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The Unstable System and Irreversible Motion in Quantum Theory

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Abstract. It has been shown by Flesia and Piron that the scattering theory of Lax and Phillips, developed primarily for the treatment of hyperbolic systems (e.g., classical scattering of electromagnetic waves on a finite target), can be applied to the quantum theory. In order to construct the correspondence, the continuously infinite family of (isomorphic) Hilbert spaces associated with each value of the time t are taken to be a direct integral representation (with a choice of measure) of a larger Hilbert space which includes the variable t in its measure space. We show that this theory, for which the evolution contracted to the subspace orthogonal to the incoming and outgoing subspaces is a contractive semigroup, provides a natural description of the irreversible motion of an unstable quantum system.

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1 Introduction

The unstable system is an important example of irreversible phenomena in nature. Such systems, ranging from excited atomic states to short-lived elementary particles, are characterized by what is generally observed to be deterministic but irreversible evolution, and also sometimes a random process. In the quantum theory, deterministic evolution has the property that if there is an experiment that separates two states of the system after some evolution, i.e., that these states are orthogonal, then there exists an experiment that can distinguish the corresponding two states before the evolution takes place. Hence, two orthogonal states cannot evolve deterministically from non-orthogonal initial states. The stronger statement, that, in addition, two orthogonal states can only evolve to orthogonal states, goes beyond deterministic evolution. This is the case for reversible evolution.

The description of irreversible evolution in the quantum theory has traditionally been described by the addition of non-Hermitian terms to the Hamiltonian[1,2]. Structures of this type were originally introduced by Gamow[3], who studied the effect of assigning complex values to the energy spectrum, and hence introduced a kind of generalized eigenvector. Weisskopf and Wigner[4], in a remarkable paper in 1930, succeeded in finding an approximate exponential decay law for an unstable system, for times not too short or too long. To obtain this result, they studied, perturbatively, what has become known as the "survival" amplitude[5]

$$A(t) = (\psi, e^{-iHt}\psi), \tag{1.1}$$

where ψ represents the state of the undecayed system, which is not an eigenstate of the Hamiltonian H. A study[6] of the reduced resolvent obtained by Laplace transform of (1.1),

$$A(z) = (\psi, \frac{1}{z-H}\psi) \tag{1.2}$$

provides a simple way to understand the exponential and long time behavior of (1.1). In fact, in the Lee-Friedrichs model[7], a closed formula may be obtained for the reduced resolvent (1.2). The condition giving the location of the pole in the second sheet of the function $A(z)$ (analytic in the cut plane, and for H semibounded) corresponds precisely to the condition for the construction of a generalized eigenfunction of (an extension of) H which is an element of $\mathcal{H}' \supset \mathcal{H} \supset \mathcal{D}$, a Gel'fand triple for which D is a subspace of $\mathcal{H}$ of functions in the multiplication representation of the unperturbed Hamiltonian which are analytic in some domain of the lower half z-plane including the pole[8]. Generalized eigenstates of this type (corresponding to the states introduced by Gamow[3]) have also played an important role in statistical mechanics where the linear vector space is taken to be the space of Hilbert-Schmidt operators on the Hilbert space of the underlying quantum theory[9]. It has, nevertheless, been difficult to understand the physical meaning of these generalized eigenstates, which are only defined through some kind of analytic continuation.

There is, furthermore, another, perhaps more fundamental problem associated with the general method of Wigner and Weisskopf;[4] this is that the expression (1.1) for the survival amplitude implicitly assumes the existence of a linear superposition

$$e^{-iHt}\psi = A(t)\psi + \chi(t), \tag{1.3}$$

where $\chi(t)$ represents the decayed system and $(\psi, \chi(t)) = 0$. In general, this linear superposition does not correspond to any physical situation; a short-lived particle, for example, is seen as either the particle before the decay, or the decay products at a certain time, which cannot be predicted. The linear superposition does not correspond to the object that we see experimentally in such a process.

It is not difficult to see that an irreversible process must be described by a semigroup (for the reversible case it is a group generated by a unitary transformation). It is well-known that the survival amplitude $A(t)$ of (1.1) cannot satisfy the product law corresponding to a contractive semigroup if the generator H is semibounded[10]. In fact, since with (1.1) the very short time decay is not exponential, but $O(t^2)$, such a theory predicts that a sequence of measurements performed on the unstable system at a sufficiently high frequency inhibits the decay (the so-called Zeno paradox[5]). These features are essentially related to the attempt to describe an unstable system in a framework appropriate to the description of reversible phenomena.

There is, however, within the scope of the general quantum theory[11], a framework more suitable to the description of deterministic, irreversible phenomena. It has been known[12] since 1952 that physical systems cannot be always described in the quantum theory by a single Hilbert space, but, in general, by a direct sum of Hilbert spaces for which the matrix elements of any observable between states belonging to different components vanish. The work of ref. 12 provided the idea of superselection rules, a notion more general than had been previously imagined. There may be many superselection rules in physics. In particular, in the description of the state of a particle, the time plays the role of an uncountable superselection rule, in fact continuous, and the system is therefore described by a family of Hilbert spaces indexed by t.[11]

It is useful to consider this family of Hilbert spaces as a functional space[13]. There is a freedom in the choice of topology in the construction of this space; we shall choose the Lebesgue measure here for the weights of the orthogonal direct sum, but the choice depends, generally, on the mathematical problem. In fact, the procedure for the realization of a Hilbert space as a space of functions with the help of a rigged Hilbert space, or Gel'fand triple, is a special case of a more general result connected with the representation of a Hilbert space as a direct integral of Hilbert spaces.[13] We briefly describe this structure.

Given a Hilbert space, one may choose a positive measure σ on some set, for example the real line, and denote by L^2_σ the space of functions $\phi(x)$ for which the norm with respect to the measure $d\sigma(x)$ converges; with the associated scalar product, one obtains a Hilbert space. The elements of L^2_σ are not separate functions $\phi(x)$ but classes of functions which differ from each other on a set of σ-measure 0. Associating with these functions their values at points , e.g., x_0(except at points of non-zero measure), one does not obtain a continuous linear functional, and one can not speak of their values at such points. In some applications, such as the theory of the spectral decomposition of self-adjoint operators, and the application we wish to treat here, it is desirable to consider the value of such functions at a point as a linear functional. To do this, consider the Gel'fand triple $\Phi \subset \mathcal{H} \subset \Phi'$; to each element of the nuclear space Φ there corresponds a function $\phi(x)$, which is naturally embedded (Φ is dense in $\mathcal{H}$) in $\mathcal{H}$. The realization $\phi \to \phi(x)$ of the space Φ is then said to be induced by the realizations of elements of $\mathcal{H}$ by functions (classes). One then has the

Theorem 1.1[13] Let $\Phi \subset \mathcal{H} \subset \Phi'$ be a rigged Hilbert space and $\phi \to \phi(x)$ be the realization of Φ, as a space of functions, induced by the realization of the Hilbert space as a space L^2_σ. Then, to each value of x one can associate a linear functional F_x on the space Φ such that for any function $\phi(x) \in \Phi$, $\phi(x_0) = F_{x_0}(\phi)$ for almost every x_0 (relative to the measure σ).

