Zeitschrift: Helvetica Physica Acta
Band: 68 (1995)
Heft: 5

Artikel: A perturbative expansion for the Hopfield model
Autor: Shcherbina, Maria / Tirozzi, Brunello
DOI: https://doi.org/10.5169/seals-116750

Helv Phys Acta
Vol. 68 (1995)

0018-0238/95/050470-22$1.50+0.20/0

A Perturbative Expansion for the Hopfield Model

By Maria Shcherbina

Mathematical Department of Low Temperature Physics Institute,
310164 Kharkov, Ukraine

and Brunello Tirozzi

Department of Mathematics, University of Rome "La Sapienza",
5-00185 Roma, Italy

(8.IX.1995)

Abstract. We construct a perturbative series for order parameters and other interesting observables of the Hopfield model. It is shown that the L^2 norm of the n-th term of the expansion is bounded by α^n. We apply this construction to the L^2 fluctuation of the order parameter r and we find that the coefficients of its expansion vanish in the thermodynamic limit. Using a recent result [1] we deduce that the replica symmetric solution of the Hopfield model holds at any order of α.

1 Introduction

The interest in the Hopfield model arose because it is the first model with an increasing number of patterns and neurons which was solved by physicists [2] and which exhibits the property of associative memory. This model was introduced in [3] many years ago in order to study its asymptotic free-energy but the properties of the associative memory of the model were not considered there and the number of patterns was finite. Since the appearence of the first solution of the Hopfield model many generalizations exhibiting interesting properties have been constructed. The solution of these kind of models is always reduced to the computation of the averaged free-energy in the $N \to \infty$ limit and to the solution of a set of equations for the order parameters which are found by applying the saddle-point method. Unfortunately this computation is done following the so called replica method which is not rigorous from the mathematical point of view. In [4] a rigorous method, called the cavity method, was proposed for finding the asymptotic mean free-energy and the saddle-point point equations for the Sherrington and Kirkpatrick (S-K) model. The main idea of this method is the same of the classical cavity method used by Parisi [5] for solving the spin-glass models but its realization in the paper of Pastur and Shcherbina is very different. This method gives the same result as the replica method in the case of replica symmetry. The replica symmetry hypothesis was substituted with the more suitable hypothesis of the self-averaging of the Edward-Anderson order parameter q. This hypothesis can be verified by numerical simulations more easily than replica symmetry. A generalized version of this approach was applied to the Hopfield model in [1] to obtain the saddle-point equations which were derived in [2] by using replica trick. In this derivation the self-averaging property of the Edward-Anderson order parameter is a sufficient condition. In [6] it was shown that, for any β, the free-energy of the Hopfield model is self-averaging in the L^2 sense. The proof of this fact is based on the method proposed in [4] to prove the selfaveraging property of the free energy of the S-K model. The main mathematical objects used in these proofs were the martingale differences. In [7] this argument was generalized in order to obtain the a.s. convergence. This approach was used also in [8] where the large deviation principle for the free-energy in a strong form was obtained. Using this property it was shown (see [8]) that there exists a ball, in the space of the overlaps, around a Mattis state such that, for α small enough, the Gibbs measure of the complement to this ball tends to zero when $N \to \infty$ with probability one. A stronger version of the s.a. property was found in the paper [9] in the high temperature region. In this paper it is also shown the central limit theorem for the free-energy of the Hopfield model in the case of α finite and when $T > 1 + \sqrt{\alpha}$. The rigorous derivation of the saddle point equations has been applied also to the model which retrieves sequences of patterns (see [10]- [12]) and for the hetero-associative model called B.A.M [13]. Thus the cavity method is a good tool for solving in a rigorous way the neural network models as well as the replica trick. But the check of the condition of the s.a. property of q remains an open question which was investigated up to now only numerically.

The other direction of rigorous investigation of the Hopfield model deals with the case when the number of patterns is finite or grows slower than the number of neurons. The first result was obtained in [3]. It was proved that, if the number of patterns is finite, then the free-energy converges, as $N \to \infty$ to the well known Curie-Weiss (mean-field) expression. Other interesting results for the finite number of patterns have been obtained in [14], [15], [16]. The result found by [3] was extended to the case $p \sim \log N$ in [17] and finally in [6] to the most general case when $p \to \infty$, $N \to \infty$, in such a way that $p/N \to 0$. The Curie-Weiss expression of the free-energy can be considered as the first term of an expansion of the asymptotic free-energy in powers of α.

In this paper we construct an asymptotic expansion in α of the order parameters of the Hopfield model which implies in a natural way the construction of the α-expansion for the free-energy. We remark that since our expansion is an asymptotic one all our estimates are true when α is small enough. The Hamiltonian considered by us is that of the Hopfield model with the addition of the usual symmetry breaking term:

$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}S_iS_j - t\sum_i \xi_i^1 S_i \tag{1.1}$$

where

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu, \qquad \alpha \equiv \frac{p}{N}. \tag{1.2}$$

The symmetry breaking term $t\sum_i \xi_i^1 S_i$ is introduced in order to study the retrieval property of the model following the approach of Amit et al. [2]. Our results hold for $t \neq 0$ but for $T > 1$ the same method can be applyed to the case $t = 0$. The series is generated trough an integration by parts. The Gibbs measure $< \ldots >$ is expressed as a measure in the space of overlaps and the Hamiltonian which generates this measure is expanded around the solution of the Curie-Weiss mean field theory. Each term of the expansion can be bounded in L^2 norm by an integer power of α. We show that, if we apply this construction to the expectation of $\Delta \equiv \sum_{\mu\nu}(< m^\mu m^\nu > - < m^\mu >< m^\nu >)^2$, then all the coefficients of the expansion of Δ converge to zero when $N \to \infty$. Since according to the results of [1] $\Delta \to 0$ implies that the replica symmetry saddle-point equations hold, we have got that the replica symmetry solution of the Hofpield model holds in any order of α.

The paper is organized as follows: in Section 2 we introduce the definitions and show three technical lemmas which are useful for estimating the L^2 norm of the generic term of the series. In Section 3 we costruct the expansion for the order parameter of the model (1.1). In Section 4 we estimate the L^2 norm of the rest of the series. In Section 5 the convergence to zero of all the coefficients of the expansion of $E\Delta$ is shown. Finally in Section 6 we give the conclusions.

2 The Model

Let $\underline{S} \equiv (S_1, \ldots, S_N)$, $S_i = \pm 1$ be the collection of the activities of a system with N neurons and $\{\xi_i^\mu\}$, $i = 1, \ldots N$, $\mu = 1, \ldots p$ the set of the patterns stored in the system through the Hebb synapsis J_{ij} of the form (1.2), where the patterns $\{\xi_i^\mu\}$ are indipendent equally distributed random variables with values ± 1 and zero mean.

Consider the Hamiltonian of the Hopfield model of the form (1.1). As it was mentioned in the first section, the second term in the Hamiltonian is introduced in order to study the retrieval of the first pattern. For $T < 1$ t will be considered fixed throughout the paper, but for $T > 1$ it can be zero. Let $< \cdot >_{\mathcal{H}}$ be the Gibbs average generated by the Hamiltonian (1.1) and let $\varphi(\underline{S})$ be an observable. Using a standard gaussian trasformation $< \varphi(\underline{S}) >_{\mathcal{H}}$ can be expressed by means of an integral on the R^p space with coordinates $\underline{\tilde{m}} \equiv (\tilde{m}^1, \ldots \tilde{m}^p)$

$$< \varphi(\underline{S}) >_{\mathcal{H}} = < \varphi_1(\underline{\tilde{m}}) >_{\tilde{H}} \equiv \frac{\int d\underline{\tilde{m}}\varphi_1(\underline{\tilde{m}})e^{-\beta\tilde{H}(\underline{\tilde{m}})}}{\int d\underline{\tilde{m}}e^{-\beta\tilde{H}(\underline{\tilde{m}})}}. \tag{2.1}$$

where $\tilde{H}(\underline{\tilde{m}})$ and $\varphi_1(\underline{\tilde{m}})$ are given by

$$\tilde{H}(\underline{\tilde{m}}) = \frac{N}{2}\sum_\mu (\tilde{m}^\mu)^2 - \frac{1}{\beta}\sum_i \log 2\cosh\beta(\sum_\mu \xi_i^\mu \tilde{m}^\mu + t\xi_i^1)$$

$$\varphi_1(\underline{\tilde{m}}) = \frac{\sum_{\underline{S}} \varphi(\underline{S}) \exp(-\beta\sum_i S_i(\sum_\mu m^\mu \xi_i^\mu + t\xi_i^1))}{\sum_{\underline{S}} \exp(-\beta\sum_i S_i(\sum_\mu m^\mu \xi_i^\mu + t\xi_i^1))} \tag{2.2}$$

