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Autor(en): **Chayes, L.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **64 (1991)**

Heft 7

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-116330

Helvetica Physica Acta
Vol. 64 (1991)

0018-0238/91/071055-16$1.50+0.20/0

On the Critical Behavior of the First Passage Time in d ≥ 3

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(21. I. 1991, revised 5. VIII. 1991)

Abstract The first passage times for the Bernoulli percolation problems on the d-dimensional hypercubic lattices are investigated. For all d (and hence for d ≥ 3) it is rigorously established that, in the critical region, the first passage times tend to zero with the same scaling behavior as the decay rate for correlations in the associated percolation problem.

* Work supported by the NSF under grant # DMS 9009049

Introduction

The problem called *first passage percolation* was invented in the 1965 paper by Hamersley and Welsh [HW] and has since, on and off, been the subject of a certain amount of attention; cf. the monographs [SW],[K_1],[CC], and [D]. Although the model was originally intended as a simplified description of fluid flow in a random medium, it has seen applications in diverse areas such as neural networks and the ever popular topic of "crack propagation" cf. above references and references therein.

Here, for the d-dimensional Bernoulli system the critical behavior of the so called first passage time is investigated. It is established that, in all dimensions, the scaling behavior of this object can be understood completely via that of the correlation length of the underlying (ordinary) percolation problem.

Such results have already been proved in two dimensions [CC], [CCD] using geometric arguments. (Although the methods used here are also valid for d = 2, the results are technically weaker than those found in [CC], [CCD].) Prior to [CCD], there had been a certain amount of confusion in the theoretical community (see, e.g. [Ker_1][Ker_2]) but, knowing that such results hold in two dimensions, there can be little doubt of the analogous result for any dimension. Nevertheless, a mathematical proof this has turned out to be elusive and ultimately, the arguments used here differ a good deal from those of [CC] and [CCD].

The space below will be devoted to a precise definition of the problem and used as an opportunity to dispense with some notational preliminaries. Then there will actually be a statement of the theorems to be proved.

To avoid excessive provisos, this note will be concerned exclusively with the d-dimensional hypercubic bond lattices $\mathbb{B}_d$; that is the set of bonds joining the neighboring pairs of $\mathbb{Z}^d$. It will always be assumed that $d \geq 2$. To each $b \in \mathbb{B}_d$, one assigns a random variable ω_b which is called the time coordinate of the bond. The ω_b are understood to be independent and identically distributed. The cases of interest here are the Bernoulli first passage problems where

$$\omega_b = \begin{cases} 1; & \text{with probability p} \\ 0; & \text{with probability (1- p)} \end{cases} \tag{1}$$

with $0 \leq p \leq 1$.

The collection of all time coordinates, $(\omega_b | b \in \mathbb{B}_d)$ will be referred to as a *realization* and, in general, will be denoted by an ω.

If $\alpha,\beta \in \mathbb{Z}^d$, and $\mathcal{p}:\alpha \to \beta$ is a self-avoiding path in $\mathbb{B}_d$ with endpoints α and β then, in a given realization, ω, we may define the path time (or length):

$$t_{\mathcal{p}}(\omega) = \sum_{b \in \mathcal{p}} \omega_b \tag{2a}$$

and the minimum time separation between α and β:

$$t_{\alpha\beta} = \inf_{\mathcal{p}:\alpha \to \beta} t_{\mathcal{p}}(\omega) . \tag{2a}$$

Using the notation $\underline{n} = (0, 0, \ldots, n)$ for the point n units along the x_d axis, the first passage time $\theta \equiv \theta(p)$ is defined via the limit

$$\theta = \lim_{n\to\infty} \frac{t_{0,\underline{n}}}{n} . \tag{3}$$

That the above is meaningful is a consequence of the Kingman subadditive theorem [Ki], from which it follows that the limit in (3) exists almost surely and is almost surely a constant independent of the realization.

The statistical behavior of the underlying configurations as a function of p is the subject called percolation. Here the necessary items from percolation will be reviewed but, in order to keep this note self contained, we will couch these in the language of the first passage problem.