We now turn to the question of the orthogonal direct sum of Hilbert spaces, generalizing it to the uncountable case. In particular, consider some set T on which there is defined a positive measure μ. Suppose that with each point $t \in T$ there is associated a separable Hilbert space $\mathcal{H}_t$. Defining a norm

$$\int_T \|h(t)\|^2 d\mu(t) < \infty, \tag{1.4}$$

for $h(t) \in \mathcal{H}_t$ and the corresponding scalar product, one obtains a Hilbert space, called the *direct integral of the Hilbert spaces $\mathcal{H}(t)$ with respect to the measure μ*. The analogous theorem to that stated above then applies, i.e.,

Theorem 1.2[13] Let $\Phi \subset \overline{\mathcal{H}} \subset \Phi'$ be a rigged Hilbert space and

$$\overline{\mathcal{H}} = \int_T \oplus \mathcal{H}_t d\mu(t) \tag{1.5}$$

be the direct integral representation of $\overline{\mathcal{H}}$. Then, for any t there exists a nuclear operator mapping Φ into $\mathcal{H}_t$ such that for $\phi \in \Phi$ the (vector valued) function $\phi(t)$ and the corresponding map of ϕ differ only on a set of μ-measure zero.

Each component in the direct integral (1.5) is a copy of a single Hilbert space $\mathcal{H}$ (as in the usual Schrödinger picture); we shall take, in our application, the embedding to be over Lebesgue measure. Scalar products in $\overline{\mathcal{H}}$ then have the form

$$(f,g) = \int_{-\infty}^{\infty} dt (f_t, g_t)_{\mathcal{H}} \tag{1.6}$$

and the norm squared is

$$\|f\|^2_{\overline{\mathcal{H}}} = \int_{-\infty}^{\infty} dt \|f_t\|^2_{\mathcal{H}}, \tag{1.7}$$

where $(f_t, g_t)_{\mathcal{H}}$ is the scalar product in the space of states $\mathcal{H}_t$ (the states at time t).

Misra, Prigogine and Courbage[14], in their effort to describe irreversible processes, have argued for the existence of a "time operator" in order to define an observable corresponding to a Lyapunov function. Such a time operator could, of course, not exist in the framework of a single Hilbert space with semi-bounded Hamiltonian[15]. Several authors[16] have therefore considerably developed the theory of evolution of quantum systems in the framework of the space of Hilbert-Schmidt operators, the so-called Liouville space, for which the generator of evolution is the Liouville operator (the prototype for this operator is the commutator with the Hamiltonian) and hence is not necessarily semi-bounded.

What we wish to show is that the structure of (1.5), within the scope of ordinary quantum theory, provides a framework for the description of the semigroup evolution of deterministic irreversible systems. This is, in fact, a solution of the problem. The direct sum space plays the role of the generalized spaces obtained in the analysis of the Wigner-Weisskopf expression (1.1); the eigenvalues of the generator of the semigroup correspond to the poles in the second sheet of the resolvent, and the eigenfunctions to the generalized eigenfunctions for the complex eigenvalue.*

The basic mathematical ideas which we shall apply to this problem were formulated by Lax and Phillips[18]; these have been adapted to the quantum theory by Flesia and Piron[19]. The essential mechanism introduced by Lax and Phillips[18] (and maintained by Flesia and Piron[19]) for achieving the semigroup property is the decomposition of the large space $\overline{\mathcal{H}}$ into incoming and outgoing subspaces, and their orthogonal complement. In the quantum case, for bound states and the problem of simple scattering, the results of the usual reversible theory are obtained. For the unstable system, which must be created (prepared) at some time, we find the semigroup behavior characteristic of deterministic irreversible evolution.

2 Lax-Phillips Theory in the Quantum Case

In this section, for the sake of completeness and to establish our notation, we review the construction of Flesia and Piron[19].

The evolution of a system, described by $\psi \in \overline{\mathcal{H}}$, is given by the family of operators $V_t(\tau)$, unitary on $\mathcal{H}_t$, according to (see also ref. 1, p.111)

$$\psi^{\tau}_{t+\tau} = V_t(\tau)\psi_t. \tag{2.1}$$

Since $V_t(\tau)$ represents an evolution, it follows that

$$V_{t+\tau_1}(\tau_2)V_t(\tau_1) = V_t(\tau_1 + \tau_2). \tag{2.2}$$

Eq.(2.1) defines a unitary evolution in $\overline{\mathcal{H}}$. Let us define $U(\tau)$ on $\overline{\mathcal{H}}$ such that $\psi^{\tau} = U(\tau)\psi$, i.e.,

$$\psi^{\tau}_{t+\tau} = (U(\tau)\psi)_{t+\tau} = V_t(\tau)\psi_t. \tag{2.3}$$

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* Note that, as pointed out in the first of refs. 8, the analytic continuation method of extending the Hamiltonian operator to obtain complex eigenvalues is not the only way to achieve such a result; for dilation analytic potentials, one may analytically continue the dilation parameter to rotate the continuous spectrum, and obtain the complex eigenvalues on the first sheet of the modified resolvent[17]. The use of the direct sum space in this way is yet another, but carries with it important physical interpretation that is lacking in the other, more *ad hoc* methods.