The average of the type (2.1) will be denoted with the symbol $< \cdot >_H$, where H is an Hamiltonian written as a function of the overlaps obtained by making the above transformations. It may be $\tilde{H}(\underline{\tilde{m}})$ or the Hamiltonian (2.6) $H(\underline{m})$ or other kind of Hamiltonians which will be introduced after. Consider the Curie-Weiss free-energy f_0:

$$f_0(x) = -\frac{1}{\beta}\log 2\cosh\beta(x+t) + x^2/2 \tag{2.3}$$

and let z be the value for which $f_0(x)$ takes its global minimum, i.e. z is the positive solution of the equation:

$$z = \tanh\beta(z+t). \tag{2.4}$$

To simplify formulae we shift the origin of the $\underline{\tilde{m}}$ variables in z:

$$m^\mu = \tilde{m}^\mu - z\delta^{\mu,1}$$

and denote by $H(m)$ the Hamiltonian obtained from $\tilde{H}(\underline{\tilde{m}})$ by using this change of variables. We introduce also notaions $C_i = \sum_{\mu=1}^p \xi_i^\mu m^\mu$ and denote by $\phi(x)$ the rest of the Taylor expansion of $f_0(x)$ around z , thus, setting $d \equiv f_0''(z) = 1 - \beta(1-z^2)$, we have:

$$f_0(x+z) = f_0(z) + \frac{d}{2}x^2 + \phi(x). \tag{2.5}$$

By using this notation we get

$$H(\underline{m}) = Nf_0(z) + \frac{N}{2}d((I-\lambda\mathcal{A})\underline{m},\underline{m}) - Nz\sum_{\mu=1}^p h^\mu m^\mu + \sum_{i=1}^N \phi(C_i) \tag{2.6}$$

where

$$h^\mu = (\frac{1}{N}\sum_i \xi_i^\mu \xi_i^1)(1-\delta^{\mu,1})$$

$$\lambda = \frac{\beta d}{1-\beta d} = \frac{1-f_0''(z)}{f_0''(z)} = \frac{\beta(1-z^2)}{1-\beta(1-z^2)}$$

$$\mathcal{A}_{\mu\nu} = \frac{1}{N}\sum_{i=1}^N \xi_i^\mu \xi_i^\nu (1-\delta^{\mu,\nu}).$$

The symbol $< \cdot >$ denotes the expectation $< \cdots >_H$ generated by $H(\underline{m})$. We are going to set up a perturbative series for the order parameters of the Hopfield model using the expansion (2.5) around solution z of the mean field equation (2.4) and the corresponding representation (2.6) for the Hamiltonian $H(\underline{m})$. The expectation of the n-th term of the expansion have been constructed is bounded by the n-th power of the parameter α. This estimate is based on three technical lemmas.

Lemma 1 *Let $\chi_\tau(x)$ be 0 , if $x \le \tau\sqrt{\alpha}$, 1 if $x > \tau\sqrt{\alpha}$ and $M = \{\sum_\mu (m^\mu)^2\}^{1/2}$ then there exixts a constant τ_0 such that for $\tau > \tau_0$*

$$< \chi_\tau(M) > \le \exp(-N(\tau - \tau_0)\alpha \,\mathrm{const}\,)$$

with probability larger than

$$P_N = 1 - \exp(-\alpha^{4/3} N^{2/3}\,\mathrm{const}\,). \tag{2.7}$$

Proof.

It is easy to see that, if $t \ne 0$ then, since z is a global minimum point, there exist two constants $0 < k_1 < k_2$ such that

$$\frac{k_1}{2}(m-z)^2 \le f_0(m) - f_0(z) \le \frac{k_2}{2}(m-z)^2$$

Let us note, that if $T = \beta^{-1} > 1$ and $t = 0$ then there exists only one minimum $z = 0$ and this inequality also can be used.

Using the definitions of f_0 and $H(\underline{m})$ we can write

$$\begin{aligned} H(\underline{m}) - Nf_0(z) &= -\frac{N}{2}\sum_{\mu\nu}\mathcal{A}_{\mu\nu}m^\mu m^\nu - Nz\sum_\mu h^\mu m^\mu + \sum_i (f_0(\sum_{\mu=1}^{p}\xi_i^\mu \xi_i^1 m^\mu + z) - f_0(z)) \\ &\le -\frac{N}{2}\sum_{\mu\nu}\mathcal{A}_{\mu\nu}m^\mu m^\nu - Nz\sum_\mu h^\mu m^\mu + \frac{k_2}{2}\sum_{\mu\nu}\sum_i \xi_i^\mu \xi_i^\nu m^\mu m^\nu \\ &\le \frac{N}{2}(k_2\sum_\mu (m^\mu)^2 + (k_2-1)(\mathcal{A}\underline{m},\underline{m}) + Nz(\sum_\mu (h^\mu)^2)^{1/2}M \end{aligned} \tag{2.8}$$

But

$$(\sum_\mu (h^\mu)^2) = (\sum_\mu (\mathcal{A}^{\mu 1})^2) \le ||\mathcal{A}||^2$$

and we have shown in a previous paper [6] that $||\mathcal{A}|| \le 3\sqrt{\alpha}$ with probability larger than P_N defined by (2.7). Then we can continue the estimates in (2.8):

$$\begin{aligned} N[\frac{k_1}{2} - 3\alpha^{1/2}\frac{(k_1-1)}{2}]M^2 - 3Nz\alpha^{1/2}M &\le H(\underline{m}) - Nf_0(z) \\ &\le [\frac{k_2}{2} + 3\alpha^{1/2}\frac{(k_2-1)}{2}]M^2 + 3Nz\alpha^{1/2}M \end{aligned} \tag{2.9}$$

We choose α small enough to provide the positivity of the coefficients in front of M^2 in the l.h.s of (2.9) and denote by k_1 this coefficient and by k_2 corresponding coefficient in the r.h.s. of (2.9). Thus we get:

$$\frac{N}{2}k_1 M^2 - 3Nz\alpha^{1/2}M \le H(\underline{m}) - Nf_0(z) \le \frac{N}{2}k_2 M^2 + 3Nz\alpha^{1/2}M$$

Now by using this inequality we can estimate the expectation of $\chi_\tau(M)$:

$$< \chi_\tau(M) >= \frac{\int d\underline{m}\chi_\tau(M)e^{-\beta(H(\underline{m})-Nf_0(z))}}{\int d\underline{m}e^{-\beta(H(\underline{m})-Nf_0(z)}}$$

$$\frac{\int_{M>\tau\sqrt{\alpha}}^{\infty} dMM^{p-1}\exp\{-\beta Nk_1M^2/2+3\beta Nz\alpha^{1/2}M\}}{\int_0^\infty dMM^{p-1}\exp\{-\beta Nk_2M^2/2-3\beta Nz\alpha^{1/2}M\}}$$

$$\equiv \frac{\int_{M>\tau\sqrt{\alpha}}^{\infty} dM\exp\{-Nf_1(M)\}}{\int_0^\infty dMM^{p-1}\exp\{-Nf_2(M)\}} \tag{2.10}$$

where

$$f_1(M) = \beta k_1M^2/2 - \beta 3z\alpha^{1/2}M + \alpha\log M$$

$$f_2(M) = \beta k_2M^2/2 + \beta 3z\alpha^{1/2}M + \alpha\log M.$$

The contribution of the integration over the angles in the numerator simplifies with the analogous term in the denominator. Let τ_1 and τ_2 be minimum points of functions $f_1(\sqrt{\alpha}\tau)$ and $f_2(\sqrt{\alpha}\tau)$ respectively. Choosing $\tau > \tau_1$ and applying the Laplace method to compute the integrals in the numerator and denominator of the r.h.s. of (2.10) we obtain

$$< \chi_\tau(M) >\leq \frac{(\pi Nf_2''(\sqrt{\alpha}\tau_2))^{1/2}}{Nf_1'(\sqrt{\alpha}\tau)}e^{-Nf_1(\sqrt{\alpha}\tau)+Nf_2(\sqrt{\alpha}\tau_2)}(1+o(1)) \tag{2.11}$$

Here we have estimated the integral in the numerator by the value of the integrand in the left endpoint since $\tau > \tau_1$. Now it is easy to obtain the statement of Lemma 1 choosing

$$\tau > \tau_0 = \max\{\tau_1, \tau^*\}$$

where τ^* is the largest solution of the equation

$$f_1(\sqrt{\alpha}\tau) = f_2(\sqrt{\alpha}\tau_2)$$

From this Lemma one gets easily:

Lemma 2 *For any constants B, n the following inequality holds :*

$$E\{< M^n e^{BM^2} >\} \leq (\tau_0\sqrt{\alpha})^n e^{B\alpha\tau_0} \tag{2.12}$$

The last lemma is

Lemma 3

$$E\{< C_i^{2n}e^{BC_i} >\} \leq KD(2n)\alpha^n \tag{2.13}$$

where the constant K does not depend on n, N and $D(2n) = (2n)^n$.

Proof.

According to the symmetry in the index i of the E expectation of the average of C_i with respect to the Gibbs measure of the Hopfield model, it is enough to prove (2.13) in the case $i = 1$. Let us define a new Hamiltonian:

$$H_1(\underline{m}) = H(\underline{m})|_{\xi_1^\mu=0,\mu=1,\dots,p}.$$

Then

$$E\{< C_1^{2n} e^{BC_1} >\} = E\{\frac{< C_1^{2n} e^{BC_1} \cosh\beta(C_1 + \xi_1^1(z+t) >_{H_1}}{< \cosh\beta(C_1 + \xi_1^1(z+t) >_{H_1}}\}$$

$$\leq E\{< C_1^{2n} e^{\beta C_1} \cosh\beta(C_1 + \xi_1^1(z+t)) >_{H_1}\} \qquad (2.14)$$

Averaging the r.h.s. of (2.14) with respect to $\xi_1^1, \dots, \xi_1^p$ and using the estimate (2.12) we get the estimate (2.13).

3 Construction of the α-expansion.

In this section we construct the α-expansion for the order parameter r which has been introduced in the Hopfield model to take into account the influence of the non-condensed patterns in the retrieval process. We start from the usual definition of r:

$$r = \frac{1}{p}\sum_{\mu>1} < \frac{1}{N}\sum_i \xi_i^\mu S_i >_{\mathcal{H}}^2$$

Since it is easy to see that for $\mu > 1$

$$< \frac{1}{N}\sum_i \xi_i^\mu S_i >_{\mathcal{H}} = < m^\mu >$$

we can construct the expansion for the quantity $\frac{1}{N}\sum_{\mu>1} < m^\mu >^2$. We will consider the contribution of terms of the order α^k. To this end we make the Taylor expansion up to the order $2k$ of $\phi'(C_i)$, the derivative of the third order term of the expansion around the mean-field value z of the function $f_0(C_i)$, see formula (2.5).

$$\phi'(C_i) = a_2 C_i^2 + \ldots + a_{2k} C_i^{2k} + R_{2k+1}(C_i) C_i^{2k+1}. \qquad (3.1)$$

Note that $|a_k| \leq (\text{ const })^k$ and $|R_{2k+1}(C_i)| \leq (\text{ const })^{k+1}$ since ϕ is an analytic function.