Foremost in importance is the critical point, $p_c \in (0,1)$. The point p_c may be described by saying that if $p > p_c$, there are, almost surely, infinite zero time paths while, with probability one, all zero time paths are finite when $p < p_c$. For $p < p_c$, one is interested in the objects

$$\tau_{\alpha\beta} = \text{Prob.}(t_{\alpha\beta} = 0) . \tag{4}$$

It is not difficult to establish that the quantity

$$-\frac{1}{\xi} = \lim_{n\to\infty} \frac{\log \tau_{0\underline{n}}}{n} \tag{5}$$

exists. From (5), it is clear that the object ξ defines a length scale and, in particular, ξ is referred to as the correlation length. In this model, ξ is known to diverge continuously at some point [H], and in fact, this point is known to coincide with the above mentioned p_c . [AB], [MMS]

Not surprisingly, p_c is also the critical point for the first passage problem. Indeed, one can easily demonstrate that $\theta(p) = 0$ when $p \geq p_c$. It was first shown in 1980 by Kesten [K_2] that $\theta(p) > 0$ when $p < p_c$. A bit later, the stronger result

$$\theta(p) \geq \sup_{\varepsilon} \frac{1}{\xi(p+\varepsilon)} \, [\log[(1-p)/\varepsilon]]^{-1} \tag{6}$$

was established, by somewhat simpler methods, in [CC].

Equation (6) provides a one-way bound relating the critical behaviors of θ and ξ. That is to say, if the limits of $|\log\xi|/|\log(p_c-p)|$ and

$|\log\theta|/|\log(p_c-p)|$ exist as $p\uparrow p_c$, then denoting these quantities by ν and (for peculiar reasons) $\nu(1-\psi)$ respectively, it is seen that $\nu \geq \nu(1-\psi)$. The essence of the result which is proved in this note amounts to a bound supplementary to (6) which is valid for all $d \geq 2$. Explicitly, the result of Theorem A is that for any δ there is a $V(\delta)$, uniform in p, such that

$$\theta(p) \leq \frac{V(\delta)}{[\xi(p)]^{1-\delta}} . \tag{7}$$

Equations (6) and (7) of course imply that "$\psi = 0$" if the existence of the appropriate limits can be established in any meaningful fashion.

An alternative "definition" of ψ pertains to the duration along the minimal time paths at p_c: When $p = p_c$, the minimal path times presumably scale sub-linearly with the distance vis-a-vis $t_{0\underline{n}} \cong n^{\Psi}$. An immediate consequence of the method used to prove Theorem A is that this latter ψ also vanishes. Such is the content of Theorem B.

Proof of Results

I) Strategic Outline

This subsection begins with a few more concepts -- and some more definitions! Let Λ_L denote the $L \times [3L]^{d-1}$ box:

$$\Lambda_L = \{b \in \mathbb{B}_d | \text{ at least one of the endpoints of b lies in the region } -\tfrac{1}{2}L < x_d < +\tfrac{1}{2}L\,;\, -\tfrac{3}{2}L < x_1, \dots, x_{d-1} < +\tfrac{3}{2}L\} . \tag{8}$$

These boxes have a short (or easy) direction; an inordinate amount of attention will be focused on paths within Λ_L which cross in this direction. The "bottom" of Λ_L, that is the sites with $x_d > -(L+1)/2$ which share a bond with a site outside Λ_L, will be denoted by $\mathfrak{B}_L$. The top of Λ_L, denoted by $\mathfrak{T}_L$, is defined analogously.

If $i \in \mathfrak{B}_L$ and $j \in \mathfrak{T}_L$ then let $t_{i,j}^{|L|}$ denote the minimum time over all paths within Λ_L which connect the sites i and j:

$$t_{i,j}^{|L|} = \text{the minimum time between i \& j along paths which lie entirely within } \Lambda_L. \tag{9}$$

Of considerable interest will be the travel times across (i.e. "up through") the boxes Λ_L:

$$T^{|L|} = \min_{\substack{i \in \mathfrak{B}_L \\ j \in \mathfrak{T}_L}} t_{i,j}^{|L|} . \tag{10}$$

The mechanism for introducing the percolation correlation length into these problems is via "finite-size scaling." In particular, when $p < p_c$ and $L \gg \xi$, the probability of observing a zero time path up through Λ_L will tend to zero exponentially fast in L/ξ. On the other hand, for $L \lesssim \xi$ one expects these probabilities to be not unreasonably small and indeed, when L is of order unity, these probabilities must themselves be of order unity. On the basis of these considerations, for any constant c, one can define $L^*(c)$ via

$$L^* = \max\{L \mid \text{Prob}(T^{|L'|} = 0) \geq c \ \forall\, L' \leq L\}. \tag{11}$$

Explicitly, on all scales L up to and including L^*, the probability of observing a zero time path across Λ_L exceeds c, but this condition fails in the box Λ_{L^*+1}.