The operators $U(\tau)$ are then unitary:

$$\begin{aligned}(U(\tau)\psi, U(\tau)\phi)_{\overline{\mathcal{H}}} &= \int_{-\infty}^{+\infty} dt (V_t(\tau)\psi_t, V_t(\tau)\phi_t)_{\mathcal{H}} \\ &= \int_{-\infty}^{+\infty} dt (\psi_t, \phi_t)_{\mathcal{H}} \\ &= (\psi, \phi)_{\overline{\mathcal{H}}}.\end{aligned}$$

Furthermore, the $U(\tau)$ form a one-parameter group, i.e.,

$$\begin{aligned}(U(\tau_2)U(\tau_1)\phi)_{t+\tau_1+\tau_2} &= V_{t+\tau_1}(\tau_2)V_t(\tau_1)\phi_t \\ &= V_t(\tau_1+\tau_2)\phi_t \\ &= (U(\tau_1+\tau_2)\phi)_{t+\tau_1+\tau_2}.\end{aligned}$$

If the action of $U(\tau)$ is continuous, it has a self-adjoint generator

$$K = s - \lim_{\tau\to 0} \frac{1}{\tau}(U(\tau) - I). \tag{2.4}$$

According to $(1.4)-(1.6)$, the space $\overline{\mathcal{H}}$ can be represented as $L^2(-\infty, \infty; \mathcal{H})$, with measure dt, where we identify the "auxiliary" space $\mathcal{H}$ with the usual Hilbert space of the quantum theory. We can then define the operator $i\partial_t$ on this space and assert the following

Theorem 2.1 If there exists a subset dense in $L^2(-\infty, \infty; \mathcal{H})$ on which the operators K,$i\partial_t$ and $i\partial_t + K$ are essentially self-adjoint, then H, defined as the self-adjoint extension of $i\partial_t + K$, is a decomposable operator ($(H\phi)_t = H_t\phi_t$). The proof is given in ref. 1, p. 112.

Theorem 2.2. If $\|V_t(\tau)\phi_t\|_{\overline{\mathcal{H}}}$ is measurable in t,τ and Theorem 2.1 applies, then K is unitarily equivalent to $-i\partial_t$ and hence its spectrum is absolutely continuous in $(-\infty, \infty)$.

The proof will be useful for the sequel, so that we shall reproduce it here (ref. 1, p.113 and ref.19). Let us define $R(t_0)$ on $\overline{\mathcal{H}}$ such that

$$\begin{aligned}(R(t_0)\phi)_t &= V_t(t_0 - t)\phi_t \\ &= V_{t_0}^{-1}(t - t_0)\phi_t.\end{aligned} \tag{2.5}$$

The second of these follows from the fact that $V_t(t_0 - t)\phi_t = \phi_{t_0}^{t_0-t}$, and, conversely, $V_{t_0}(t - t_0)\phi_{t_0}^{t_0-t} = \phi_t$. Then,

$$\begin{aligned}(R(t_0)e^{-iK\tau}R^{-1}(t_0)\phi)_t &= V_{t_0}^{-1}(t - t_0)(e^{-iK\tau}R^{-1}(t_0)\phi)_t \\ &= V_{t_0}^{-1}(t - t_0)V_{t-\tau}(\tau)(R^{-1}(t_0)\phi)_{t-\tau} \\ &= V_{t_0}^{-1}(t - t_0)V_{t-\tau}(\tau)V_{t_0}(t - \tau - t_0)\phi_{t-\tau}.\end{aligned}$$

According to the rule of composition (2.2), this is

$$(R(t_0)e^{-iK\tau}R^{-1}(t_0)\phi)_t = \phi_{t-\tau},$$

so that

$$R(t_0)KR^{-1}(t_0) = -i\partial_t, \tag{2.6}$$

where we have used the notation "$-i\partial_t$" to stand for the operator with the property

$$(-i\partial_t\phi)_t = -i\partial_t\phi_t.$$

Hence, $R(t_0)$ is the unitary transformation on $\overline{\mathcal{H}}$ which describes the usual connection between the Schrödinger picture and the Heisenberg picture (for which evolution is trivial, i.e., translation in time). It intertwines

$$K = H - i\partial_t \tag{2.7}$$

and

$$K_0 = -i\partial_t. \tag{2.8}$$

We remark that since H is decomposable, we have

$$\begin{aligned} i\frac{\partial}{\partial\tau}(U(\tau)\phi)_t &= (U(\tau)K\phi)_t \\ &= V_{t-\tau}(\tau)(K\phi)_{t-\tau} \\ &= V_{t-\tau}(\tau)(H_{t-\tau}\phi_{t-\tau} - i\partial_t\phi_{t-\tau}). \end{aligned}$$

The Schrödinger equation (for a possibly time-dependent Hamiltonian as well),

$$i\partial_t\phi_t = H_t\phi_t \tag{2.9}$$

then corresponds to a ϕ which is an eigenvector of K with zero eigenvalue (generalized, in the sense of Gel'fand[13], since K has continuous spectrum *). Under this condition, somewhat analogous to the notion of equilibrium, there is no real evolution of the wave function in $\overline{\mathcal{H}}$.

Lax and Phillips[12] study the properties of a system described by $\overline{\mathcal{H}} = L^2(-\infty,\infty;\mathcal{H})$, with evolution defined by a one-parameter unitary group $U(\tau)$. With the application to scattering of classical electromagnetic waves on a finite target in mind, they assume the

* It is clear that if the set of functions $\{\phi_t\}$ is related by the Schrödinger equation, their $\mathcal{H}$-norm is invariant along t, and hence their $\overline{\mathcal{H}}$ norm, defined in (1.7) does not exist.

existence of two subspaces D_- (incoming) and D_+ (outgoing), for which D_+ is orthogonal to D_-,

$$\begin{aligned} U(\tau)D_+ &\subset D_+ \qquad \tau \geq 0 \\ U(\tau)D_- &\subset D_- \qquad \tau \leq 0, \end{aligned}$$

$$\bigcap_\tau U(\tau)D_- = \bigcap_\tau U(\tau)D+ = \{0\}$$

and

$$\overline{\bigcup_\tau U(\tau)D_-} = \overline{\bigcup_\tau U(\tau)D_+} = \overline{\mathcal{H}}. \tag{2.10}$$

The subspaces $D_\pm$ are to be associated with functions which are not affected by the target (respectively, before and after the scattering) and hence can be identified with free waves that would pass if the target were not there. For the incoming or outgoing subspaces, there are mappings into representations for which the evolution is just translation.