We construct the α-expansion of $< m^\mu >$ by using the following formula of integration by parts which is valid for any differentiable function $F(\underline{m})$

$$< m^\mu F(\underline{m}) > = \frac{z}{d}\sum_\nu G^{\mu\nu} h^\nu < F > + \frac{1}{\beta N d}\sum_\nu G^{\mu\nu} < \frac{\partial}{\partial m^\nu} F(\underline{m}) >$$

$$- \frac{1}{Nd}\sum_{\nu i} G^{\mu\nu} \xi_i^\nu < F(\underline{m}) \cdot (\sum_{l=2}^{2k} a_l C_i^l + R_{2k+1}(C_i) C_i^{2k+1}) > \qquad (3.2)$$

Here $G = (1 - \lambda\mathcal{A})^{-1}$ and according to [6] we have that

$$||G|| \leq (1 - 3\lambda\sqrt{\alpha})^{-1} \tag{3.3}$$

with probability P_N. We denote by symbol $<>_0$ the average with respect to the Hamiltonian $H_0(\underline{m})$ containing the linear and quadratic part of the Taylor expansion of the Hamiltonian $H(\underline{m})$ around z:

$$H_0(\underline{m}) = \frac{N}{2}d((I - \lambda\mathcal{A})\underline{m}, \underline{m}) - Nz\sum_\mu h^\mu m^\mu$$

Applying the formula (3.2) to the variable m^μ we get

$$< m^\mu > = < m^\mu >_0 - \frac{1}{Nd}\sum_{i_1}(G\xi_{i_1})^\mu < (\sum_{l=2}^{2k} a_l C_{i_1}^l + R_{2k+1}(C_{i_1})C_{i_1}^{2k+1}) >$$

Now we construct the second order perturbation term by applying the formula (3.2) to all terms which contain an expectation $<>$ with the exception of that containing the rest $R_{2k+1}(C_i)C_{i_1}^{2k+1}$, because it follows from Lemma I.2 that its order is $\alpha^{2k+1/2}$. We write here the next iteration:

$$< m^\mu > = < m^\mu >_0 - \frac{1}{Nd}\sum_{i_1}(G\xi_{i_1})^\mu < C_{i_1} >_0 < \sum_{n=1} a_n C_{i_1}^{n-1} >$$

$$- \frac{1}{Nd}\sum_{i_1}(G\xi_{i_1})^\mu < (\dot{C}_{i_1})^2 >_0 < \sum_{l=1} a_n(n-1)C_{i_1}^{n-2} > - \frac{1}{dN}\sum_{i_1}(G\xi_{i_1})^\mu < R_{2k+1}C_{i_1}^{2k+1} >$$

$$+ \frac{1}{dN}\sum_{i_1 i_2}(G\xi_{i_1})^\mu < \dot{C}_{i_1}\dot{C}_{i_2} > \cdot \{ < \sum_{n_1 n_2} a_{n_1} a_{n_2} C_{i_1}^{n_1 - 1} C_{i_2}^{n_2} >$$

$$+ < \sum_{n_1} a_{n_1} C_{i_1}^{n_1 - 1} C_{i_2}^{2k+1} R_{2k+1}(C_{i_2}) > \} \tag{3.4}$$

where we introduced the notation $\dot{C}_i = C_i - < C_i >_0$. We remark, that every time we obtain the term containing the coefficient $a_{n_1} \cdot \ldots a_{n_M}$ such that $\sum_{i=1}^M (n_i - 1) \geq 2k + 1$ or the term containing $R_{2k+1}(C_i)C_i^{2k+1}$, we stop the procedure of integration by parts for this term, because it will be proved below that all of them have an order larger than $2k$. By repeating the procedure described above $2k$ times we get

$$< m^\mu > = < m^\mu >_0 - \frac{1}{Nd}\sum_{i_1}(G\xi_{i_1})^\mu \sum_{n_1=2}^{2k} a_{n_1} < C_{i_1}^{n_1} >_0$$

$$+ \frac{1}{dN}\sum_{i_1 i_2}(G\xi_{i_1})^\mu \sum_{n_1 n_2} a_{n_1} a_{n_2} < \dot{C}_{i_1}\dot{C}_{i_2} >_0 < C_{i_1}^{n_1 - 1} C_{i_2}^{n_2} >_0$$

$$- \frac{1}{Nd}\sum_{i_1 i_2 i_3}(G\xi_{i_1})^\mu \sum_{n_1 n_2 n_3} a_{n_1} a_{n_2} a_{n_3} < \dot{C}_{i_1}\dot{C}_{i_2} >_0 < \dot{C}_{i_2}\dot{C}_{i_3} >_0 < C_{i_1}^{n_1 - 1} C_{i_2}^{n_2 - 1} C_{i_3}^{n_3} >_0$$

$$+ \ldots + R^\mu \tag{3.5}$$

Here the terms of the type $< C_{i_1}^{n_1-1} C_{i_2}^{n_2-1} C_{i_3}^{n_3} >_0$ come from the application of the part of formula (3.2) which doesn't contain the derivative of $\phi(C_i)$. In fact if we iterate terms of the type $< C_{i_1}^{n_1-1} C_{i_2}^{n_2-1} C_{i_3}^{n_3} >$ by using this part of formula (3.2) then according to the Wick theorem we obtain $< C_{i_1}^{n_1-1} C_{i_2}^{n_2-1} C_{i_3}^{n_3} >_0$. R^μ is some linear combination of terms which have the following form

$$t^\mu = \frac{1}{Nd} \sum_{i_1,\dots i_M} (G\xi_{i_1})^\mu < \dot{C}_{i_1} \dot{C}_{i_2} >_0 < \dot{C}_{i_2} \dot{C}_{i_3} >_0 \cdot \dots < \dot{C}_{i_{M-1}} \dot{C}_{i_M} >_0$$

$$\cdot < C_{i_1}^{l_1} \cdot \dots C_{i_M}^{l_M} >_0 \cdot < C_{i_1}^{n_1-1-l_1} \cdot \dots C_{i_{M-1}}^{n_{M-1}-1-l_{M_1}} C_{i_M}^{n_M-l_M} \tilde{R}_{n_M}(C_{i_M}) > \tag{3.6}$$

where $1 \le M \le 2k$, $n_1, \dots, n_M$, $l_1, \dots, l_M$ are some integers, satisfying conditions

$$2 \le n_i \le 2k, \qquad 0 \le l_i \le n_i - 1 \qquad i = 1, \dots, M-1$$

$$2 \le n_M \le 2k+1, \qquad 0 \le l_M \le n_M$$

and

$$\sum_{i=1}^{M} (n_i - 1) \ge 2k + 1 \tag{3.7}$$

and $\tilde{R}_n(C_i) = 1$ for $n \le 2k$ and $\tilde{R}_{2k+1}(C_i) = R_{2k+1}(C_i)$. We omit here for simplicity the coefficients $a_{n_1} \dots a_{n_M}$ since as it was remarked after formula (3.1) all of them are bounded.

Since all terms in the r.h.s. of (3.5) except R^μ can be calculated exactly as a Gaussian integrals, our main goal is to show that $E\{\sum_\mu (R^\mu)^2\}$ has an order α^{k+1}.

Theorem 1 *The rest $\sum_\mu (R^\mu)^2$ is bounded in L^2 norm by the $(k+1)$th power of α:*

$$E\{\sum_\mu (R^\mu)^2\} \le \text{const}\, \alpha^{k+1}$$

Remark. The quantity $\sum_\mu (R^\mu)^2$ can be understood as the rest of the expansion of the parameter $r = \frac{1}{p} \sum_\mu (m^\mu)^2$ which is considered as the order parameter of the non-condensed patterns.

In Section 5 we will use bounds on the generic term of this series in order to show that the replica symmetric saddle-point equations of the Hopfield model, found using the replica trick in [2] hold at any order of the perturbative expansion in α.

4 Proof of Theorem 1

In order to estimate the rest of the expansion of r it is sufficient to bound the quantity

$$T \equiv E\{\sum_\mu (t^\mu)^2\}.$$

Note that in the definition (3.6) there are two different expectations: one with respect to the Gibbs measure generated by $H(\underline{m})$ and the other one with respect to the Gibbs measure associated with $H_0(\underline{m})$, which contains only the linear and quadratic terms of the expansion of $H(\underline{m})$ around the mean-field value z. The strategy for finding a good estimate of the right hand side of T is to make upper bounds of the single factors before doing the expectations E, $<>$, $<>_0$. We start from replacing the product of the expectations in (3.6) by the expectation of a single factor. To this end we define a set of duplicate variables:

$$C^*_{i_m} = \sum_\mu (m^*)^\mu \xi^\mu_{i_m} \tag{4.1}$$

where $(m^*)^\mu$ are random variables distributed according to the Gibbs measure generated by $H_0(\underline{m}^*)$, the symbol $<>_*$ will be used for indicating the averages with respect to this measure. Then we define the quantity

$$F_{i_1} = \sum_{i_2,\dots i_M} \left\langle \dot{C}_{i_1} \dot{C}_{i_2} \right\rangle_0 \left\langle \dot{C}_{i_2} \dot{C}_{i_3} \right\rangle_0 \cdot \ldots \left\langle \dot{C}_{i_{M-1}} \dot{C}_{i_M} \right\rangle_0$$

$$\cdot\, C^{l'_1}_{i_1} (C^*_{i_1})^{l_1} \ldots C^{l'_M}_{i_M} (C^*_{i_M})^{l_M} \tilde{R}_{l'_M + l_M}(C_{i_M}). \tag{4.2}$$