According to the result of [CCF], for a suitable choice of the constant c, the length scale $L^*(c)$ may be identified -- in the scaling sense -- with the correlation length for percolation:

Proposition 1 *For any* d *there is a constant* c(d) *such that the correlation length* ξ *and the finite-size scaling length* L^* *obey the inequalities*

$$a_1 L^* \geq \xi \geq \frac{L^*}{a_2 + a_3 \log L^*}$$

with the constants a_1, a_2 *and* a_3 *independent of* p.

Proof. See [CCF] Proposition 3.2.

The situation as it now stands is as follows: for all scales L up to L^*, one can find zero time paths up through Λ_L with probabilities of order unity. It is in fact easy to show (C.f. Proposition 2 below) that with probability exceedingly close to one there are "short time" paths up through Λ_L. Although this implies the existence of sites in $\mathbf{T}_L$ connected to sites in $\mathbf{B}_L$ by short paths, this is not particularly useful for an estimate of first passage times. Conversely, if a fixed pair of sites -- say the centers of $\mathbf{T}_L$ and $\mathbf{B}_L$ -- were connected by a (sufficiently) short path with (sufficiently) high probability an estimate on the first passage time is almost immediate.

The key issue is therefore the deterministic localization of short time paths. The first stage in this localization is accomplished in Proposition 3 where short paths in Λ_L get localized to some scale $m < L$. At this point an inductive assumption is made about the (probabilistic) presence of point localized paths up through the Λ_m with times shorter than a power of m. In Lemma 4 it is shown, using these smaller scale paths, that a point localized path for Λ_L with an improved power estimate for the crossing time occurs with high probability. Following this are a few unenlightening geometric constructions which are used to regenerate the inductive hypothesis with the improved power estimate.

Hence, path times at length scales smaller than $L^*(p)$ enjoy a sequence of upper bounds which are constants times smaller and smaller powers of the distance. It should be emphasized that the constants are uniform in p hence one only need investigate the limit of the sequence of powers. Since this limit is readily shown to be zero, Equation (7) follows immediately.

II) Preliminary Results

Proposition 2 *For any* $L \leq L^*(p)$

$$\mathrm{Prob}[T^{|L|} > n] \leq (1 - c)^n \equiv e^{-\lambda n}.$$

Proof. This result is easily established by looking to the dual model. Placing a dual (d-1)-cell traversal to each bond which carries a unit time coordinate, it is a standard result (see, e.g. [K_1]) that $T^{|L|}$ is exactly the number of independent surfaces which separate $\mathbf{T}_L$ from $\mathbf{B}_L$. The above estimate is obtained by conditioning to the lowest (n-1) disjoint surfaces and (paying homage to the Harris-FKG inequality [Har], [FKG]) obtaining the n^{th} surface at the additional cost of at most (1-c). Alternatively, one can apply the van den Berg - Kesten [vBK] inequality n times.

Now consider a tiling of the regions $\mathbf{T}_L$ and $\mathbf{B}_L$ by disjoint hyper squares of side m -- using smaller hyper rectangles to fit the boundaries and corners if, perchance, m does not divide 3L. Label the tiles on $\mathbf{B}_L$ and $\mathbf{T}_L$ by indices which run from 1 to K where it is observed that $K \leq [\frac{3L}{m}$

$+ 1]^{d-1}$. Let $T^{|L|,m}_{I,J}$ denote, in a configuration ω, the minimum time that it takes to get from the I^{th} tile on $\mathfrak{B}_L$ to the J^{th} tile on $\mathfrak{T}_L$:

$$T^{|L|,m}_{I,J} = \min_{\substack{i \in I \\ j \in J}} t^{|L|}_{i,j} . \tag{12}$$

The following constitutes the first step in the localization of short time paths:

Proposition 3 *For at least one particular (i.e. deterministic) pair of tiles, denoted by the* I^{*th} *on* $\mathfrak{B}_L$ *and the* J^{*th} *on* $\mathfrak{T}_L$ *the times* $T^{|L|,m}_{I^*,J^*}$ *enjoy the bound*

$$\mathrm{Prob}[T^{|L|,m}_{I^*,J^*} > n] \leq \exp-\left[\frac{\lambda n}{K^2}\right] .$$

Proof. Observe that the event $T^{|L|} > n$ is equivalent to the statement that the minimum path time from every tile on $\mathfrak{B}_L$ to every tile on $\mathfrak{T}_L$ must also exceed n. Thus one can write

$$T^{|L|} > n = \bigcap_{\substack{J \leq K \\ I \leq K}} T^{|L|,m}_{I,J} > n . \tag{13}$$

Using the FKG inequality on equation 13, one has

$$\mathrm{Prob}[T^{|L|} > n] \geq \prod_{\substack{J \leq K \\ I \leq K}} \mathrm{Prob}[T^{|L|,m}_{I,J} > n]. \tag{14}$$

Denote by I^*,J^* the pair of tiles corresponding to the term in the above product which is smallest. Then, the desired result follows immediately from the statement of Proposition 2.