Theorem 2.3 (Lax and Phillips[18]) There is a Hilbert space (in our notation the space of states $\mathcal{H}$) called an auxiliary space[18], and $W_\pm$ such that

$$\begin{aligned} W_+D_+ &= L^2(\rho_+, \infty; \mathcal{H}) \\ W_-D_- &= L^2(-\infty, \rho_-; \mathcal{H}), \end{aligned} \tag{2.11}$$

for some finite $\rho_+ \geq 0, \rho_- \leq 0$; in these representations, $U(\tau)$ acts everywhere as translation, and trivially on $\mathcal{H}$, i.e.,

$$((W_\pm U(\tau)W_\pm^{-1})\chi)_t = \chi_{t-\tau}. \tag{2.12}$$

We can clearly identify $W_\pm$ with $R(\rho_\pm)$;[19] the latter operators map the Schrödinger functions to the translations of ϕ_{ρ_+} or ϕ_{ρ_-} (where the two representations coincide, since $V_t(0) = I$). We emphasize that these transformations do not correspond to modified laws of evolution on correspondingly altered wave functions, as in the usual form of quantum theory on a single Hilbert space, but simply to a change of representation in $\overline{\mathcal{H}}$.

Furthermore, in the quantum mechanical case, the interacting scattering waves can be put into correspondence with free waves only asymptotically, so that in general $\rho_\pm$ may have to taken very large in order that the Lax-Phillips S-matrix, defined as $S^{LP} = W_+W_-^{-1}$, or correspondingly,

$$S^{LP}(\rho_-, \rho_+) = R(\rho_+)R(\rho_-)^{-1}, \tag{2.13}$$

coincides with the S-matrix of quantum mechanical scattering theory. The S-matrix defined by (2.13) is always decomposable. For a homogenenous system, for which H is independent of t, one finds (independently of t)

$$S^{LP}(\rho_-, \rho_+) = e^{iH_0\rho_+}e^{-iH\rho_+}e^{iH\rho_-}e^{-iH_0\rho_-}. \tag{2.14}$$

In the limit $\rho_+ \to +\infty, \rho_- \to -\infty$, this becomes $\Omega_+^\dagger \Omega_-$, in the sense of a bilinear form on a dense set, which defines the S-matrix of the usual quantum mechanical scattering theory. The necessity of taking the limits can be understood physically, since the S-matrix is a relation between asymptotically free waves. The translation representations for D_+ or D_- can be interpreted as the free motion only asymptotically, as in the usual theory. The properties (2.10) are maintained for $\rho_\pm$ large, since the asymptotic waves span the space (when asymptotic completeness holds) of the continuous spectrum of $\mathcal{H}$.

The physical interpretation of this structure is that at each t, one specifies a state in $\mathcal{H}$ which is a possible initial condition. In the absence of interaction, the evolution translates this set of initial conditions along the t-axis without change. With interaction, the entire set of initial conditions is shifted along with the continuous mappings that correspond to the effect of the dynamics. The $\mathcal{H}$-norm of these states must be taken to vanish sufficiently rapidly for $t \to \pm\infty$ so that this set may be considered a proper element of $\overline{\mathcal{H}}$. As we have pointed out, the set of initial conditions that are related by Schrödinger evolution does not constitute an element of $\overline{\mathcal{H}}$, but can only be described as a generalized function.

3 The Unstable System

In the evolution of an unstable system which we assume to be deterministic but irreversible, there is an earliest time beyond which it is inconceivable that the system existed in its characteristic form. One may consider that the system was prepared at this time, that is, by some mechanism for which there is no deterministic model, or that some idealized structure existed in a persistent configuration, and at this time an interaction was "turned on" (smoothly but in arbitrarily short time) which puts the system into its usual configuration, i.e., its normal unstable mode. It is convenient for our mathematical treatment to take this latter point of view, but the physical interpretation and predictions are concerned only with observations that can be made after this initial time.

Let us consider a Hamiltonian H of the form

$$H = H_0 + V(t), \tag{3.1}$$

where H_0 is the Hamiltonian of an ideal undecayed persistent system and

$$V(t) = \begin{cases} 0 & t \leq 0; \\ V & t > 0, \end{cases} \tag{3.2}$$

is the interaction which causes the system to decay; the onset of the interaction, here described as instantaneous, must be taken as smooth but over as small an interval as we wish. Under the evolution

$$U(\tau) = e^{-i(H - i\partial_t)\tau}, \tag{3.3}$$

where H is the decomposable operator defined in Theorem (2.1), we may carry out the (pointwise) transformation

$$\psi_t = e^{-iH_0 t}\phi_t, \tag{3.4}$$

inducing a change in basis in $\overline{\mathcal{H}}$. In this new basis, H is replaced by[19]

$$v(t) = e^{iH_0 t} V(t) e^{-iH_0 t}. \tag{3.5}$$

The evolution $V_t(\tau)$ is then defined by

$$\left(e^{-i(v(t)-i\partial_t)\tau}\phi\right)_{t+\tau} = V_t(\tau)\phi_t. \tag{3.6}$$

The evolution operator can, by definition, be represented by the Trotter formula

$$\begin{aligned} e^{-i(v(t)-i\partial_t)\tau} &= \lim_{n\to\infty}\left(e^{-iv(t)\tau/n}e^{-\partial_t\tau/n}\right)^n \\ &= \lim_{n\to\infty}\left(e^{-\partial_t\tau/n}e^{-iv(t)\tau/n}\right)^n, \end{aligned} \tag{3.7}$$

where $v(t)$ has a common domain with $-i\partial_t$ (it is sufficiently smooth in t, as we have assumed above). Then, by considering successive terms from left to right of (3.7) in (3.6), one can show that for $v(t') = 0, t' \in [t, t+\tau], V_t(\tau) = 1$. Since $v(t) = 0$ for $t \leq 0$, we have

$$V_t(\tau) = 1 \qquad t \leq 0, t+\tau \leq 0, \tag{3.8}$$

and the evolution is translation (the evolved function at $t+\tau$ is equal to the initial function at t).

