$
$$+ \frac{1}{\pi i} \int_{\sqrt{E}}^\infty \frac{e^{-ik^2 t}}{(b^2 + \kappa^2)^2} \frac{-4ib\kappa|c|^4 - 4ibk|c|^2(b^2 + \kappa^2)}{|c|^4 - 2|c|^2(ab - k\kappa) + (a^2 + k^2)(b^2 + \kappa^2)}\, k\, dk . \tag{34}$$

The first integral is for small $|c|$ expected to be dominated by the pole contribution; recall that the "discriminant" D is contained in the denominator. The corresponding residuum equals

$$\frac{1}{\pi i} \lim_{k \to k_2} (k - k_2) k \frac{|c|^2 + (a - ik)(b + i\kappa)}{(b + i\kappa)^2\left[|c|^2 - (a - ik)(b - i\kappa)\right]}\, e^{-ik^2 t}$$
$$= \frac{1}{\pi i} \frac{2bk_2\kappa_2(a - ik_2)}{(b + i\kappa_2)^2\left[E + i(ak_2 - b\kappa_2)\right]}\, e^{-ik_2^2 t} ,$$

where $\kappa_2 := \sqrt{E - k_2^2}$. Using the expansion (17), we get after a tedious but straightforward calculation that the preexponential factor equals $-1/2\pi i$ up to higher-order terms, *i.e.*, that the pole contribution to $v_u(t)$ is $e^{-ik_2^2 t}\left[1 + \mathcal{O}(|c|^2)\right]$ as expected.

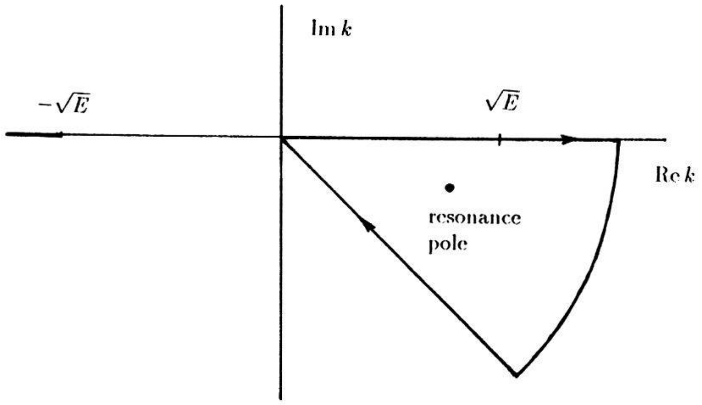

Figure 3

Evaluation of the backgroud term

It remains to determine the background term. In the first integral, we close the integration curve as indicated on Fig.3 ; taking the segment radius to infinity, the contribution to the background consists of the integrals over $(\sqrt{E},\infty)$ and the fourth-quadrant axis. The first of them, however, can be checked easily to cancel up to higher order terms in $|c|^2$ with the second integral in (34) so we get finally for the reduced evolution and the corresponding decay law $P_u(t) := |v_u(t)|^2$ the following result.

5.1 Theorem. Assume $a \neq 0$ and $-\sqrt{E} < b < 0$. The reduced evolution and the decay law of the resonant state (29) corresponding to the Hamiltonian h_u are given by

$$v_u(t) = \Big\{ e^{-ik_2^2 t} - |c|^2 \Big[\frac{2(|a|-a)b}{(a^2-b^2+E)^2}\, e^{-ik_1^2 t} + \frac{ib}{\sqrt{E-b^2}(a-i\sqrt{E-b^2})^2}\, e^{-ik_2^2 t} + \frac{4b}{\pi}\, e^{-\pi i/4} \int_0^\infty \frac{z^2 e^{-z^2 t}\, dz}{(z^2+ia^2)(z^2-i(E-b^2))^2} \Big] \Big\} \left(1+\mathcal{O}(|c|^2)\right) \tag{35}$$

and

$$P_u(t) = \Big\{ e^{2(\mathrm{Im}\, e_2)t} - 2|c|^2 \mathrm{Re} \Big[\frac{2(|a|-a)b}{(a^2-b^2+E)^2}\, e^{-i(k_1^2-\bar{k}_2^2)t} + \frac{ib}{\sqrt{E-b^2}(a-i\sqrt{E-b^2})^2}\, e^{2(\mathrm{Im}\, e_2)t} + \frac{4b}{\pi}\, e^{i(\bar{k}_2^2 t-\pi/4)} \int_0^\infty \frac{z^2 e^{-z^2 t}\, dz}{(z^2+ia^2)(z^2-i(E-b^2))^2} \Big] \Big\} \left(1+\mathcal{O}(|c|^2)\right), \tag{36}$$

where $e_j = k_j^2$ are given by the relations (7) and (18).

Hence we have obtained the expected result, namely the exponential decay law with the lifetime (25) and correction terms of order of $|c|^2$. A closer inspection reveals, however, that the long-time behaviour of the decay law depends substantially on the spectrum of the unperturbed Hamiltonian : if it has an eigenvalue in the first channel (so h_u has an eigenvalue too for a sufficiently weak coupling) then the decay law contains a term of order of $|c|^4$ *which does not vanish* as $t \to \infty$ and therefore dominates the expression for large times.

It is clear that this term comes from the component of the first-channel bound state contained in the resonant state (29). Its time dependence e^{ia^2t} in the non-decay amplitude is up to higher order terms exactly what one expects from the bound state contribution to the three-dimensional point-interaction Green's function – *cf., e.g.*, the explicit expression for its kernel derived recently by Scarlati and Teta [36]. Let us remark in this connection that in the earlier results concerning the one-dimensional resolvent kernel [29, 31] the difference between the bound- and antibound-state cases was overlooked (it was noticed, however, in [24]) so the corresponding formulae are only formal in the first named case.

Finally, let us comment on the choice of the resonant state (29). Hunziker [27] has argued recently that in general one can use the formally obtained perturbed eigenstate, however, he pointed out that in the embedded-eigenvalue case only the zero order is applicable since the higher terms in the perturbation series diverge.

The explicit form of our results allows us to see in the model under consideration why it is so. The perturbed eigenfunction conventionally obtained from the residuum at the shifted pole of the resolvent – *cf.*[28], Chap.II. For the second sheet pole (18) the residuum can be formally written as $\mathcal{R} = (f, \cdot)g$, where f, g are up to higher order terms equal

$$f(r) = \sqrt{-2b} \begin{pmatrix} -\frac{c}{a+i\sqrt{E-b^2}}\, e^{-i\sqrt{E-b^2}} + \mathcal{O}(|c|^3) \\ e^{br} + \mathcal{O}(|c|^2) \end{pmatrix}$$

and

$$g(r) = \sqrt{-2b} \begin{pmatrix} -\frac{c}{a-i\sqrt{E-b^2}}\, e^{i\sqrt{E-b^2}} + \mathcal{O}(|c|^3) \\ e^{br} + \mathcal{O}(|c|^2) \end{pmatrix}.$$

We see that $\mathcal{R}$ is not even no projection, but f, g are not normalizable vectors unless, of course, $c = 0$.

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