for some nonnegative integers $l_1, \dots, l_M$, $l'_1, \dots, l'_M$, that according to (3.7) satisfy the condition

$$\sum_{i=1}^{M} (l_i + l'_i) \geq 2k + 1. \tag{4.3}$$

Using all these definitions T can be bounded from above using a compact expression

$$T \leq \frac{1}{Nd^2} E\{\langle\langle \sum_{i_1,j_1} \frac{(G^2 \underline{\xi}_{i_1}, \underline{\xi}_{j_1})}{N} F_{i_1} F_{j_1} \rangle_* \rangle\}$$

$$\leq \frac{1}{d^2} E\{\langle\langle ||B|| \frac{1}{N} \sum_i F_i^2 \rangle_* \rangle\} \leq \frac{1}{d^2} E\{\langle\langle ||B|| |F_1|^2 \rangle_* \rangle\} \tag{4.4}$$

where we used the symmetry in the index i of the E expectation and we have introduced the notation $\underline{\xi}_i$ for the set $\xi^1_i, \dots, \xi^p_i$ and the symbol $(G^2 \underline{\xi}_{i_1}, \underline{\xi}_{j_1})$ for the sum $\sum_{\mu\nu} (G^2)_{\mu\nu} \xi^\mu_{i_1} \xi^\nu_{j_1}$. $||B||$ is the norm of the $N \times N$ matrix:

$$B_{ij} = \frac{1}{Nd} (G^2 \underline{\xi}_i, \underline{\xi}_j) \tag{4.5}$$

Now we look for a bound for $||F_1||$, and we need to introduce new notations. First we decompose the matrix $\mathcal{A}^{\mu\nu}$ as the sum of another matrix $\overline{\mathcal{A}}^{\mu\nu}$ and a projector $P^{\mu\nu}$:

$$\mathcal{A}^{\mu\nu} = \overline{\mathcal{A}}^{\mu\nu} + \frac{\alpha}{p} \xi^\mu_1 \xi^\nu_1$$

where the matrix $P^{\mu\nu} = \frac{1}{p} \xi^\mu_1 \xi^\nu_1$ is a projector. We will apply the simple identity:

$$(1 - \lambda \mathcal{A})^{-1}_{\mu\nu} = (1 - \lambda \overline{\mathcal{A}})^{-1}_{\mu\nu} + \frac{\lambda}{p} \frac{\alpha((1 - \lambda \overline{\mathcal{A}})^{-1} \underline{\xi}_1)^\mu ((1 - \lambda \overline{\mathcal{A}})^{-1} \underline{\xi}_1)^\nu}{1 - \frac{\alpha\lambda}{p}((1 - \lambda \overline{\mathcal{A}})^{-1} \underline{\xi}_1), \underline{\xi}_1)} \tag{4.6}$$

We use the notations:

$$G_{\mu\nu} = (1 - \lambda\mathcal{A})^{-1}_{\mu\nu}, \qquad \overline{G}_{\mu\nu} = (1 - \lambda\overline{\mathcal{A}})^{-1}_{\mu\nu},$$

$$B^0_{ij} = \frac{1}{\beta N d}(G\underline{\xi}_i, \underline{\xi}_j) = \left\langle \dot{C}_i \dot{C}_j \right\rangle_0,$$

$$B^1_{ij} = \frac{1}{\beta N d}(\overline{G}\underline{\xi}_i, \underline{\xi}_j) = \left\langle \dot{C}_i \dot{C}_j \right\rangle_1.$$

in the last formula the quantity $< \dot{C}_i \dot{C}_j >_1$ and B^1_{ij} are defined by the equality with $\frac{1}{\beta N d}(\overline{G}\underline{\xi}_i, \underline{\xi}_j)$. The identity (4.6) can be written now in a shorter form which will be helpful for our estimates:

$$B^0_{ij} = B^1_{ij} + \frac{\beta d}{\lambda^{-1} - \delta} B^1_{1i} B^1_{1j} \tag{4.7}$$

with $\delta = \frac{1}{N}(\overline{G}\underline{\xi}_1, \underline{\xi}_1) = \frac{1}{N}\mathrm{Tr}\overline{G} + \mathrm{o}(1)$ and $\mathrm{o}(1) \longrightarrow 0_{N\to\infty}$. Finally let us introduce some diagonal matrices which will allow us to write F_1 in a more compact form:

$$(D_k)_{ij} = \delta_{ij}(C^*_i)^{l_k} C_i^{l'_k} \quad \text{for} \quad 1 \le k \le M-1$$

$$(D_M)_{ij} = \delta_{ij}(C^*_i)^{l_M} C_i^{l'_M} \tilde{R}_{l'_M + l_M}(C_{i_M}) \quad \text{for} \quad k = M$$

Finally F_1 can be written in a suitable form :

$$F_1 = \sum_i (D_1 B^0 D_2 B^0 \dots B^0)_{1i}(D_M)_{ii} \tag{4.8}$$

Let $\rho = \frac{\beta d}{\lambda^{-1} - \delta}$ and let us subsitute each B^0 in F_1 with its "expansion" $B^0_{ij} = B^1_{ij} + \rho B^1_{1i} B^1_{1j}$. We obtain a sum of terms in which some of the B^0_{ij} are replaced by $\rho B^1_{1j} B^1_{1i}$ and the others by B^1_{ij}. Let $\tilde{F}^l_{1,k_1,\dots,k_l}$ be the term obtained by making l of the replacements $B^0_{ij} \to \rho B^1_{1j} B^1_{1i}$ in F_1 at the places $k_1, \dots, k_l$ then we can write:

$$\tilde{F}^l_{1,k_1,\dots,k_l} = \rho^l \sum_{i_{k_1},\dots,i_{k_l+1},i_M} (D_1 B^1 \dots D_{k_1})_{1,i_{k_1}} B^1_{i_{k_1},1} B^1_{i_{k_1+1},1}$$

$$\cdot (D_{k_1+1} B^1 \dots D_{k_2})_{i_{k_1}+1, i_{k_2}} \cdot B^1_{i_{k_2},1} B^1_{1,i_{k_2}+1}$$

$$\cdot\, (D_{k_l-1} B^1 \dots D_{k_l})_{i_{k_l-1}, i_{k_l}} B^1_{i_{k_l},1} B^1_{i_{k_l+1},1} \cdot (D_{k_l+1} B^1 \dots D_M)_{i_{k_l+1}, i_M}. \tag{4.9}$$

In writing the equation (4.9) we assumed that the "substitution" $B^0_{ij} \to \rho B^1_{1j} B^1_{1i}$ has been done respectively for the B^0 matrix which is between D_{k_1} and D_{k_1+1}, for the B^0 matrix which is between D_{k_2} and D_{k_2+1} and so on up to the B^0 matrix which is between D_{k_l} and D_{k_l+1}, in all the other terms $B^0 \to B^1$. After having found the estimate of $\tilde{F}^l_{1,k_1,\dots,k_l}$, a sum over all the possible values of l and of $k_1, \dots, k_l$ must be done which gives a constant depending on M only, since the upper bound that we find is uniform on $l, k_1, \dots, k_l$. Making the sum in (4.9) over all the indices $i_{k_1}, \dots, i_{k_l}, i_{k_l+1}$ we get:

$$\tilde{F}^l_{1,k_1,\dots,k_l} = \rho^l \sum_{i_M} (D_1 B^1 \dots D_{k_1} B^1)_{1,1} (B^1 D_{k_1+1} B^1 \dots D_{k_2} B^1)_{1,1}$$

$$\cdot (B^1 D_{k_l+1} B^1 \dots D_M)_{1,i_M} \tag{4.10}$$

We observe that if one of the indices inside one of the groups

$$(B^1 D_{k_i} B^1 \dots D_{k_j})_{1,1}$$

is equal to 1, then we can consider this group as the product of some smaller groups of the same kind. Thus without limitation of generality we can consider all the sums in the matrix products inside these groups starting from 2 in all the formula which appear after (4.10). In the estimate of $E\{<<||B|||F_1|^2>_*>\}$ there will be a sum over two sets of indices $k_1, \dots, k_l$, $m_1, \dots, m_{l'}$ corresponding to the pair $\tilde{F}^l_{1,k_1,\dots,k_l}$, $\tilde{F}^{l'}_{1,m_1,\dots,m_{l'}}$ coming from the $|F_1^2|$ term but the same argument as before holds. The final expression that we have to consider is:

$$\tilde{F}^l_{1,k_1,\dots,k_l} \tilde{F}^{l'}_{1,m_1,\dots,m_{l'}} = \rho^{l+l'} \sum_{i_M, j_M} (D_1 B^1 \dots D_{k_1} B^1)_{1,1} \dots (B^1 D_{k_l+1} B^1 \dots D_M)_{1,i_M}$$

$$\cdot (D_1 B^1 \dots D_{m_1} B^1)_{1,1} \dots (B^1 D_{m_{l'}+1} B^1 \dots D_M)_{1,j_M} \tag{4.11}$$

In the formula (4.11) the terms of the type $(D_1 B^1 \dots D_{k_1} B^1)_{1,1}$ are estimated differently from the terms of the type $(B^1 D_{k_l+1} B^1 \dots D_M)_{1,i_M}$ as we will show now.