The intertwining operator $R(\rho_-)$, for any $\rho_- \leq 0$ and $t \leq 0$ is therefore unity, since

$$(R(\rho_-)\phi)_t = V_t(\rho_- - t)\phi_t, \tag{3.9}$$

and $\rho_- - t + t = \rho_-$ (the alternative expression, the second of (2.5), $V_{\rho_-}^{-1}(t-\rho_-)$, satisfies, as it must, the same condition). We shall therefore take $\rho_- = 0$ in the sequel, and call $\rho_+ \equiv \rho$. The intertwining operator $R(\rho)$, for $\rho > 0$, acting as

$$(R(\rho)\phi)_t = V_t(\rho - t)\phi_t, \tag{3.10}$$

is not, in general, trivial for any t, since $\rho - t + t = \rho$. We then have

$$\begin{aligned} (S(0,\rho)\phi)_t &= (R(\rho)R(0)^{-1}\phi)_t \\ &= V_t(\rho - t)(R(0)^{-1}\phi)_t \\ &= V_t(\rho - t)V_0(t)\phi_t. \end{aligned}$$

We thus obtain, for $t = 0$,

$$\begin{aligned} (S(0,\rho)\phi)_0 &= V_0(\rho)\phi_0 \\ &= \phi_\rho^\rho, \end{aligned} \tag{3.11}$$

the state of the system evolved from $t = 0$ to $t = \rho$.

During the time $t \leq 0$, external and internal interactions are in the process of forming what is to become the unstable system; we do not attempt to describe this process. In our model, we shall consider the evolution of the system in this domain to be simple translation of a set of states characteristic of the ideally defined initial system before the interaction which causes the decay is turned on, taking $R(0) = 1$. Hence, we consider our basic representation as the one which is translation for D_-. The evolution of the system in the domain $t \geq 0$ reflects the action of the instability inducing interaction, and it is this evolution which we shall make an effort to characterize. For some very long time after this interaction is introduced, almost all of the evolution becomes that of the free decay products of the unstable system, which are no longer affected by the interaction; it then becomes again simple translation. We shall associate this domain with the outgoing subspace D_+. The description of the evolution of the system in the intermediate domain, which is of essential physical interest, corresponds to the Lax-Phillips semigroup, to be discussed in the next section.

We denote the functions corresponding to D_- in the incoming representation by $f_- \in L^2(-\infty, 0; \mathcal{H})$. Then, the functions $\hat{f}^{(-)}$ taking into account all interactions are given by

$$\hat{f}_t^{(-)} = R(0)^{-1} f_-. \tag{3.12}$$

According to (2.5) and (3.8), $\hat{f}^{(-)} = (f_-)_t$ for $t \leq 0$, and $\hat{f}_t^{(-)} = 0$ for $t > 0$. The mapping $R(0)$ of $\hat{f} \in \overline{\mathcal{H}}$ to the translation representation of D_- is, however, defined for all t ($V_t(-t)$ is, in general, not the unit operator for $t > 0$).

In the outgoing representation, the functions $f_+ \in L^2(\rho, \infty; \mathcal{H})$ with evolution represented as translation, are related to the "interacting waves" $\hat{f}^{(+)} \in D_+$ by

$$\hat{f}^{(+)} = R(\rho)^{-1} f_+,$$

which, according to (2.5), is zero for $t < 0$. The mapping $R(\rho)$ of $\hat{f} \in \overline{\mathcal{H}}$ to the outgoing representation is also defined for all t.

The mapping from the outgoing representation to the incoming representation is therefore given by

$$S(0, \rho)^{-1} = R(0) R(\rho)^{-1} \tag{3.13}$$

and D_+ is represented in the incoming representation as (pointwise)

$$S(0, \rho)^{-1} L^2(\rho, \infty; \mathcal{H}), \tag{3.14}$$

i.e., each function $f_+ \in L^2(\rho, \infty; \mathcal{H})$, in the outgoing representation, corresponds to a function $S(0, \rho)^{-1} f_+$ in the incoming representation. We shall use these representations in the discussion of the evolution of an unstable system in the next section.

4 The Lax-Phillips Semigroup

Lax and Phillips[18] define the operator on $\overline{\mathcal{H}}$

$$Z(\tau) = P_+ U(\tau) P_-, \tag{4.1}$$

where P_- is the projection on the orthogonal complement of D_- and P_+ the projection on the orthogonal complement of D_+. This operator vanishes on D_+ and D_- and maps the subspace

$$\mathcal{K} = \overline{\mathcal{H}} \ominus (D_+ \oplus D_-) \tag{4.2}$$

into itself[18]. The mapping is a semigroup

$$Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2), \tag{4.3}$$

and is strongly contractive, i.e., there exists a τ_ϕ for any $\phi \in \mathcal{K}$ such that

$$\|Z(\tau)\phi\|_{\overline{\mathcal{H}}} < \epsilon \tag{4.4}$$

for any $\tau \geq \tau_\phi$ and any $\epsilon > 0$.

According to (3.14) and (4.2), the functions of the subspace $\mathcal{K}$ in the incoming representation are in

$$\mathcal{K} = L^2(-\infty, \infty; \mathcal{H}) \ominus \{L^2(-\infty, 0; \mathcal{H}) \oplus S^{-1}(0,\rho)L^2(\rho, \infty; \mathcal{H})\}$$

or

$$\mathcal{K} = L^2(0, \infty; \mathcal{H}) \ominus S^{-1}(0,\rho)L^2(\rho, \infty; \mathcal{H}). \tag{4.5}$$

As pointed out in Section 3, one may regard the structure of the subspace $\mathcal{K}$ as referring to a system created at $t = 0$; no explicit reference is made to functions with support on the negative half line. Implicitly, however, we make use of the incoming representation as the natural description giving identity to the initial system before the interaction causing instability is turned on.

The essential nature of the truncation at $t = 0$ is illustrated by the fact that every semigroup of contraction operators can be represented as a translation followed by a truncation acting on a translation invariant subspace, provided it tends strongly to zero as $t \to \infty$ (see Lax and Phillips[18], p. 67; one can work equally well with $L^2(0, \infty; \mathcal{H})$).

From (3.11), for $t \geq 0$, we see that the S-matrix is independent of t‡ and

$$S^{-1}(0,\rho) = V_0(\rho)^{-1} = V_\rho(-\rho). \tag{4.6}$$

The latter form provides an interpretation of the structure of the subspace $\mathcal{K}$. Since

$$(U(-\rho)\phi)_{t-\rho} = V_t(-\rho)\phi_t, \tag{4.7}$$

it follows from (4.6) that

$$(U(-\rho)\phi)_0 = V_\rho(-\rho)\phi_\rho, \tag{4.8}$$

i.e., the functions ϕ_t of the outgoing representation, when transformed to the incoming representation, can be put into correspondence with a set of elements of $\overline{\mathcal{H}}$, $\{U(-\rho)\phi\}$, which have support in $t \in (0, \infty)$, but are modified by the interaction.