Now for the inductive hypothesis. Let $c\mathfrak{T}_L$ and $c\mathfrak{B}_L$ denote the central portions of the top and bottom of Λ_L:

$$c\mathfrak{B}_L = \{x \in \mathfrak{B}_L | -\tfrac{1}{2}L < x_1, \dots , x_{d-1} < +\tfrac{1}{2}L\} \tag{15a}$$

$$c\mathfrak{T}_L = \{x \in \mathfrak{T}_L | -\tfrac{1}{2}L < x_2, \dots , x_{d-1} < +\tfrac{1}{2}L\}. \tag{15b}$$

It is supposed that:

$\mathbf{H}_b$ *For every* $L \leq L^*(p)$ *and for every* $i \in c\mathfrak{B}_L$ *and* $j \in c\mathfrak{T}_L$, *there are constants* s_b, c_b, *and* k_b *independent of* L *or* p *such that*

$$t^{|L|}_{i,j} \leq s_b L^b$$

with probability exceeding $1 - c_b \exp-[k_b L^{b/2}]$.

Remark. It is obvious that $\mathbf{H}_b$ is satisfied for any b≥1 since $t_{i,j}^{|L|}$ is, in the worst case, a constant times L. Thus $\mathbf{H}_b$ is only interesting when b < 1. However, as we will be demonstrated shortly, $\mathbf{H}_b$ implies $\mathbf{H}_{b'}$ with a b' strictly less than 1.

Lemma 4. *Under the hypothesis* $\mathbf{H}_b$, *for any* $L \leq L^*(p)$ *there is a (deterministic) pair of points* $i^* \epsilon \mathbf{B}_L$ *and* $j^* \epsilon \mathbf{T}_L$ *such that*

$$t_{i^*,j^*}^{|L|} \leq s^\dagger_{b'} L^{b'}$$

with probability larger than $1 - c^\dagger_{b'} \exp\text{-}[k^\dagger_{b'} L^{b'/2}]$. *In the preceding, the constants* $s^\dagger_{b'}$, $c^\dagger_{b'}$, *and* $k^\dagger_{b'}$ *are independent of* L *or* p *and* b' *is given by*

$$b' = \frac{ab}{a+b}$$

where, for the purpose of typographical ease, we have defined $a \equiv 4(d\text{-}1)$.

Proof. Let $L \leq L^*(p)$ and let m be an integer smaller than L/2 which, for the moment, is kept unspecified. Focusing attention on the box Λ_{L-2m}, consider a tiling of the top and bottom which is of scale m. For the privileged pair of tiles, I* and J*, the event of a connecting path, inside Λ_{L-2m}, with path time shorter that n enjoys the probabilistic estimate

$$\text{Prob}[\ T_{I^*,J^*}^{|L-2m|,m} \leq n] \geq 1 - \exp - \left[\frac{\lambda n}{\left(\frac{3(L-2m)}{m} + 1\right)^{a/2}}\right]$$

$$\geq 1 - \exp - \lambda n \left[\frac{m}{3L-5m}\right]^{a/2}. \tag{16}$$

(The extra 5m in the denominator is being held in reserve to nullify future inconsequentials.)

Consider now the box Λ_L. Since $\mathbf{B}_L$ and $\mathbf{B}_{L-2m}$ are separated by a layer of thickness m, the m × m tiles on $\mathbf{B}_{L-2m}$ may be regarded as translates of $\mathbf{cB}_m$; the corresponding translate of $\mathbf{cB}_m$ forms a portion of $\mathbf{B}_L$. (See Figure 1 for further clarification.)

Given that the event $T_{I^*,J^*}^{|L-2m|,m} \leq n$ has occurred, *some* site on the I*th tile is connected to *some* site on the J*th tile by a short path inside of Λ_{L-2m}. What is now desired is to attach this path to fixed (deterministic) sites on the top and bottom of Λ_L. This will be achieved (shortly) by using translates of events in Λ_m. The obstruction to the immediate accomplishment of this deed is that there is no natural partition of the event $T^{|L-2m|,m}_{I^*,J^*} \leq n$ according to which sites on the I^{*th} and J^{*th} tiles are "the ones" that are connected. However, we have at our disposal the fact that the events above $\mathbf{T}_{L-2m}$ and below $\mathbf{B}_{L-2m}$ are independent of the configuration inside Λ_{L-2m}; this permits the luxury of an essentially arbitrary partition of $T^{|L-2m|,m}_{I^*,J^*} \leq n$.