Part A. *Estimate of* $(D_{k_l+1} B^1 \dots D_{k_{l+1}} B^1)_{1,1}$ *terms.*

$$(D_1 B^1 D_2 B^1 D_3 \dots B^1 D_{k_1})_{1,1} = \sum_{ij} (D_1)_{11} B^1_{1i}$$

$$\cdot (D_2 B^1 D_3 \dots B^1 D_{k_1})_{ij} B^1_{j1} = (D_1)_{11} \sum_{ij} \frac{\overline{G}^{\mu\nu} \xi_1^\mu \xi_i^\nu}{\beta N d} \cdot (\dots)_{ij} \frac{\overline{G}^{\gamma\rho} \xi_j^\gamma \xi_1^\rho}{\beta N d}$$

$$= (D_1)_{11} (\sum_{\mu=\rho} + \sum_{\mu\neq\rho}) \sum_{ij} \frac{\overline{G}^{\mu\nu} \xi_1^\mu \xi_i^\nu}{\beta N d} (\dots)_{ij} \frac{\overline{G}^{\gamma\rho} \xi_j^\gamma \xi_1^\rho}{\beta N d} == (D_1)_{11} (U_1 + U_2) \tag{4.12}$$

Since in the product of the matrices $\overline{G}$, B, D_i does not appear ξ_1^μ, averaging with respect to $\underline{\xi}_1$ we get that the L_2 norm of U_2 is negligible. In fact:

$$E\{U_2^2\} = \frac{1}{(\beta N d)^2} E(\sum_{i,j,i_1,j_1} \sum_{\mu\neq\rho, \mu_1\neq\rho_1} \overline{G}^{\mu\nu} \xi_1^\mu \xi_i^\nu \cdot (\dots)_{ij} \overline{G}^{\gamma\rho} \xi_j^\gamma \xi_1^\rho$$

$$\cdot \overline{G}^{\mu_1\nu_1} \xi_1^{\mu_1} \xi_{i_1}^{\nu_1} \cdot (\dots)_{i_1 j_1} \overline{G}^{\gamma_1\rho_1} \xi_{j_1}^{\gamma_1} \xi_1^{\rho_1})$$

$$= \frac{2}{(\beta N d)^2} E(\mathrm{Tr}(\overline{B}(\dots)\overline{B}(\dots))) = O(\frac{1}{N})$$

where $\overline{B}$ is defined analogously to B:

$$(\overline{B})_{ij} = \frac{1}{\beta N d} ((\overline{G})^2 \underline{\xi}_i, \underline{\xi}_j).$$

We can finally write :

$$|(D_1 B^1 D_2 B^1 D_3 \dots B^1 D_{k_1})_{1,1}| = (D_1)_{11} U_1 + O(N^{-1/2})$$

$$= (D_1)_{11} \sum_{i,j,\mu,\nu,\gamma} \frac{\overline{G}^{\mu\nu}\overline{G}^{\mu\gamma}\xi_i^\nu \xi_j^\gamma}{(\beta N d)^2}(D_2 B^1 D_3 \dots B^1 D_{k_1})_{ij} + O(N^{-1/2})$$

$$= \frac{1}{\beta N d}(D_1)_{11}\mathrm{Tr}(\overline{B} D_2 B^1 D_3 \dots B^1 D_{k_1})$$

After having reduced the form of the term (4.12) to the simpler form written above we can apply our strategy for bounding the terms of F_1 before making the expectations with respect to the patterns. We will use repeatedly the following inequality for matrices A, B and C, with C being a positive matrix:

$$\frac{1}{N}\mathrm{Tr} C A^\dagger B \leq ||C||(\frac{1}{N}\mathrm{Tr} A A^\dagger)^{1/2}(\frac{1}{N}\mathrm{Tr} B B^\dagger)^{1/2} \tag{4.13}$$

where $A^\dagger$ is the adjoint of the matrix A. In order to show how formula (4.13) is applied we take the example $k_1 = 4$ and we call

$$R_4 = \frac{1}{N}\mathrm{Tr}(\overline{B} D_2 B^1 D_3 B^1 D_4)$$

the term which we will analyze, the $(D_1)_{11}$ factor will be substituted in the final estimate with a term of the form $C_1^{2l'_1}(C_1^*)^{2l_1}$ which is obtained by the same arguments we are going to show. We can use the inequality (4.13) by setting :

$$A^\dagger = D_2 B^1 D_3, \quad B = B^1 D_4, \quad C = \overline{B},$$

Then we get

$$|R_4| \leq ||\overline{B}|| \cdot |\frac{1}{N}\mathrm{Tr} B^1 D_2^2 B^1 D_3^2|^{1/2} |\frac{1}{N}\mathrm{Tr} B_1^2 D_4^2|^{1/2}$$

Let us apply (4.13) again to

$$|\frac{1}{N}\mathrm{Tr} B^1 D_2^2 B^1 D_3^2| \leq |\frac{1}{N}\mathrm{Tr}(B^1)^2 D_2^4|^{1/2} |\frac{1}{N}\mathrm{Tr}(B^1)^2 D_3^4|^{1/2}$$

For the other factor we get:

$$|\frac{1}{N}\mathrm{Tr} B_1^2 D_4^2|^{1/2} \leq ||B_1|| |\frac{1}{N}\mathrm{Tr} D_4^4|^{1/4}.$$

Collecting these estimates we obtain the inequality:

$$|R_4| \leq ||\overline{B}|| \cdot ||B^1||^2 |\frac{1}{N}\mathrm{Tr} D_2^4|^{1/4}| \cdot |\frac{1}{N}\mathrm{Tr} D_3^4|^{1/4} ||B^1|| |\frac{1}{N}\mathrm{Tr} D_4^4|^{1/4} \tag{4.14}$$

These arguments can be repeated for all the factors of the formula (4.11) of the type $(\dots)_{11}$. The final term which we get from these estimates should be then of the form of the product of traces of diagonal matrices D_k each to a certain power. In order to apply the Lemmas of section I to these terms we will use Hölder and Schwartz inequalities. In fact the generic term $|\frac{1}{N}\mathrm{Tr} D_k^8|^{1/8}$ will be estimated using its definition:

$$(D_k)_{ij} = (C_i^*)^{l_k}(C_i)^{l'_k}\delta_{ij}$$

from which we get:

$$|\frac{1}{N}\mathrm{Tr}D_k^4|^{1/4} \leq |\frac{1}{N}\sum_i (C_i^*)^{8l_k}|^{1/8}|\frac{1}{N}\sum_i C_i^{8l'_k}|^{1/8}$$

Thus

$$|\frac{1}{N}\mathrm{Tr}D_2^4|^{1/4}|\frac{1}{N}\mathrm{Tr}D_3^4|^{1/4}|\frac{1}{N}\mathrm{Tr}D_4^4|^{1/4}$$

$$\leq |\frac{1}{N}\sum_i (C_i^*)^{8l_2}\frac{1}{N}\sum_i (C_i^*)^{8l_3}\frac{1}{N}\sum_i (C_i^*)^{8l_4}|^{1/8} \cdot |\frac{1}{N}\sum_i C_i^{8l'_2}\frac{1}{N}\sum_i C_i^{8l'_3}\frac{1}{N}\sum_i C_i^{8l'_4}|^{1/8} \quad (4.15)$$

Then by the inequality

$$\frac{1}{N}\sum_i a_i^p \frac{1}{N}\sum_i a_i^q \leq \frac{1}{N}\sum_i a_i^{p+q}$$

with $a_i \geq 0 \quad \forall i$, which follows from the Hölder inequality $|(a,b)| \leq ||a^\gamma||^{1/\gamma}||b^\delta||^{1/\delta}$ with $\gamma = \frac{p+q}{p}$ and $\delta = \frac{p+q}{p}$ we have

$$|(D_1B^1D_2B^1D_3B^1D_4)_{1,1}| \leq ||\overline{B}||||B^1||^2$$

$$\cdot(\frac{1}{N}\sum_i (C_i^*)^{8(l_2+l_3+l_4)})^{1/8}(\frac{1}{N}\sum_i C_i^{8(l'_2+l'_3+l'_4)})^{1/8}$$

Applying these estimates for the $(\ldots)_{11}$ terms we get the bound for T:

$$T \leq \mathrm{const}E\langle\langle(C_1)^{2p'_1}(C_1^*)^{2p_1}(\frac{1}{N}\sum_i C_i^{2r})^q(\frac{1}{N}\sum_i (C_i^*)^{2r'})^{q'}|\overline{F}_1||\overline{F}'_1|\rangle_*\rangle \quad (4.16)$$

Here $\overline{F}_1$ and $\overline{F}'_1$ are the terms that remain in the r.h.s. of (4.10) after estimating all $(\ldots)_{11}$ by the above procedure and p_1, p'_1, q, q', r, r' are some integers. Since C_1 and C_1^* come from $(D_1)_{11}$ but can come also from other terms of the same type, then $p'_1 \geq l'_1$ and $p_1 \geq l_1$.