‡ A generalization of the evolution law (2.1) for which $V(\tau)$ acts as a non-trivial integral kernel on the t variable (as it can in a different representation of t) will be discussed in a succeeding publication. In this case, the S- matrix is a (operator-valued) kernel of the form $S(t - t')$, and its Fourier transform is singular on the spectrum of the generator of the Lax-Phillips contractive semigroup, as in the general Lax-Phillips theory.

Note that if $v(t) \equiv 0$, $U(\tau)$ takes the form $e^{-\tau\partial_t}$, so that

$$(U(\tau)\phi)_{t'+\tau} = e^{-\tau\partial_t}\phi_t|_{t'+\tau} = \phi_{t-\tau}|_{t'+\tau} = \phi_{t'}; \tag{4.9}$$

hence $V_t(\tau) \equiv 1$. In this case $(U(-\rho)\phi)_0 = \phi_\rho$, i.e., the correspondence is just by translation.

Since in this limit $S(0,\rho) = 1$, the subspace $\mathcal{K}$ reduces to

$$\begin{aligned} \mathcal{K}^0 &= L^2(0,\infty;\mathcal{H}) \ominus L^2(\rho,\infty;\mathcal{H}) \\ &= L^2(0,\rho;\mathcal{H}), \end{aligned} \tag{4.10}$$

the set of L^2 functions with support in $t = (0,\rho)$.

For ρ sufficiently large, the functions in D_+ can be put into correspondence with the asymptotic states of scattering theory ($S(0,\rho)$ can be expected to converge, as in (2.14)). If we are concerned only with the behavior of the system for $t > 0$, the projection operator P_-, in the construction of $Z(\tau)$, acts as unity. If ρ becomes very large, so that D_+ corresponds to asymptotic states, the operator $Z(\tau)$ corresponds to the evolution of the system in the interval $(0,\rho)$. Its contractive semigroup behavior follows from the restriction placed on the full unitary evolution in $\overline{\mathcal{H}}$ by P_- and P_+, i.e., that we are concerned only with the behavior of the system after it is created at $t = 0$, and after some large time ρ, the "unstable system" is essentially gone, i.e., only asymptotic states remain, corresponding to a new equilibrium state (for the decaying particle, for example, only the decay products).

As shown by Lax and Phillips[18], the strongly contractive property of $Z(\tau)$ follows directly from the fact that the left (dynamical, i.e., under $U(\tau)$) translates of D_+ are dense in $\overline{\mathcal{H}}$. If the left translates of the asymptotic states are dense in $\mathcal{K}$ (consistent with asymptotic completeness[20]), the strong contractive property remains for any large ρ.

5 Imprimitivity of the Full Evolution

In this section we study the Lax-Phillips theory from a different point of view. The theorems of Lax and Phillips[18] follow by direct construction in terms of the spectral family of an operator on $\overline{\mathcal{H}}$, conjugate to the full evolution K, which we shall call T, corresponding to a "time operator." Some of the essential properties of the Lax-Phillips construction are clarified in this way.

We start by postulating the existence of a self-adjoint operator T on $\overline{\mathcal{H}}$ (with absolutely continuous spectrum) with spectral representation[21],

$$T = \int_{-\infty}^{\infty} t\,dE(t), \tag{5.1}$$

and a unitary family $U(\tau) = e^{-iK\tau}$ such that

$$U(\tau)TU(\tau)^{-1} = T - \tau I, \tag{5.2}$$

i.e., on a suitable domain, $[T, K] = i$. In the spectral representation corresponding to the operator T, the generator K has the form (2.8), i.e., just translation. This structure can be realized by transforming the operator conjugate to (2.7) by the same wave operators that bring (2.7) to (2.8). The dynamical information for the system is thus contained in the spectral family for the new operator T.

By the uniqueness of the spectral family for a self-adjoint operator, we have the imprimitivity relation[23]

$$U(\tau)E(t)U(\tau)^{-1} = E(t+\tau). \tag{5.3}$$

Under these conditions, $\overline{\mathcal{H}}$ can be represented as $L^2(-\infty, \infty; \mathcal{H})$ such that $U(\tau)$ corresponds to a right translation by τ and T to multiplication by the independent variable (see ref. 18, p. 35).

For our example, which is, in effect, equivalent to the so-called "free representation"[18], we take D_- to be the subspace defined by

$$D_- : E(0) = \int_{-\infty}^{0} dE(t) \tag{5.4}$$

and

$$D_+ : I - E(\rho) = \int_{\rho}^{\infty} dE(t) \tag{5.5}$$

It is clear from the properties of a spectral family that

$$E(0) \supset E(-\tau) \to 0$$

and

$$I - E(\rho) \supset I - E(\rho+\tau) \to 0 \tag{5.6}$$

for $\tau \to \infty$, and that the requirements (2.10) are satisfied.

In this case, according to (4.2), the projection to $\mathcal{K}$ is given by

$$\begin{aligned} P_{\mathcal{K}} &= I - (I - E(\rho)) - E(0) \\ &= E(\rho) - E(0). \end{aligned} \tag{5.7}$$

Since

$$\begin{aligned} P_+ &= E(\rho) \\ P_- &= I - E(0), \end{aligned} \tag{5.8}$$

it follows that

$$\begin{aligned} Z(\tau) &= P_+ U(\tau) P_- \\ &= E(\rho)U(\tau)(I - E(0)) \\ &= (E(\rho) - E(\rho)E(\tau))U(\tau) \\ &= \begin{cases} (E(\rho) - E(\tau))U(\tau) & \tau \le \rho \\ 0 & \tau > \rho. \end{cases} \end{aligned} \tag{5.9}$$

An alternative form can be found for $Z(\tau)$ by bringing $U(\tau)$ to the left, instead of to the right, i.e.,

$$Z(\tau) = \begin{cases} U(\tau)(E(\rho-\tau) - E(0)) & \tau \leq \rho \\ 0 & \tau > \rho. \end{cases} \tag{5.10}$$

The semigroup property of $Z(\tau)$ follows directly from (for $0 \leq \tau_1, \tau_2 \leq \rho$)

$$\begin{aligned} Z(\tau_1)Z(\tau_2) &= (E(\rho) - E(\tau_1))U(\tau_1)(E(\rho) - E(\tau_2))U(\tau_2) \\ &= (E(\rho) - E(\tau_1))(E(\rho+\tau_1) - E(\tau_1+\tau_2))U(\tau_1+\tau_2) \\ &= \begin{cases} (E(\rho) - E(\tau_1+\tau_2))U(\tau_1+\tau_2) & \tau_1+\tau_2 \leq \rho \\ 0 & \tau_1+\tau_2 \geq \rho. \end{cases} \end{aligned} \tag{5.11}$$