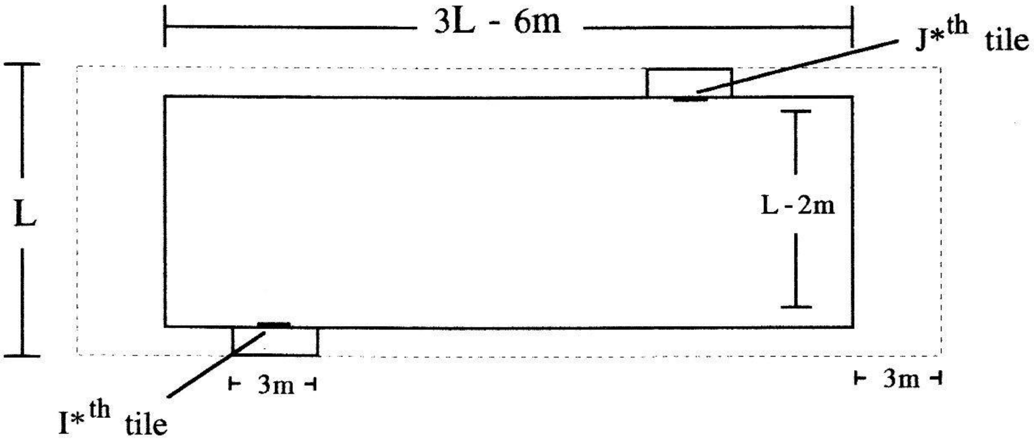

Figure 1

It is thus claimed that there is an $i^* \in \mathbf{B}_L$ and a $j^* \in \mathbf{T}_L$ such that with probability exceeding $(1 - c_b \exp-[k_b m^{b/2}])^2 (1 - \exp-\lambda n[m/(3L-5m)]^{a/2})$, the event $t^{|L|}_{i^*,j^*} \leq n + 2s_b m^b$ occurs.

Let $\Lambda_m(I^*)$ be that translate of Λ_m for which the I^{*th} tile of Λ_{L-2m} serves as "$\mathbf{cT}_m$." (In case the I^{*th} tile is near an edge or corner, the above sentence should be replaced by " ... serves as part of '$\mathbf{cT}_m$.'" As the reader can easily glean from Figure 1, there is still ample room to fit the relevant translate of Λ_m inside Λ_L.) Pick any site i^* on the translate of $\mathbf{cB}_m$ corresponding to $\Lambda_m(I^*)$. Pick a similarly defined site $j^* \in \mathbf{T}_L$ from the top of $\Lambda_m(J^*)$.

Order the sites on the I^{*th} tile 1, 2, ... , k, ... and those on the J^{*th} tile by 1, 2, ... , h, ... Denote by A_{kh} the event

$A_{kh} = \{\omega \in T^{|L-2m|,m}_{I^*,J^*} \leq n$; The k^{th} site on the I^{*th} tile is the "earliest" site which is connected to the J^{*th} tile by a path inside Λ_{L-2m} which is shorter than n. Of all the sites on the

J^{*th} tile to which it is connected by this sort of path, the h^{th} is the "earliest."} (17)

The above partition is manifestly a disjoint partition of $T^{|L-2m|,m}_{I^*,J^*} \leq n$ (and otherwise has very little intrinsic value) but it is now possible to state that

$$\mathrm{Prob}[T^{|L-2m|,m}_{I^*,J^*} \leq n] = \sum_{k,h} \mathrm{Prob}[A_{k,h}] \ . \tag{18}$$

Using the notation $\tilde{t}^{|m|}_{a,b}$ for path times in the translate of Λ_m near $\mathbf{B}_L$ and $\tilde{\tilde{t}}^{|m|}_{a,b}$ for path times in the translate of Λ_m near $\mathbf{T}_L$ it is not awfully difficult to see that

$$\mathrm{Prob}\left[(t^{|L|}_{i^*,j^*} \leq n + 2\, s_b m^b)\right] \geq$$

$$\mathrm{Prob}\left[\bigcup_{k,h} (\tilde{t}^{|m|}_{i^*,k} \leq s_b m^b) \cap A_{k,h} \cap (\tilde{\tilde{t}}^{|m|}_{j^*,h} \leq s_b m^b)\right] . \tag{19}$$