Part B. *Estimate of the* $(B^1D_{k_l+1}\ldots D_M)_{i,i_M}$ *terms.*

The rest of our estimates will be concentrated from now on the terms $|\overline{F}_1|$, $|\overline{F}'_1|$ in the formula (4.16). We will analyze them in the case which is the "worst", i.e. in the case in which all the terms of the expansion belong to them or in other words we consider the case when there are no diagonal terms $(B^1D_{k_1+1}B^1\ldots D_{k_2})_{1,1}$. We are going to average first with respect to $\{\underline{\xi}_1\}$. Then we obtain an expression similar to the last factor in the r.h.s. of formula (4.8) but with at least one product B^1D less. We now explain in detail how this program is realized. First we note that, since one can drop from the sums $\frac{1}{N}\sum_i C_i^{2r}$ the terms with $i = 1$, we can consider these sums as independent of $\underline{\xi}_1$. Let ψ be any function of the patterns $\underline{\xi}_1$ then its double expectation $<< \psi >_*>$ can be written in a form useful for our program in which we use the following symbols and notations: $E_{\underline{\xi}_1}$ is the expectation with respect the random variables $\underline{\xi}_1$, H_1 (already introduced in section 1) is the full Hamiltonian of the system in which all the variables $\underline{\xi}_1$ are put equal to zero, H_{01}^* is the Hamiltonian containing only the quadratic and linear terms of the expansion around the mean field solution of $H(m^*)$ in which all the variables $\underline{\xi}_1$ are put equal to zero. Then we write

$$E_{\underline{\xi}_1}\langle\langle\psi(\underline{\xi}_1)\rangle\rangle_* =$$
$$= E_{\underline{\xi}_1}\left\{\frac{\langle\langle\psi\cosh(\beta(C_1+\xi_1^1(t+z))\exp\{\beta d\lambda(C_1^*)^2+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu\}\rangle_{H_1}\rangle_{H_{01}^*}}{\langle\cosh\beta(C_1+\xi_1^1(t+z))\rangle_{H_1}\langle\exp\{\beta d\lambda(C_1^*)^2+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu\}\rangle_{H_{01}^*}}\right\}$$
$$\leq E_{\underline{\xi}_1}\langle\langle\psi\cosh(\beta(C_1+\xi_1^1(t+z))\exp\{\beta d\lambda(C_1^*)^2+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu\}\rangle_{H_1}\rangle_{H_{01}^*}$$
$$\langle\exp\{-\beta z\sum_\mu\xi_1^\mu\xi_1^1\overline{m}^\mu\}\rangle_{H_{01}^*}. \tag{4.17}$$

In the above calculations we bounded the $\cosh\beta(C_1+\xi_1^1(t+z))$ in the denominator from below by 1, the other expectation

$$\langle\exp\{\beta d\lambda(C_1^*)^2+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu\}\rangle_{H_{01}^*}$$

has been bounded using Jensen's inequality $< A >^{-1}\leq< A^{-1} >$ and dropping the exponent $e^{-\beta d\lambda(C_1^*)^2}$. A third Hamiltonian $\overline{H}_{01}$ which is defined as H_{01}^* for variables $\overline{m}$ has been introduced in order to write the product of the two expectations appearing in (4.17) with only one average. Finally we get:

$$E_{\underline{\xi}_1}\langle\langle\psi(\underline{\xi}_1)\rangle_H\rangle_{H_0^*} = \langle\langle\langle\psi(\underline{\xi}_1)\cosh(\beta(C_1+\xi_1^1(t+z))$$
$$\exp\{\beta d\lambda(C_1^*)^2+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu-\beta z\sum_\mu\xi_1^\mu\xi_1^1\overline{m}^\mu\}\rangle_{H_1}\rangle_{H_{01}^*}\rangle_{\overline{H}_{01}}. \tag{4.18}$$

Since it is easier to average exponents which contain only linear terms we make a gaussian integral in order to linearize the quadratic term $\beta\lambda d(C_1^*)^2$. We use the formula:

$$E_{\underline{\xi}_1}\langle\langle\psi(\underline{\xi}_1)\rangle_{H_0^*}\rangle_H = \int\frac{dte^{-t^2/2}}{\sqrt{2\pi}}\cdot\langle\langle\langle\psi(\underline{\xi}_1)\cosh\beta(C_1+\xi_1^1(t+z))$$
$$\exp\{t\sqrt{\beta d\lambda}C_1^*+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu-\beta z\sum_\mu\xi_1^\mu\xi_1^1(\overline{m})^\mu\}\rangle_{\overline{H}_{01}(\overline{m})}\rangle_{H_{01}^*(m^*)}\rangle_{H_1(m)} \tag{4.19}$$

Applying these manipulations to our case we get that inside the triple average (4.19) we will have the expression:

$$U = F_1^2\cosh\beta(C_1+\xi_1^1(t+z))C_1^{2l_1'}(C_1^*)^{2l_1}$$
$$\exp\{t\sqrt{\beta d\lambda}C_1^*+\beta z\sum_\mu\xi_1^\mu\xi_1^1(m^*)^\mu-\beta z\sum_\mu\xi_1^\mu\xi_1^1\overline{m}^\mu\}$$
$$= (\sum_\mu\xi_1^\mu F^\mu)^2C_1^{2l_1'}(C_1^*)^{2l_1}\exp\{\sum_\mu\xi_1^\mu f^\mu\}$$

where we introduced the notations:

$$F^\mu = \frac{1}{\beta Nd}\sum_{i,i_M}(\overline{G}\underline{\xi}_i)^\mu(D_{k_l+1}B^1\dots D_M)_{ii_M}$$

$$f^\mu = \pm\beta(m^\mu + \delta_{\mu,1}(z+t)) + t\sqrt{\beta d\lambda}C_1^* + \beta z\sum_\mu \xi_1^\mu\xi_1^1(m^*)^\mu + \beta\sum_\mu \xi_1^\mu\xi_1^1\overline{m}^\mu$$

By the help of these symbols we write (4.19) in a shorter form:

$$E_{\underline{\xi}_1}(U) \le E_{\underline{\xi}_1}\{\sum_\mu (F^\mu)^2 C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\}$$

$$+ E_{\underline{\xi}_1}\{\sum_{\mu\nu} F^\mu F^\nu \frac{d^2}{d\xi_1^\mu d\xi_1^\nu} C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\} \tag{4.20}$$

where the derivative $\frac{d^2}{d\xi_1^\mu d\xi_1^\nu}$ stands for indicating all the possible way to extract the coefficients of the product $\xi_1^\mu\xi_1^\nu$ from the expression $C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})$. Calculating this term we get the sum

$$E_{\underline{\xi}_1}\{\sum_{\mu\nu} F^\mu F^\nu \exp(\underline{\xi}_1, \underline{f})(m^\mu m^\nu 2l_1'(2l_1'-1)C_1^{2l_1'-2}(C_1^*)^{2l_1}$$

$$+ (m^*)^\mu(m^*)^\nu 2l_1(2l_1-1)C_1^{2l_1'}(C_1^*)^{2l_1-2} + f^\mu f^\nu C_1^{2l_1'}(C_1^*)^{2l_1})\ldots\} \tag{4.21}$$

Each term in (4.20) can be estimated from above by means of the Schwarz inequality as in the example:

$$|\sum_{\mu\nu} F^\mu F^\nu f^\mu m^\nu| \le \sum_\mu (F^\mu)^2 M(\sum_\mu (f^\mu)^2)^{1/2}$$

Since by definition f^μ is a linear combination of $m^\mu, (m^*)^\mu$ and $\overline{m}^\mu$ we have to estimate expressions of the form:

$$E_{\underline{\xi}_1}\{\sum_\mu (F^\mu)^2 M^2 C_1^{2l_1'-2}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\}$$

$$E_{\underline{\xi}_1}\{\sum_\mu (F^\mu)^2 (M^*)^2 C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\}$$

$$E_{\underline{\xi}_1}\{\sum_\mu (F^\mu)^2 (\overline{M})^2 C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\} \tag{4.22}$$

Note that $F^\mu, (M^*)^2, (\overline{M})^2, M^2$ do not depend on $\underline{\xi}_1$ thus inside (4.22) the only terms which we have really to average are of the type:

$$E_{\underline{\xi}_1}\{C_1^{2l_1'}(C_1^*)^{2l_1}\exp(\underline{\xi}_1, \underline{f})\} \le$$

$$\le E_{\underline{\xi}_1}^{1/3}\{C_1^{6l_1'}\}E_{\underline{\xi}_1}^{1/3}\{(C_1^*)^{6l_1}\}(E_{\underline{\xi}_1})^{1/3}\{\exp 3(\underline{\xi}_1, \underline{f})\}$$

$$\le D^{1/3}(6l_1)D^{1/3}(6l_2)M^{2l_1'}(M^*)^{2l_1}\exp\{3\sum_\mu (f^\mu)^2\}$$

$$= D^{1/3}(6l_1)D^{1/3}(6l_2)M^{2l_1'}(M^*)^{2l_1}\exp\{K(M^2 + (M^*)^2t^2 + (\overline{M})^2)\} \tag{4.23}$$

where $D(n)$ is the function introduced in Lemma I.3. In the first of the inequality of (4.23) we used again Schwarz inequality using the first factor as a weight and then we used Holder inequality. We show here the intermediate steps of this calculation using the auxiliary variables x, y and z for simplicity:

$$E(xyz) \leq E^{1/2}(xy^2)E^{1/2}(xz^2) \leq E^{1/6}(x^3)E^{1/3}(y^3)E^{1/6}(x^3)E^{1/3}(z^3)$$
$$\leq E^{1/3}(x^3)E^{1/3}(y^3)E^{1/3}(z^3) \tag{4.24}$$

We used Holder inequality with $p = 3$, $q = 3/2$ in the second line of (4.24). We now estimate $\sum_\mu (F^\mu)^2$:

$$\sum_\mu (F^\mu)^2 = \frac{1}{\beta N d}\sum_\mu \sum_{i,i_M,j,j_M} (\overline{G}\underline{\xi}_i)^\mu(\overline{G}\underline{\xi}_j)^\mu (D_{k_l+1}\dots D_M)_{ii_M}(D_{k_l+1}B^1\dots D_M)_{jj_M}$$
$$= \frac{1}{\beta N d}\sum_{i,i_M,j,j_M} (B^1)^2_{ij}(D_{k_l+1}\dots D_M)_{ii_M}(D_{k_l+1}B^1\dots D_M)_{jj_M}$$
$$\leq \frac{1}{\beta N d}||B^1||^2\sum_{i=2}^{N}\Big(\sum_{i,i_M=2}(D_{k_l+1}B^1\dots D_M)_{ii_M}\Big)^2. \tag{4.25}$$

After the average over the $\underline{\xi}_1$ variables we get:

$$E\{T\} \leq \text{const}\,||B^1||^2 E\{(\frac{1}{N}\sum_i C_i^{2r})^q(\frac{1}{N}\sum_i (C_i^*)^{2r'})^{q'}$$
$$\cdot M^{2l_1'}(M^*)^{2l_1}\exp\{K(M^2+(M^*)^2t^2+(\overline{M})^2)\}$$
$$\cdot \sum_{i_M,j_M}(B^1D_{k_l+1}\dots D_M)_{2,i_M}(B^1D_{k_l+1}\dots D_M)_{2,j_M}\} \tag{4.26}$$