These structures are of the form of the dilation construction of Sz.-Nagy and Foias.[21,22]

The contractive property follows just from the support on $L^2(-\infty,\infty;\mathcal{H})$, i.e., for $x \in \mathcal{K}$, and $\tau \leq \rho$ (P_- acts as the identity on $\mathcal{K}$), *

$$\begin{aligned} \|Z(\tau)x\|^2 &= \|(E(\rho) - E(\tau))U(\tau)x\|^2 \\ &= \int_\tau^\rho dt \|(U(\tau)x)_t\|^2_{\mathcal{H}} \\ &= \int_\tau^\rho dt \|x^\tau_{t-\tau}\|^2_{\mathcal{H}} \\ &= \int_0^{\rho-\tau} dt \|x^\tau_t\|^2_{\mathcal{H}} \leq \|x\|^2. \end{aligned} \tag{5.12}$$

We see that the rate of decrease of $\|Z(\tau)x\|$ in $0 \leq \tau \leq \rho$ depends on the support of x in $\mathcal{K}$. For $\rho \to \infty$, and any fixed τ, $\|Z(\tau)x\|^2 = \|x\|^2$, and the contractive semigroup appears to become a group. However, for any fixed ρ, there is a τ sufficiently large so that the left hand side of (5.11) is as small as we wish. To resolve this ambiguity, we calculate the generator of the semigroup explicitly in the following.

The generator for the semigroup can be calculated explicitly from (5.9). We do this here at $\tau = 0$. Taking the derivative with respect to τ and setting $\tau = 0$, we obtain

$$i\frac{dZ(\tau)}{d\tau}|_{\tau=0} = E(\rho)K(I - E(0) \equiv A. \tag{5.13}$$

* In the proof of Lax and Phillips[18] that the semigroup is strongly contractive, the fact that $\{U(\tau)D_+\}$ is dense in $\overline{\mathcal{H}}$ is used (leading to the result (4.4)) in the following way. There is a $y \in D_+$ and a τ_0 such that for any $x \in \mathcal{K}$,

$$\|x - U(-\tau_0)y\| < \epsilon.$$

It then follows (since $Z(\tau)U(-\tau_0)y = P_+U(\tau-\tau_0)y = 0$ for $\tau > \tau_0$) that $\|Z(\tau)x - Z(\tau)U(-\tau_0)y\| = \|Z(\tau)x\| \leq \|x - U(-\tau_0)y\| < \epsilon$ for $\tau > \tau_0$; hence the semigroup is strongly contractive. Since we are dealing in this case with simple translation, we may set, for $x \in \mathcal{K}$, $y = U(\tau_0)x \in D_+$ for $\tau_0 \geq \rho$. It then follows from this argument that $\|Z(\tau)x\| = 0$ for all $\tau \geq \rho$, as we have concluded directly above.

The imprimitivity relation (5.3) implies (on a suitable domain)

$$[K, E(\tau)] = i\frac{dE(\tau)}{d\tau}, \tag{5.14}$$

so that (5.13) is equivalent to

$$A = P_{\mathcal{K}}K - i\frac{dE(0)}{d\tau}$$

or

$$A = KP_{\mathcal{K}} - i\frac{dE(\rho)}{d\tau}. \tag{5.15}$$

These relations also follow from the alternative forms (5.9) and (5.10). Since K and $P_{\mathcal{K}}$ are self-adjoint in $\overline{\mathcal{H}}$, the anti-Hermitian part of the generator is

$$A - A^{\dagger} = -i(\frac{dE(0)}{d\tau} + \frac{dE(\rho)}{d\tau}). \tag{5.16}$$

Since $dE(\tau)/d\tau \geq 0$ for all τ, this is a non- positive imaginary operator. It also follows[24] from this result that the semigroup is contractive[25].

This conclusion remains valid for ρ very large. In this case, one sees from (5.9) that if $E(\rho)$ is close to the unity, the principal mechanism for the contraction is the action of P_{-}, the projection which removes the elements of $L^2(-\infty, 0; \mathcal{H})$.

It is rather remarkable that the "free case" contains so much structure. The non-trivial content of this structure is governed by the spectral family $\{E(t)\}$, on which we have imposed no restriction but absolute continuity.

6 Conclusions

We have extended the work of Flesia and Piron[19], which demonstrated that quantum theory can be embedded in a theory of the type developed by Lax and Phillips[18]. This was done by constructing a larger Hilbert space as a direct integral of quantum mechanical Hilbert spaces over time.

As we have seen, the interaction picture of quantum theory provides concrete models for which the interaction term of the Lax-Phillips formulation (i.e., the first term of the right hand side of Eq.(2.7)) acts decomposably (pointwise on the t-axis). There are many such applications; for example, the Lee-Friedrichs model[7], a soluble model in quantum mechanical scattering theory, also provides a soluble model in this framework. We also point out that the fact that the transformation between the Heisenberg and Schrödinger pictures is, in the Lax-Phillips formulation, a single global unitary transformation, can be exploited to study the relation between these pictures (in some cases, as in quantum field theory in the usual framework, this is a non-trivial mathematical question).

The larger Hilbert space carries a description of the global motion of the system. The physical states, for which the time evolution is governed by the Schrödinger equation, emerge as generalized states (elements of a Gel'fand triple[13]) associated with the larger space. The vectors of the larger space therefore do not correspond to states, but to an auxiliary structure in which the dynamical evolution of the system is described.

Since the generator of unitary evolution in the larger space has continuous spectrum in $(-\infty,\infty)$, a conjugate "time" operator can be defined. We show in the simple example for which incoming and outgoing subspaces of the larger space are determined by projections from the spectral family of this time operator, that the contraction semigroup of Lax and Phillips has a generator whose non-self-adjoint part is proportional to the derivatives of the spectral family of the time operator at the end points of the interval between the support bounds of the incoming and outgoing subspaces. These are projections to the defect subspaces of the operator $-i\partial_t$ restricted to the interval. More generally, if the generator contains an additional term which is self-adjoint when restricted to this interval (as for an operator which acts decomposably on the larger space and is self-adjoint on the quantum mechanical Hilbert space), this conclusion remains valid. It therefore is a general property of the application of Lax-Phillips theory to quantum mechanics.