Now the rhs of equation 19 can be expressed as the following a sum:

$$\mathrm{Prob}\left[(t^{|L|}_{i^*,j^*} \leq n + 2\, s_b m^b)\right] \geq$$

$$\sum_{h,k} \mathrm{Prob}\left[(\tilde{t}^{|m|}_{i^*,k} \leq s_b m^b) \cap A_{k,h} \cap (\tilde{\tilde{t}}^{|m|}_{j^*,h} \leq s_b m^b)\right] . \tag{20}$$

Now it is observed the terms inside the square brackets are independent (since they take place on disjoint regions) so each term inside the sum can be factored. The quantities $\mathrm{Prob}[\tilde{t}^{|m|}_{i^*,k} \leq s_b m^b]$ and $\mathrm{Prob}[\tilde{\tilde{t}}^{|m|}_{i^*,k} \leq s_b m^b]$ are uniformly estimated by invoking the inductive hypothesis. The estimate then simply multiplies the sum which is seen to be a round about expression for $\mathrm{Prob}[T^{|L-2m|,m}_{I^*,J^*} \leq n]$ (c.f. equation (18). Using equation (16), the above mentioned claim, namely that

$$\mathrm{Prob}[t^{|L|}_{i^*,j^*} \leq n + 2s_b m^b] \geq$$

$$(1 - c_b \exp\text{-}[k_b m^{b/2}])^2 (1 - \exp{-\lambda n}[m/(3L-5m)]^{a/2}) \tag{21}$$

has finally been established. Now, to finish the lemma, all that remains is to optimize over the choices of m and n.

The procedure starts by choosing n to be the largest integer smaller than $(3L/m)^a$ so, explicitly,

$$\left(\frac{3L}{m}\right)^a \geq n \geq \left(\frac{3L}{m}\right)^a - 1. \tag{22}$$

A dreary calculation can then be performed which shows that

$$n\left[\frac{m}{3L - 5m}\right]^{a/2} \geq \left(\frac{3L}{m}\right)^{a/2} \tag{23}$$

which can temporarily be inserted into the estimate in equation (16). Next, m is chosen to be the largest integer which satisfies

$$s_b m^b \leq \left(\frac{3L}{m}\right)^a. \tag{24a}$$

Using (24a) and the alternate bound

$$s_b(m + 1)^b \geq \left(\frac{3L}{m + 1}\right)^a, \tag{24b}$$

one can quickly obtain the estimates

$$g_{b'}L^{b'} \geq m^b \geq h_{b'}L^{b'} \tag{25}$$

where the constants $g_{b'}$ and $h_{b'}$ are messy but do not depend on L or p. A similar endeavor will also produce estimates which bound $(L/m)^a$ from above and below by constants times $L^{b'}$. Substituting the appropriate bounds into equation (21) -- so that all quantities are now expressed in terms of $L^{b'}$ -- it is seen that the desired result has been obtained. ▨

The geometric lemmas must now be dispensed with

Lemma 5. *Under the condition* $\mathbf{H}_b$, *there are constants* $s^\bullet_{b'}$, $c^\bullet_{b'}$, *and* $k^\bullet_{b'}$ *such that for all* $L \leq L^*(p)$ *the centers of* $\mathbf{B}_L$ *and* $\mathbf{T}_L$ *(that is the points nearest to* $(0,0, \dots, \pm L/2)$*) are connected inside* Λ_L *by a path shorter than* $s^\bullet_{b'}L^{b'}$ *with probability exceeding* $1 - c^\bullet_{b'}\exp\text{-}[k^\bullet_{b'}L^{b'/2}]$.

Proof. To simplify the exposition, it will be assumed that L is even. (The proof for odd L cannot be too different since a careful look at the definition of Λ_L will reveal that $\Lambda_{2N+1} = \Lambda_{2N}$) Consider two copies of the box $\Lambda_{L/2}$: one translated $L/2$ units in the positive x_d direction and the other $L/2$ units in the negative x_d direction. Suppose that in the bottom translate of $\Lambda_{L/2}$, the associated translate of the event $t^{|L/2|}_{i^*,j^*} \leq s^\dagger_{b'}(L/2)^{b'}$ occurs. Meanwhile, in the top copy of $\Lambda_{L/2}$, suppose that a translate of the reflection of this event across the $x_d = 0$ hyperplane occurs. Then, denoting the (untranslated) coordinates of i* by

$$i^* = (i^*_1, i^*_2, \dots, i^*_{d-1}, -L/2) \tag{26}$$

it is seen that a path connecting the point $(i^*_1, \ldots, i^*_{d-1}, -L)$ to the point $(i^*_1, \ldots, i^*_{d-1}, +L)$ has occurred at a probabilistic cost of no more than $(1 - c^\dagger_{b'}\exp-[k^\dagger_{b'}(L/2)^{b'/2}])^2$.