In (4.26) the factor $\frac{1}{N}$ which was in the last line of (4.25) has been cancelled together with the $\sum_i$ using the symmetry of E averages wih respect to the neuronal indices. Now we have got an expression which is of the same form of the last factor in (4.11), which is the goal that we were looking for, there is just one const in front of the term, but with no dependence on $\underline{\xi}_1$ and a B^1D product less. The terms which disappeared are substituted by new term which are not depending on $\underline{\xi}_1, \dots, \underline{\xi}_N$ like the factors $M^{2l_1'}(M^*)^{2l_1}\exp(K(M^2+(M^*)^2t^2+(\overline{M})^2))$. Since we got the same form we can apply again the same estimates M times and get at the end of all

$$E\{T\} \leq \text{const}\int\prod_{i=1}^{M}\frac{dt_ie^{-t_i^2/2}}{\sqrt{2\pi}}\cdot E\langle\langle\langle(\frac{1}{N}\sum_i C_i^{2r})^q(\frac{1}{N}\sum_i(C_i^*)^{2r'})^{q'}M^{2(l_1'+\dots+l_M')}(M^*)^{2(l_1+\dots+l_M)}$$
$$\exp\{K(M^2+(\overline{M})^2)+K(M^*)^2(t_1^2\dots+t_M^2)\}\rangle\rangle\rangle \tag{4.27}$$

In (4.27) the triple expectations $<<<\dots>>>$ is respect to the Gibbs measures associated respectively to the Hamiltonians H_M, H^*_{0M}, $\overline{H}_{0M}$ which are obtained from the Hamiltonians H, H_0 by dropping the terms containing $\xi_1,\dots,\xi_M$ according to our iterative procedure. We can then apply Lemma I.2, formula (2.12):

$$\left\langle M^n e^{BM^2}\right\rangle_{H_0} \leq (\tau_0\sqrt{\alpha})^n e^{B\alpha\tau_0}$$

putting it directly into the formula (4.27) from which we get the last inequality

$$E\{T\} \leq \text{const} \int \prod_{i=1}^{M} \frac{dt_i e^{-t_i^2/2}}{\sqrt{2\pi}} (\tau_0\sqrt{\alpha})^{2(l'_1+\ldots+l'_M+l_1+\ldots+l_M)}$$

$$\exp\left\{\alpha\tau_0 K(t_1^2\ldots+t_M^2)\right\} \leq \text{const}\,(\alpha)^{k+1} \tag{4.28}$$

Which is the statement that we wanted to proof. For completeness we give show also how to bound the norm of the $(B^1)^2$ operator.

$$||(B^1)^2|| \leq \sup_{||x||=1} \frac{1}{\beta N d}(\overline{G}^2 \sum_i \underline{\xi}_i x_i, \sum_i \underline{\xi}_i x_i) \leq \frac{1}{\beta d}||\overline{G}^2|| \sup_{||x||=1} \frac{1}{N}\sum_\mu (\sum_i x_i \xi_i^\mu)^2$$

$$\leq \frac{1}{\beta d}||\overline{G}^2|| \sup_{||x||=1} \sum_{ij} x_i x_j \frac{1}{N}\sum_\mu \xi_i^\mu \xi_j^\mu \leq \frac{1}{\beta d}||\overline{G}^2|| \, ||J||$$

5 α-expansion for Δ

In this section we apply the expansion that we have found in the previous section to the variance of r, the order parameter of the Hopfield model.

In fact we show that all the coefficients of the α expansion of the quantity Δ defined by

$$\Delta = E\left\{\sum_{\mu\nu}(< m^\mu m^\nu > - < m^\mu >< m^\nu >)^2\right\} \tag{5.1}$$

are vanishing in the thermodynamic limit. In [1] we have shown that the selfaveraging of the Edwards-Anderson parameter q implies the selfaveraging of the parameter r and $\Delta \to 0$. It is simple to invert these derivations and show that $\Delta \to 0$ implies selfaveraging of q and r. The Hamiltonian which has been used in [1] for this proof has an additional term $\epsilon\sqrt{N}\sum \gamma^\mu m^\mu$ where $\gamma^1, \ldots, \gamma^p$ are independent Gaussian variables with zero mean and variance 1. After the thermodynamical limit was done we send $\epsilon \to 0$. But if we add this term in our Hamiltonian then all derivations of Sections II and III remain valid since this term is linear in m^μ and vanishing when $\epsilon \to 0$. Thus if we prove that all coefficients of α expansion for Δ tend to zero, we can conclude that all coefficients of variance of parameters q and r tend to zero and then the result [1] gives us that replica symmetry equations are true when $\alpha \to 0$ in any order of α. This conclusion is important for the rigorous foundation of the replica solution of the Hopfield model. We state the main result of this section in a theorem:

Theorem 2 *All the coefficient of the α expansion of Δ vanish when $N \to \infty$.*

Proof.

We use the expansion (3.5) and its analog for $< m^\mu m^\nu >$, but since now we are considering only terms of order less then α^k we will not consider the term R^μ and therefore only averages $< \ldots >_0$ will be used in this Section. Thus

$$< m^\mu m^\nu > = < m^\mu >_0 < m^\nu >_0 + \frac{1}{Nd\beta} G^{\mu\nu} - \frac{1}{Nd} < m^\nu >_0 \sum_{i_1} (G\xi_{i_1})^\mu \sum_{l_1=2}^{2k} a_{l_1} \left\langle C_{i_1}^{l_1} \right\rangle_0$$

$$- \frac{1}{Nd} < m^\mu >_0 \sum_{i'_1} (G\xi_{i'_1})^\nu \sum_{l'_1=2}^{2k} a_{l'_1} \left\langle C_{i'_1}^{l'_1} \right\rangle_0$$

$$+ \frac{1}{N^2 d^2} \sum_{i_1 i'_1} (G\xi_{i_1})^\mu (G\xi_{i'_1})^\nu \sum_{l_1 l_2} a_{l_1} a_{l_2} \left\langle C_{i_1}^{l_1-1} C_{i_2}^{l_2-1} \right\rangle_0 + \ldots \tag{5.2}$$

Comparing with the expression (3.5) for $< m^\mu >$ one can conclude that up to terms of order greater than k :

$$\langle (m^\mu - < m^\mu >)(m^\nu - < m^\nu >) \rangle = \frac{1}{Nd\beta} G^{\mu\nu} + L^{\mu\nu}$$

where $L^{\mu\nu}$ is a linear combination of terms the type:

$$t^{\mu\nu} \le \frac{1}{(\beta N d)^2} \sum_{i_1, \ldots i_M, j_1, \ldots, j_S} (G\underline{\xi}_{i_1})^\mu (G\underline{\xi}_{j_1})^\nu \langle \dot{C}_{i_1} \dot{C}_{i_2} \rangle_0 \langle \dot{C}_{i_2} \dot{C}_{i_3} \rangle_0 \cdot \ldots \langle \dot{C}_{i_{M-1}} \dot{C}_{i_M} \rangle_0$$

$$\cdot \langle \dot{C}_{j_1} \dot{C}_{j_2} \rangle_0 \langle \dot{C}_{j_2} \dot{C}_{j_3} \rangle_0 \cdot \ldots \langle \dot{C}_{j_{S-1}} \dot{C}_{j_S} \rangle_0 \cdot [\langle C_{i_1}^{l_1} \cdot \ldots C_{i_M}^{l_M} C_{j_1}^{l'_1} \cdot \ldots C_{j_S}^{l'_S} \rangle_0$$

$$- \langle C_{i_1}^{l_1} \cdot \ldots C_{i_M}^{l_M} \rangle_0 \langle C_{j_1}^{l'_1} \cdot \ldots C_{j_S}^{l'_S} \rangle_0] \tag{5.3}$$

According to the Wick theorem the difference in the square parenthesis is the sum of terms of such a form:

$$\langle \dot{C}_{i_r} \dot{C}_{j_v} \rangle_0 \langle C_{i_1}^{l_1} \ldots C_{i_r-1}^{l_r-1} \ldots C_{i_M}^{l_M} C_{j_1}^{l'_1} \ldots C_{j_v}^{l'_v-1} \ldots C_{j_S}^{l'_S} \rangle_0$$

Therefore in order to estimate $E\left\{ \sum_{\mu\nu} (L^{\mu\nu})^2 \right\}$ it is enough to estimate:

$$T_{rv} = \frac{1}{(\beta N d)^4} E \sum_{i_1, \ldots i_M, j_1, \ldots, j_S} \sum_{i'_1, \ldots i'_M, j'_1, \ldots, j'_S}$$

$$(G^2 \underline{\xi}_{i_1}, \underline{\xi}_{j_1})(G^2 \underline{\xi}_{i'_1}, \underline{\xi}_{j'_1}) \langle \dot{C}_{i_1} \dot{C}_{i_2} \rangle_0 \langle \dot{C}_{i_2} \dot{C}_{i_3} \rangle_0 \cdot \ldots \langle \dot{C}_{i_{M-1}} \dot{C}_{i_M} \rangle_0$$

$$\cdot \langle \dot{C}_{j_1} \dot{C}_{j_2} \rangle_0 \langle \dot{C}_{j_2} \dot{C}_{j_3} \rangle_0 \cdot \ldots \langle \dot{C}_{j_{S-1}} \dot{C}_{j_S} \rangle_0 \langle \dot{C}_{i'_1} \dot{C}_{i'_2} \rangle_0 \langle \dot{C}_{i'_2} \dot{C}_{i'_3} \rangle_0 \cdot \ldots \langle \dot{C}_{i'_{M-1}} \dot{C}_{i'_M} \rangle_0$$

$$\cdot \langle \dot{C}_{j'_1} \dot{C}_{j'_2} \rangle_0 \langle \dot{C}_{j'_2} \dot{C}_{j'_3} \rangle_0 \cdot \ldots \langle \dot{C}_{j'_{S-1}} \dot{C}_{j'_S} \rangle_0$$

$$\langle \dot{C}_{i_r} \dot{C}_{j_v} \rangle_0 \langle C_{i_1}^{l_1} \ldots C_{i_r-1}^{l_r-1} \ldots C_{i_M}^{l_M} C_{j_1}^{l'_1} \ldots C_{j_v}^{l'_v-1} \ldots C_{j_S}^{l'_S} \rangle_0$$

$$\langle \dot{C}_{i'_r} \dot{C}_{j'_v} \rangle_0 \langle C_{i'_1}^{l_1} \ldots C_{i'_r-1}^{l_r-1} \ldots C_{i'_M}^{l_M} C_{j'_1}^{l'_1} \ldots C_{j'_v}^{l'_v-1} \ldots C_{j'_S}^{l'_S} \rangle_0 \tag{5.4}$$