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References

1. C. Piron,*Foundations of Quantum Physics*, Benjamin/Cummings, Reading, Mass. (1976).
2. T.T. Wu and C.N. Yang, Phys. Rev. Lett.**13**, 380 (1964); T.D. Lee, R. Oehme and C.N. Yang, Phys. Rev. **106**, 340 (1957).
3. G. Gamow, Zeits. f. Phys. **51**, 204 (1928).
4. V.F. Weisskopf and E.P. Wigner, Zeits. f. Phys. **63**, 54 (1930); **65**, 18 (1930).
5. C. Chiu, E.C.G. Sudarshan and B. Misra, Phys. Rev. **D16**, 5201 (1979). See also, L.A. Khalfin, Dokl. Akad. Nauk. USSR, **115**, 277 (1957) and Zh. eksper. teor. fiz. **33**, 1371 (1957), for early studies of the long and short time behavior.
6. L.P. Horwitz and J.-P. Marchand, Rocky Mtn. Jour. Math. **1**, 225 (1973).
7. T.D. Lee, Phys. Rev. **95**, 1329 (1956); K.O. Friedrichs, Comm. Pure Appl. Math **1**, 361 (1950).
8. L.P. Horwitz and I.M. Sigal, Helv. Phys. Acta **51**, 685 (1978); W. Baumgartel, Math. Nachr. **75**, 133 (1978). See also, G. Parravicini, V. Gorini and E.C.G. Sudarshan,

Jour. Math. Phys. **21**, 2208 (1980); A. Bohm, *The Rigged Hilbert Space and Quantum Mechanics*, Springer Lecture Notes on Phys. **78**, Berlin (1978); A. Bohm, *Quantum Mechanics:Foundations and Applications*, Springer, Berlin (1986); A. Bohm, M. Gadella and G.B. Mainland, Am. Jour. Phys. **57**, 1103 (1989); T. Bailey and W.C. Schieve, Nuovo Cim. **47A**, 231 (1978).

9. I. Prigogine, C. George and F. Henin, Proc. Nat. Aca. Sci. **65**, 789 (1972); R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics*, Wiley, New York (1977);

10. L.P. Horwitz, J.-P. Marchand, and J. LaVita Jour. Math. Phys. **12**, 2537 (1971); D. Williams, Comm. Math. Phys. **21**, 314 (1971).

11. C. Piron, *Mecanique Quantique, Bases et Applications*, p.61, Presses polytechnique et universitaires romandes, Lausanne (1991). See also, C. Piron, Helv. Phys. Acta **42**, 330 (1969).

12. G.C. Wick, A.S. Wightman and E.P. Wigner, Phys. Rev **88**, 101 (1952).

13. I.M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, vol. 4, p. 110 ff., Academic Press, N.Y. (1964).

14. B. Misra, Proc. Nat. Aca. Sci. **75**, 1627 (1978); B. Misra, I. Prigogine and M. Courbage, Proc. Nat. Aca. Sci. **76**, 4768 (1979). See also, ref. 11.

15. For example, K. Gottfried, *Quantum Mechanics, vol. 1, Fundamentals*, p. 248, Benjamin/Cummings, Reading (1966).

16. C. George, Physica **65**, 277 (1973); I. Prigogine, C. George, F. Henin and L. Rosenfeld, Chemica Scripta **4**, 5(1973); T. Petrosky, I. Prigogine and S. Tasaki, Physica **A173**, 175 (1991); T. Petrosky and I. Prigogine, Physica **A175**, 146 (1991), and references therein.

17. J.M. Combes, *Proc. of Int. Congress of Math.*, Vancouver (1974); J. Aguilar and J.M. Combes, Comm. Math. Phys. **22**, 269 (1971); E. Baslev and J.M. Combes, Comm. Math. Phys. **22**, 280 (1971); B. Simon, Ann. of Math. **97**, 247 (1973).

18. P.D. Lax and R.S. Phillips, *Scattering Theory,* Academic Press , N.Y. (1967).

19. C. Flesia and C. Piron, Helv. Phys. Acta **57**, 697 (1984).

20. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, vol. 3, Scattering Theory*, p.4, Academic Press, N.Y. (1979).

21. F. Riesz and B. Sz. Nagy, *Lecons d'analyse fonctionelle*, p. 452, Gauthier-Villars, Paris (1965).

22. C. Foias and B. Sz.-Nagy, Acta. Sci. Math. **23**, 106 (1962); **25**, 38 (1964).

23. G.W. Mackey, *Induced Representations of Groups and Quantum Mechanics*, Benjamin, N.Y. (1968).

24. E.B. Davies, *Quantum Theory of Open Systems*, Academic Press, London (1976).

25. This example has been studied from a different point of view by Reed and Simon (ref.

20, p. 236), by using the result[18]

$$Z(\tau) = P_{\mathcal{K}}U(\tau)P_{\mathcal{K}},$$

and defining the semigroup, for the case in which $U(\tau)$ is pure translation, as an operator restricted to the subspace $\mathcal{K} = \overline{\mathcal{H}} \ominus D_+ \ominus D_-$. They show that since *in* this subspace, $Z(\tau) = e^{-B\tau}$, and it must vanish for $\tau \geq \rho$, its Laplace transform,$(s+B)^{-1}$, must be an entire function of s. Hence the spectrum of B is empty. The relation $Z(\tau) = P_{\mathcal{K}}U(\tau)P_{\mathcal{K}}$ follows, in fact, directly from the imprimitivity relation (5.3) since, for $0 \leq \tau \leq \rho$,

$$\begin{aligned}(E(\rho) - E(0))U(\tau)(E(\rho) - E(0)) &= (E(\rho) - E(0))(E(\rho+\tau) - E(\tau))U(\tau) \\ &= (E(\rho) - E(\tau))U(\tau),\end{aligned}$$

in agreement with (5.9). In this form, the derivative at $\tau = 0$ is, however, not well-defined; after differentiation and setting $\tau = 0$, one obtains a formal result which is not in agreement with (5.15). The derivatives obtained from (5.10) and the two forms of (5.9) (which are well-defined at $\tau = 0$) all yield (5.15). With suitable boundary conditions in $\mathcal{K}$, the derivatives of the spectral family at the endpoints of the interval $(0, \rho)$ do not contribute to the restricted operator, and the defect spectrum evident in (5.15), (5.16) is removed. We thank B. Simon for a communication on this point.