Noting that all of the $|i^*_1|, |i^*_2|, \ldots, |i^*_{d-1}|$ are smaller than $3L/4$, it is observed that if a translate of the above described event by the vector $(-i^*_1, -i^*_2, \ldots, -i^*_{d-1}, 0)$ occurs, then the desired connection between $(0, \ldots, -L/2)$ and $(0, \ldots, +L/2)$ does not wander outside of the confines of Λ_L. See figure 2 for further clarification.

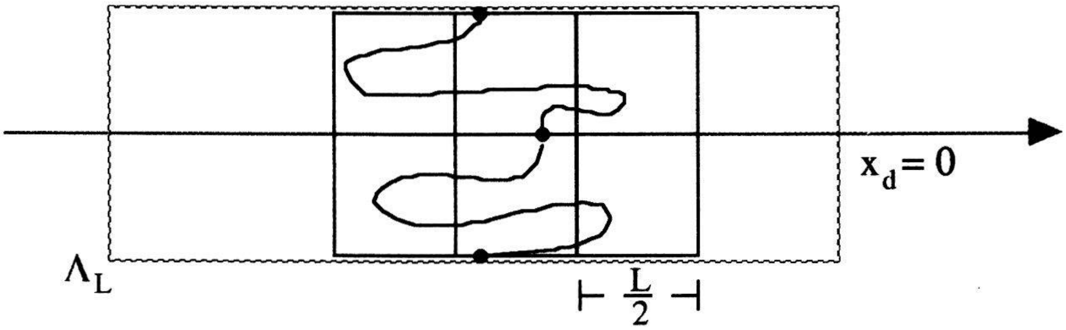

Figure 2

Lemma 6. Under $\mathbf{H}_b$, the condition $\mathbf{H}_{b'}$ holds where, explicitly,

$$b' = \frac{4(d-1)b}{4(d-1)+b}.$$

Remark. It is perhaps useful to first illustrate the procedure in $d = 2$ after which the details in $d \geq 3$ need only be sketched. The basic scheme, for Λ_L, is to use the "center to center" events constructed in Lemma 5 in boxes of scale smaller than L. Unfortunately, one cannot use scales *very* small compared to L without spoiling the probabilistic estimates. Thus, constrained to use only events on scales of order L, there will be a certain amount of wastage as well as seemingly unnecessary tedium.

Proof. (*For* $d = 2$) Let $L \leq L(p)$ and let $i = (i_1, i_2) \in \mathbf{c}\mathfrak{B}_L$. Consider the points placed as near as possible to the points $(L,0)$, $(L/6,0)$, $(-L/6,0)$ and $(-L,0)$. These will be denoted as "anchor points." The primary goal is to hook i up to one of these anchor points by a short path residing inside Λ_L. Having done so, a similar procedure will be invoked to connect any $j \in \mathbf{cT}_L$ up to another anchor point after which the anchor points will be strung together via events which are translations and 90° rotations of the "center to center" events (c.f. Lemma 5) in boxes of scale $L/6$.

First, consider the box $\Lambda_{L/2}$ (or at worst $\Lambda_{(L+1)/2}$) translated so that the bottom center lies at the site i. Then, with probability no less than 1 - $c^{\bullet}_{b'}\exp\text{-}[k^{\bullet}_{b'}(L/2)^{b'/2}]$, this site is connected to its partner at $(i_1,0)$ by a path which is shorter than $s^{\bullet}_{b'}(L/2)^{b'}$.

Now the end of this path, that is the site $(i_1,0)$, is somewhere between two of the anchor points. Let $(i_1,0)$ and the further of these two points, (P,0) constitute the "top and bottom" centers of a rotation and translation of a Λ_N with $N = |P - i_1|$. Observe that $L/3 \geq N \geq L/6$; thus it is certainly the case that a path of length shorter than $k^{\bullet}_{b'}(L/3)^{b'}$ connecting $(i_1,0)$ and (P,0) occurs with probability no smaller than 1 - $c^{\bullet}_{b'}\exp\text{-}[k^{\bullet}_{b'}(L/6)^{b'/2}]$. Good. Now consider the same type of procedure with regards to an arbitrary $j \in \mathbf{c}\mathbf{T}_L$; this gets us a path from j to another (ostensibly different) anchor point (P',0). Finally, suppose that there are paths connecting all adjacent anchor points -- the paths lie inside Λ_L and all three of them are shorter than $s^{\bullet}_{b'}(L/3)^{b'}$. Using Lemma 5 on boxes which look like $\Lambda_{L/3}$, this can be accomplished at a cost of no more than $(1 - c^{\bullet}_{b'}\exp\text{-}[k^{\bullet}_{b'}(L/3)^{b'/2}])^3$. The intersection of the above described events certainly implies the much desired path from i to j , and since they are all positive FKG events, there is no difficulty in multiplying the lower bounds on the probabilities. The desired result for two dimensions has been established.