Following the ideas and notations of Section 4 we introduce the matrix B^0, already used in the preceeding section, and the matrix $B^{(2)}$. Then we need also the diagonal matrices D, similar to the matrix D_k of Section 4 but with a simpler structure, and the matrices $F^{(1)}$, $F^{(2)}$. These matrices are defined in such a way that we can apply the inequality (4.13) to the quantity T_{rv}.

$$B^0_{ij} = \frac{1}{\beta N d}(G\underline{\xi}_i, \underline{\xi}_j) = \langle \dot{C}_i \dot{C}_j \rangle_0$$

$$B^{(2)}_{ij} = \frac{1}{\beta N d}(G^2\underline{\xi}_i, \underline{\xi}_j)$$

$$D_{ij} = \delta_{ij} C_i F^{(1)}_{ij} = \delta_{ij} \sum_l (B^0 D^{l_r-1} B^0 \ldots D^{l_M-1} B^0)_{il} C_l^{l_M}$$

$$F^{(2)}_{ij} = \delta_{ij} \sum_l (B^0 D^{l'_v-1} B^0 \ldots D^{l'_S-1} B^0)_{il} C_l^{l'_S} \tag{5.5}$$

In fact we can rewrite (5.4) in the more suitable form:

$$T_{rv} = \frac{1}{(\beta N d)^2} E\{\mathrm{Tr} F^{(1)} D^{l_{r-1}} B^0 \ldots D^{l_1} B^{(2)} D^{l_1} \ldots D^{l_{r-1}}) F^{(1)} B^0$$

$$\cdot F^{(2)} (D^{l'_{v-1}} B^0 \ldots D^{l'_1} B^{(2)} D^{l'_1} B^0 \ldots D^{l'_{v-1}}) F^{(2)} B^0\} \tag{5.6}$$

Now we are able to apply the above mentioned identities and get

$$T_{rv} \leq \text{const}\, \frac{1}{N} E^{1/2}\{\frac{1}{N}\mathrm{Tr}\langle D^{2(l_{r-1}+\ldots+l_1)} (F^{(1)})^4 \rangle_0\}$$

$$\cdot E^{1/2}\{\frac{1}{N}\mathrm{Tr}\langle D^{2(l'_{v-1}+\ldots+l'_1)} (F^{(2)})^4 \rangle_0\}$$

Repeating almost literally the arguments of the previous section it is easy to show that the quantities

$$E\{\frac{1}{N}\mathrm{Tr}\langle D^{2L}(F^{(1)})^4\rangle_0\} = E\{\langle D^{2L}(F^{(1)})^4_{11}\rangle_0\} E\{\frac{1}{N}\mathrm{Tr}\langle D^{2L}(F^{(2)})^4\rangle_0\}$$

$$= E\{\langle D^{2L}(F^{(2)})^4_{11}\rangle_0\}$$

are uniformly bounded in N. Then we got that

$$T_{rv} \leq \text{const}\, \frac{1}{N}$$

and therefore it is proven that all the coefficients of the α expansion of $\Delta \longrightarrow 0_{N\to\infty}$. Finally, if we use the result of [1] according to which the replica symmetric saddle-point equations are true up to terms of the order $\mathrm{O}(\Delta^{1/4})$, we can conclude that the replica symmetric solution is true in any order of α and that it gives us all the coefficients of the α expasion of the order parameters r, q and m^1.

6 Conclusions

As it was mentioned above, our results imply that the replica simmetric solution for the Hopfield model [2] is true at any order of α. Since one can see that the rest of our expansion satisfies the inequality

$$ET_n^2 \leq n!(c\alpha)^n$$

then we cannot derive from it the convergence of the series. Of course, having the analiticity for $\alpha \neq 0$ of our observables we could get the Borel summability, unfortunately it is not possible to have this property because α is defined only as a rational number and doesn't appear explicitely in the formulae. From the other side we cannot expect to have analytic behaviour directly from the solution of the saddle point equations in the replica symmetry case:

$$m^\nu = E\int \frac{dze^{-z^2/2}}{\sqrt{2\pi}}\xi^\nu \tanh\beta(\sqrt{\alpha r}z + \sum_{\mu=1}^{s}(m^\mu + h^\mu)\xi^\mu)$$

$$q = E\int \frac{dze^{-z^2/2}}{\sqrt{2\pi}}\tanh^2\beta(\sqrt{\alpha r}z + \sum_{\mu=1}^{s}(m^\mu + h^\mu)\xi^\mu)$$

$$r = \frac{q}{(1-\beta(1-q))^2}$$

In fact from these equations we evidently see the non-analyticity at the point $\alpha = 0$ even if we take as independent variable $\sqrt{\alpha}$ and so we cannot expect to have convergent series or Borel summability expanding directly. But it is interesting to note that if we add to the Hamiltonian (4.1) a term $\sum_i h_i S_i$ with h_i independent random gaussian r.v with zero mean and variance h^2 then the above system becomes:

$$m^\nu = E\int \frac{dze^{-z^2/2}}{\sqrt{2\pi}}\xi^\nu \tanh\beta(\sqrt{\alpha r + h^2}z + \sum_{\mu=1}^{s}(m^\mu + h^\mu)\xi^\mu)$$

$$q = E\int \frac{dze^{-z^2/2}}{\sqrt{2\pi}}\tanh^2\beta(\sqrt{\alpha r + h^2}z + \sum_{\mu=1}^{s}(m^\mu + h^\mu)\xi^\mu)$$

$$r = \frac{q}{(1-\beta(1-q))^2}$$

which now has analytic solutions for $\alpha = 0$. Therefore we can hope to construct for such a model a convergent series for m^ν, q, r. The starting point in this direction is to have an analogue of the result [6] for this model with Gaussian random terms, i.e. to find the limit when $\alpha \to 0$ of the limit free-energy $f(\alpha, h)$. This program will be completed in our following papers.

References

[1] Pastur, L., Shcherbina, M., Tirozzi, B.: *The replica symmetric solution of the Hopfield model without replica trick.*, J. of Stat. Phys., **74, 5/6**, (1994), pp 1167-1183

[2] Amit, D., Gutfreund, H., Sompolinsky, H.: *Statistical mechanics of neural networks near saturation.*, Ann. Phys. **173**, (1987), pp. 30.

[3] Figotin,A., Pastur, L.: *On the theory of disordered spin system.*, Theor. Math. Phys. **35**, (1978), pp. 193-204.

[4] Pastur, L., Shcherbina, M.: *The absence of self-averageness of the order parameter in the Sherrington -Kirkpatrick model.*, J. Stat. Phys., **62**, (1991), pp.1-19.

[5] Mezard, M., Virasoro, M., A., Parisi, G.: *Spin glass theory and beyond.*, World Scientific, Singapore(1987)

[6] Shcherbina, M., Tirozzi, B.: *The free-energy of a class of Hopfield models.*, J. of Stat. Phys., **72, 1/2**, (1993), pp 113-125.

[7] Feng J., Tirozzi B.: *The SLLN for the free-energy of a class of Neural Networks.*, to be published in Helvetica Physica Acta

[8] Bovier, A., Gayrard, G., Picco, P.,: *Gibbs states of the Hopfield model with extensively many patterns.*, (1994), preprint

[9] Scacciatelli, E., Tirozzi, B.: *Fluctuations of the free- energy in the Hopfield model.*, J. of Stat. Phys., **67**, (1991), pp 981-1008

[10] Shcherbina, M., Tirozzi, B.: *Saddle point equations of a neural network with correlated attractors.*, J. Phys. I France, **4**, (1994), pp 629-633.

[11] Amit, D., Griniasty, M., Tsodyks, M.: *Conversion of temporal correlations between stimuli to spatial correlations between attractors.*, Neural Computation, **6**, (1994) p. 220.

[12] Cugliandolo, L.F., Tsodyks, M.: *Capacity of networks with correlated attractors.*, J. Phys. A (1993), p. 367.

[13] Englisch, H., Mastropietro, H., Tirozzi, B.: *The B. A. M. storage capacity.*, J. Phys. I France, **5**, (1995), pp 1-12.

[14] Van Hemmen, J. L.: *Spin-glass methods of neural networks.*, Phys. Rev., **A 34**, (1986), pp 3435-3445.

[15] Albeverio, S., Tirozzi, B., Zegarlinski, B.: *Rigorous results for the free-energy in the Hopfield model.*, Comm. Math. Phys., **150**, (1992), pp 337-373.

[16] Amit, D., Gutfreund, H., Sompolinsky, H.: *Spin glass models of neural networks.*, Phys. Rev **A 32**, (1985), pp. 1007-1018.

[17] Koch, H., Piasko, J.: *Some rigorous results on the Hopfield neural network model.*, J. of Stat. Phys., **55**, (1993), pp 903-928.