Proof. (*For* $d > 2$) A similar procedure can be employed in $d \geq 3$. First, a grid of scale $L/3$ must be set up in the middle portion of the central hyperplane ($x_d = 0$, $-\frac{1}{2}L < x_2, \dots, x_{d-1} < +\frac{1}{2}L$) amounting, e.g., to a total of 4^{d-1} anchor points. The site $i \in \mathbf{c}\mathbf{B}_L$ is hooked to one of these anchor points in 1 + (d-1) steps; the first taking it to the central hyperplane and then, cutting down one dimension at a time, a series of d-1 moves each time going to the grid (point - line - square etc) which of all the (2 - 4 - 8 etc) nearest ones is the *furthest* away (to safeguard the probabilistic estimate.) When i is finally tacked on to an anchor point, the same must be done for the arbitrary $j \in c\mathbf{T}_L$; then the two relevant anchor points must be "tied" which necessitates the use of no more than 3(d-1) additional events. When all is said and done, there is a path between i and j -- within Λ_L -- of length no longer than

$$s^{\bullet}_{b'}[3(d-1)(L/3)^{b'} + 2((L/2)^{b'}+3(d-1)(L/3)^{b'})] \equiv s_{b'}L^{b'} \tag{27a}$$

at a cost of no more than

$$\begin{aligned}&(1 - c^{\bullet}_{b'}\exp-[k^{\bullet}_{b'}(L/3)^{b'/2}])^{3(d-1)} \times \\ &\times [(1 - c^{\bullet}_{b'}\exp-[k^{\bullet}_{b'}(L/2)^{b'/2}])^2(1 - c^{\bullet}_{b'}\exp-[k^{\bullet}_{b'}(L/6)^{b'/2}])^{3(d-1)}]^2 \\ &\geq 1 - c_{b'}\exp-[k_{b'}(L)^{b'/2}]\end{aligned} \tag{27b}$$

for some constants $c_{b'}$ and $k_{b'}$. Thus, finally, the inductive hypothesis has been regenerated. ▨

III) Final Proofs

Theorem A. *For any* $\delta > 0$, *there is a* $V(\delta)$ *such that for all* p *sufficiently close to* p_c,

$$\theta(p) \leq \frac{V(\delta)}{[\xi(p)]^{1-\delta}}$$

Proof. In light of Proposition 1, it suffices to establish the above stated with ξ replaced by $L^*(p)$. Starting at $b = 1$, (where $\mathbf{H}_b$ is manifestly satisfied) and noting that $ab/(a + b) < b$ whenever $b > 0$, it is seen that -- after the order of $1/\delta$ iterations -- $\mathbf{H}_{\delta'}$ is satisfied for some $\delta' < \delta$. Consider a sequence of times $t_{0,\underline{N}}$ where $\underline{N} = (0, \ldots nL^*(p))$. Using translates of the box $\Lambda_{L^*(p)}$ stacked up along the x_d axis (so that successive tops and bottoms coincide) a worst case scenario -- and the law of large numbers -- puts

$$\begin{aligned}\theta(p) = \lim_{N\to\infty} \frac{t_{0,\underline{N}}}{N} \leq \frac{1}{L^*(p)}[s_{\delta'}(L^*(p))^{\delta'} + \\ + (L^*(p))c_{\delta'}\exp-[k_{\delta'}(L^*(p))^{\delta'/2}]]\end{aligned} \tag{28}$$

with probability one. For p close to p_c one can rest assured that

$$s_{\delta'}L^{*\delta'} > L^*c_{\delta'}\exp-[k_{\delta'}L^{*\delta'/2}]$$

from which the statement of Theorem A follows. ▨

Theorem B. *At* $p = p_c$, *for any* $\delta > 0$,

$$\lim_{N\to\infty} \frac{t_{0,N}}{N^{\delta}} = 0$$

with probability one.

Proof. At $p = p_c$, the condition $L < L^*(p_c)$ is enjoyed for every finite L. Again, the "b" in $\mathbf{H}_b$ is whittled down to any δ' below δ. Then, using a single Λ_N box event, the direct estimate described by $\mathbf{H}_{\delta'}$ leads immediately to the above stated result. ▨

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